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Introduction to Mathematical Logic

Hyper-textbook for students

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Sections 1, 2, 3 of this book represent an extended translation of the corresponding chapters of the book: **V. Detlovs, Elements of Mathematical Logic,** Riga, University of Latvia, 1964, 252 pp. (in Latvian). With kind permission of Dr. Detlovs.

Vilnis Detlovs. Memorial Page

In preparation – forever (however, since 2000, used successfully in a real logic course for computer science students).

This hyper-textbook contains links to:
Wikipedia, the free encyclopedia;
MacTutor History of Mathematics archive
of the University of St Andrews;
MathWorld of Wolfram Research.

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References

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1. Introduction. What Is Logic, Really?

WARNING! In this book,

predicate language is used as a synonym of **first order language**,

formal theory – as a synonym of **formal system, deductive system,**

constructive logic – as a synonym of **intuitionistic logic,**

algorithmically solvable – as a synonym of **recursively solvable,**

algorithmically enumerable – as a synonym of **recursively enumerable.**

What is logic?

See also Factasia Logic by Roger Bishop Jones.

In a sense, logic represents the most general **means of reasoning** used by people and computers.

Why are means of reasoning important? Because any body of data may contain not only facts visible directly. For example, assume the following data: the date of birth of some person X is January 1, 2000, and yesterday, September 14, 2010 some person Y killed some person Z. Then, most likely, X did not kill Z. This conclusion is not represented in our data directly, but can be derived from it by using some means of reasoning – **axioms** ("background knowledge") and **rules of inference**. For example, one may use the following statement as an axiom: "Most likely, a person of age 10 can´t kill anybody".

There may be means of reasoning of different levels of generality, and of different ranges of applicability. The above "killer axiom" represents the lowest level – it is a very specific statement. But one can use laws of physics to derive conclusions from his/her data. Theories of physics, chemistry, biology etc. represent a more general level of means of reasoning. But can there be means of reasoning applicable in almost every situation? This – the most general – level of means of reasoning is usually regarded as **logic**.

Is logic absolute (i.e. unique, predestined) or relative (i.e. there is more than one kind of logic)? In modern times, an absolutist position is somewhat inconvenient – you must defend your "absolute" concept of logic against heretics and dissidents, but very little can be done to exterminate these people. They may freely publish their concepts on the Internet.

So let us better adopt the relativist position, and define logic(s) as **any common framework for building theories**. For example, the so-called

absolute geometry can be viewed as a common logic for both the Euclidean and non-Euclidean geometry. Group axioms serve as a common logic for theories investigating mathematical structures that are subtypes of groups. And, if you decide to rebuild all mathematical theories on your favorite set theory, then you can view set theory as your logic.

Can there be a common logic for the entire mathematics? To avoid the absolutist approach let us appreciate all the existing concepts of mathematics – classical (traditional), constructivist (intuitionist), New Foundations etc. Of course, enthusiasts of each of these concepts must propose some specific common framework for building mathematical theories, i.e. some specific kind of logic. And they do.

Can set theory (for example, currently, the most popular version of it – Zermelo-Fraenkel's set theory) be viewed as a common logic for the classical (traditional) mathematics? You may think so, if you do not wish to distinguish between the first order notion of natural numbers (i.e. discrete mathematics) and the second order notion (i.e. "continuous" mathematics based on set theory or a subset of it). Or, if you do not wish to investigate in parallel the classical and the constructivist (intuitionist) versions of some theories.

1.1. Total Formalization is Possible!

Gottlob Frege (1848-1925)
Charles S. Peirce (1839-1914)
Bertrand Russell (1872-1970)
David Hilbert (1862-1943)

How far can we proceed with the mathematical rigor – with the axiomatization of some theory? Complete elimination of intuition, i.e. full reduction of all proofs to a list of axioms and rules of inference, is this really possible? The work by Gottlob Frege, Charles S. Peirce, Bertrand Russell, David Hilbert and their colleagues showed how this can be achieved even with the most complicated mathematical theories. All mathematical theories were indeed reduced to systems of axioms and rules of inference without any admixture of sophisticated human skills, intuitions etc. Today, the logical techniques developed by these brilliant people allow ultimate axiomatization of any theory that is based on a stable, self-consistent system of principles (i.e. of any mathematical theory).

What do they look like – such "100% rigorous" theories? They are called **formal theories** (the terms “formal systems” and “deductive systems” also are used) emphasizing that no step of reasoning can be done without a reference to an exactly formulated list of axioms and rules of inference. Even the most "self-evident" logical principles (like, "if *A* implies *B*, and *B* implies *C*, then *A* implies *C*") must be either formulated in the list of axioms and rules explicitly,

or derived from it.

What is, in fact, a mathematical theory? It is an **"engine" generating theorems**. Then, a formal theory must be an "engine" generating theorems without involving of human skills, intuitions etc., i.e. by means of a precisely defined algorithm, or a computer program.

The first distinctive feature of a formal theory is a **precisely defined** ("formal") **language** used to express its **propositions**. "Precisely defined" means here that there is an algorithm allowing to determine, is a given character string a correct proposition, or not.

The second distinctive feature of a formal theory is a precisely defined ("formal") notion of **proof**. Each proof proves some proposition, that is called (after being proved) a **theorem**. Thus, theorems are a subset of propositions.

It may seem surprising to a mathematician, but the most general exact definition of the "formal proof" involves neither axioms, nor inference rules. Neither "self-evident" axioms, nor "plausible" rules of inference are distinctive features of the "formality". Speaking strictly, "self-evident" is synonymous to "accepted without argumentation". Hence, axioms and/or rules of inference may be "good, or bad", "true, or false", and so may be the theorems obtained by means of them. The only definitely verifiable thing is here the fact that some theorem has been, indeed, **proved** by using some set of axioms, and by means of some set of inference rules.

Thus, the second distinctive feature of "formality" is the possibility to **verify the correctness of proofs** mechanically, i.e. without involving of human skills, intuitions etc. This can be best formulated by using the (since 1936 – precisely defined) notion of algorithm (a "mechanically applicable computation procedure"):

A theory *T* is called a formal theory, if and only if there is an algorithm allowing to verify, is a given text a correct proof via principles of *T*, or not. If somebody is going to publish a "mathematical text" calling it "*proof of a theorem in theory T*", then we must be able to verify it mechanically whether the text in question is really a correct proof according to the standards of proving accepted in theory *T*. Thus, in a formal theory, the standards of reasoning should be defined precisely enough to enable verification of proofs by means of a precisely defined algorithm, or a computer program. (Note that we are discussing here **verification of ready proofs**, and not the much more difficult problem – is some proposition provable in *T* or not, see Exercise 1.1.5 below and the text after it).

Axioms and rules of inference represent only one (but the most popular!) of the possible techniques of formalization.

As an unpractical example of a formal theory let us consider the *game of chess*, let us call this "theory" *CHESS*. Let 's define as **propositions** of *CHESS* all the possible positions – i.e. allocations of some of the pieces (kings included) on a chessboard – plus the flag: "whites to move" or "blacks to move". Thus, the set of all the possible positions represents the **language** of *CHESS*. The only **axiom** of *CHESS* is the initial position, and the **rules of inference** – the rules of the game. Rules allow passing from some propositions of *CHESS* to some other ones. Starting with the axiom and iterating moves allowed by the rules we obtain **theorems** of *CHESS*. Thus, theorems of *CHESS* are defined as all the possible positions (i.e. propositions of *CHESS*) that can be obtained from the initial position (the axiom of *CHESS*) by moving pieces according to the rules of the game (i.e. by using the inference rules of *CHESS*).

Exercise 1.1.1 (optional). Could you provide an unprovable proposition of *CHESS*?

Why is *CHESS* called a formal theory? When somebody offers a "mathematical text" *P* as a proof of a theorem *A* in *CHESS*, this means that *P* is a record of some chess-game stopped in the position *A*. Checking the correctness of such "proofs" is a boring, but an easy task. The rules of the game are formulated precisely enough – we could write a computer program that will execute the task.

Exercise 1.1.2 (optional). Try estimating the size of this program in some programming language.

Our second example of a formal theory is only a bit more serious. It was proposed by Paul Lorenzen, so let us call this theory *L*. **Propositions** of *L* are all the possible "words" made of letters *a, b,* for example: *a, b, aa, aba, baab*. Thus, the set of all these "words" is the **language** of *L*. The only **axiom** of *L* is the word *a*, and *L* has two **rules of inference**: $X \vdash Xb$, and $X \vdash aXa$. This means that (in *L*) from a proposition *X* we can infer immediately the propositions *Xb* and *aXa*. For example, the proposition *aababb* is a **theorem** of *L*:

$$a \vdash ab \vdash aaba \vdash aabab \vdash aababb$$
$$\text{rule1} \quad \text{rule2} \quad \text{rule1} \quad \text{rule1}$$

This fact is expressed usually as $L \vdash aababb$ ("L proves *aababb*", $\vdash$ being a "fallen T").

Exercise 1.1.3. a) Verify that *L* is a formal theory. (Hint: describe an algorithm allowing to determine, is a sequence of propositions of *L* a correct proof, or not.)

b) (P. Lorenzen) Verify the following property of theorems of L: for any *X*,

if $L \vdash X$, then $L \vdash aaX$.

One of the most important properties of formal theories is given in the following

Exercise 1.1.4. Show that the **set of all theorems of a formal theory is algorithmically enumerable**, i.e. show that, for any formal theory T, a algorithm A_T can be defined that prints out on an (endless) paper tape all theorems of this theory (and nothing else). (Hint: we will call *T* a formal theory, if and only if we can present an algorithm for checking texts as correct proofs via principles of reasoning of *T*. Thus, assume, you have 4 functions: *GenerateFirstText()* – returns *Text*, *GenerateNextText()* – returns *Text*, *IsCorrectProof(Text)* – returns *true* or *false*, *ExtractTheorem(Text)* – returns *Text*, and you must implement the functions *GenerateFirstTheorem()* – returns *Text*, *GenerateNextTheorem()* – returns *Text*).

Unfortunately, **such algorithms and programs cannot solve the problem that the mathematicians are mainly interested in: is a given proposition *A* provable in *T* or not?** When, executing the algorithm A_T, we see our proposition *A* printed out, this means that *A* is provable in *T*. Still, in general, until that moment we cannot know in advance whether *A* will be printed out some time later or it will not be printed at all.

Note. According to the official terminology, algorithmically enumerable sets are called "recursively enumerable sets", in some texts – also "listable sets".

Exercise 1.1.5. a) Describe an algorithm determining whether a proposition of *L* is a theorem or not.

b) (optional) Could you imagine such an algorithm for *CHESS*? Of course, you can, yet... Thus you see that even, having a relatively simple algorithm for *checking the correctness* of proofs, the problem of *provability* can be a very complicated one.

T is called a **solvable theory** (more precisely – **algorithmically solvable theory**), if and only if there is an algorithm allowing to check whether some proposition is *provable* by using the principles of *T* or not. In the Exercise 1.1.5a you proved that *L* is a solvable theory. Still, in the Exercise 1.1.5b you established that it is hard to state whether *CHESS* is a "feasibly solvable" theory or not. **Determining the provability of propositions is a much more complicated task than checking the correctness of ready proofs.** It can be proved that **most mathematical theories are unsolvable**, the elementary (first order) arithmetic of natural numbers and set theory included (see, for example, Mendelson [1997], or Podnieks [1997], Section 6.3). I.e. there is no algorithm allowing to determine, is some arithmetical proposition provable from the axioms of arithmetic, or not.

Note. According to the official terminology, algorithmically solvable sets are called "recursive sets".

Normally, mathematical theories contain the negation symbol ***not***. In such theories solving the problem stated in a proposition *A* means proving either *A,* or proving ***not****A* ("disproving *A*", "refuting *A*"). We can try to solve the problem by using the enumeration algorithm of the Exercise 1.1.4: let us wait until *A* or ***not****A* is printed. If *A* and ***not****A* will be printed both, this will mean that *T* is an **inconsistent theory** (i.e. using principles of *T* one can prove some proposition and its negation). In general, we have here 4 possibilities:

a) *A* will be printed, but ***not****A* will not (then the problem *A* has a positive solution),

b) ***not****A* will be printed, but *A* will not (then the problem *A* has a negative solution),

c) *A* and ***not****A* will be printed both (then *T* is an inconsistent theory),

d) neither *A*, nor ***not****A* will be printed.

In the case d) we may be waiting forever, yet nothing interesting will happen: using the principles of *T* one can neither prove nor disprove the proposition *A,* and for this reason such a theory is called an **incomplete theory**. The famous incompleteness theorem proved by Kurt Gödel in 1930 says that **most mathematical theories are either inconsistent or incomplete** (see Mendelson [1997] or Podnieks [1997], Section 6.1).

Exercise 1.1.6. Show that any (simultaneously) consistent and complete formal theory is solvable. (Hint: use the algorithm of the Exercise 1.1.4, i.e. assume that you have the functions *GenerateFirstTheorem()* − returns *Text*, *GenerateNextTheorem()* − returns *Text*, and implement the function *IsProvable(Text)* – returns *true* or *false*). Where the consistency and completeness come in?

Exercise 1.1.7 (optional). a) Verify that "fully axiomatic theories" are formal theories in the sense of the above general definition. (Hint: assume, that you have the following functions: *GenerateFirstText()* − returns *Text*, *GenerateNextText()* − returns *Text*, *IsPropositon(Text)* − returns *true* or *false*, *IsAxiom(Proposition)* − returns *true* or *false*, there is a finite list of inference rule names: $\{R_1, ..., R_n\}$, function *Apply(RuleName, ListOfPropositions)* − returns *Proposition* or *false*, and you must implement the functions *IsCorrectProof(ListOfPropositions)* − returns *true* or *false*, *ExtractTheorem(Proof)* − returns *Proposition*).

b) (for smart students) What, if, instead of $\{R_1, ..., R_n\}$, we would have an infinite list of inference rules, i.e. functions GenerateFirstRule(),

GenerateNextRule() returning RuleName?

1.2. Predicate Languages

History

For a short overview of the history, see Quantification.

See also:

Aristotle (384-322 BC) – in a sense, the "first logician", "... was not primarily a mathematician but made important contributions by systematizing deductive logic." (according to MacTutor History of Mathematics archive).

Gottlob Frege (1848-1925) – "In 1879 Frege published his first major work *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (Conceptual notation, a formal language modelled on that of arithmetic, for pure thought). A.George and R Heck write: ... *In effect, it constitutes perhaps the greatest single contribution to logic ever made and it was, in any event, the most important advance since* Aristotle. ... In this work Frege presented for the first time what we would recognise today as a logical system with negation, implication, universal quantification, essentially the idea of truth tables etc." (according to MacTutor History of Mathematics archive).

Charles Sanders Peirce (1839-1914): "... He was also interested in the Four Colour Problem and problems of knots and linkages... He then extended his father's work on associative algebras and worked on mathematical logic and set theory. Except for courses on logic he gave at Johns Hopkins University, between 1879 and 1884, he never held an academic post." (according to MacTutor History of Mathematics archive).

Hilary Putnam. Peirce the Logician. *Historia Mathematica*, Vol. 9, 1982, pp. 290-301 (an online excerpt available, published by John F. Sowa).

Richard Beatty. Peirce's development of quantifiers and of predicate logic. *Notre Dame J. Formal Logic*, Vol. 10, N 1 (1969), pp. 64-76.

Geraldine Brady. From Peirce to Skolem. A Neglected Chapter in the History of Logic. Elsevier Science: North-Holland, 2000, 2000, 625 pp. (online overview at http://www.elsevier.com/wps/find/bookdescription.cws_home/621535/description#description).

When trying to formalize some piece of our (until now – informal) knowledge, how should we proceed? We have an informal vision of some domain consisting of “objects”. When speaking about it, we are uttering various *propositions*, and some of these propositions are regarded as “true” statements about the domain.

Thus, our first formalization task should be defining of some *formal language*, allowing to put all our propositions about the domain in a uniform and precise

way.

After this, we can start considering propositions that we are regarding as "true" statements about the domain. There may be an infinity of such statements, hence, we can't put down all of them, so we must organize them somehow. Some minimum of the statements we could simply declare as *axioms*, the other ones we could try to derive from the axioms by using some *rules of inference*.

As the result, we could obtain a *formal theory* (in the sense of the previous Section).

In mathematics and computer science, the most common approach to formalization is by using of the so-called **predicate languages**, first introduced by G. Frege and C. S. Peirce.

(In many textbooks, they are called **first order languages**, see below the warning about second order languages.)

Usually, linguists analyze the sentence "John loves Britney" as follows: *John* – subject, *loves* – predicate, *Britney* – object. The main idea of predicate languages is as follows: instead, let us write *loves(John, Britney)*, where *loves(x, y)* is a two-argument predicate, and *John*, and *Britney* both are objects. Following this way, we could write *=(x, y)* instead of *x=y*. This approach – reducing of the human language sentences to variables, constants, functions, predicates and **quantifiers** (see below), appears to be flexible enough, and it is much more uniform when compared to the variety of constructs used in the natural human languages. A unified approach is much easier to use for communication with computers.

Another example: "Britney works for BMI as a programmer". In a predicate language, we must introduce a 3-argument predicate "x works for y as z", or works(x, y, z). Then, we may put the above fact as: works(Britney, BMI, Programmer).

Language Primitives

Thus, the informal vision behind the notion of predicate languages is centered on the so-called "domain" – a (non-empty?) collection of "objects", their "properties" and the "relations" between them, that we wish to "describe" (or "define"?) by using the language. This vision serves as a guide in defining the language precisely, and further – when selecting axioms and rules of inference.

Object Variables

Thus, the first kind of language elements we will need are **object variables** (sometimes called also individual variables, or simply, variables). We need an unlimited number of them):

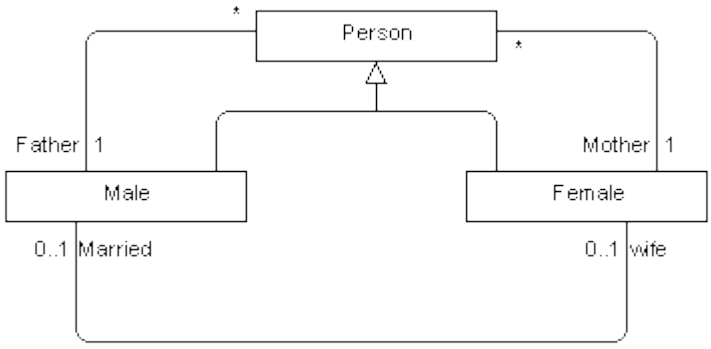
$$x, y, z, x_1, y_1, z_1, ...$$

The above-mentioned "domain" is the intended "range" of all these variables.

Examples. 1) Building a language that should describe the "domain of people", we must start by introducing "variables for people": x denotes an arbitrary person.

2) Building the language of the so-called **first order arithmetic**, we are thinking about "all natural numbers" as the range of variables: 0, 1, 2, 3, 4, ... (x denotes an arbitrary natural number).

3) Building the language of **set theory**, we think about "all sets" as the range of variables: x denotes an arbitrary set.



"*Domain of people" represented as UML class diagram*

Note. Since our screens and printers allow only a limited number of pixels per inch, in principle, we should generate variable names by using a finite set of characters, for example, by using a single letter x:

x, xx, xxx, xxxx, xxxxx, ...

Object Constants

The next possibility we may wish to have in our language are the so-called **object constants** (sometimes called individual constants, constant letters, or simply, constants) – names or symbols denoting some specific "objects" of our "domain".

Examples. 1) In our "language for people" we may introduce constants denoting particular people: John, Britney etc.

2) In the language of first order arithmetic, we may wish to introduce two constants – 0 and 1 to denote "zero" and "one" – two natural numbers having specific properties.

3) In the language of set theory, we could introduce a constant denoting the empty set, but there is a way to do without it as well (for details, Podnieks [1997], Section 2.3).

Function Constants

In some languages we may need also the so-called **function constants** (sometimes called function letters) – names or symbols denoting specific functions, i.e. mappings between "objects" of our "domain", or operations on these objects.

Examples. 1) In our "language for people" we will not use function constants.

2) In the language of first order arithmetic, we will introduce only two function constants "+" and "*" denoting the usual addition and multiplication of natural numbers, i.e. the two-argument functions x+y and x*y.

3) In the language of set theory, we could introduce function constants denoting set intersections $x \cap y$, unions $x \cup y$, set differences x–y, power sets P(x) etc., but there is a way to do without these symbols as well (for details, Podnieks [1997], Section 2.3).

In mathematics, normally, we are writing *f(x, y)* to denote the value of the function *f* for the argument values x, y. This (the so-called "prefix" notation) is a uniform way suitable for functions having any number of arguments: *f(x), g(x, y), h(x, y, z)* etc. In our everyday mathematical practice some of the two-argument functions (in fact, operations) are represented by using the more convenient "infix" notation (x+y, x*y instead of the uniform +(x, y), *(x, y), etc.).

Note. In a sense, object constants can be viewed as a special case of function constants – an object constant is a “zero-argument function”.

Predicate Constants

The last (but the most important!) kind of primitives we need in our language are the so-called **predicate constants** (sometimes called predicate letters) – names or symbols denoting specific **properties** (of) or **relations** between "objects" of our "domain".

Note. Using "predicate" as the unifying term for "property" and "relation" may seem somewhat unusual. But some kind of such unifying term is necessary. Properties are, in fact, unary (i.e. one-argument) "predicates", for example, "x is red". Relations are, two- or more-argument "predicates", for example, "x is better than y", or "x sends y to z".

Examples. 1) In our "language for people" we will use the following predicate constants (see the class diagram above):

Male(x) − means "x is a male";

Female(x) − means "x is a female";

Mother(x, y) − means "x is mother of y";

Father(x, y) − means "x is father of y";

Married(x, y) − means "x and y are married, y being wife";

x=y − means "x an y are the same person".

The first two constants represent, in fact, "properties" (or, "classes") of our objects. The other 4 constans represents "relations" between our objects. The term "predicate" is used to include both versions. We do not introduce *Person(x)* as a predicate because our domains consists of persons only.

2) It may seem strange to non-mathematicians, yet the most popular relation of objects used in most mathematical theories, is **equality** (or identity). Still, this is not strange for mathematicians. We can select an object x in our "domain" by using a very specific combination of properties and relations of it, and then – select another object y – by using a different combination. And after this (sometimes it may take many years to do) we prove that x=y, i.e. that these two different combinations of properties and relations are possessed by a single object. Many of discoveries in mathematics could be reduced to this form.

In the language of first order arithmetic, equality "=" is the only necessary predicate constant. Other"basic" relations must be reduced to equality. For example, the relation x<y for natural numbers x, y can be reduced to equality by using the addition function and the formula ∃z(x+z+1=y).

3) In the language of set theory a specific predicate constant "in" denotes the set membership relation: "x in y" means "x is a member of y". The equality predicate x=y also will be used – it means "the sets x an y possess the same members".

The uniform way of representation suitable for predicates having any number of arguments is again the "prefix" notation: *p(x), q(x, y), r(x, y, z)* etc. In the real mathematical practice, some of the two-argument predicates are represented by using the "infix" notation (for example, x=y instead of the uniform =(x, y), etc.).

Zero-argument predicate constants? In an interpretation, each such predicate must become either "true", or "false". Hence, paradoxically, zero-argument predicate constants behave like "propositional variables" – they represent assertions that do not possess a meaning, but possess a "truth value".

Summary of Language Primitives

Thus, the specification of a predicate language includes the following **primitives**:

1) A countable set of object variable names (you may generate these names, for example, by using a single letter "x": x, xx, xxx, xxxx, ...).

2) An empty, finite, or countable set of object constants.

3) An empty, finite, or countable set of function constants. To each function constant a fixed argument number must be assigned.

4) A finite, or countable set of predicate constants. To each predicate constant a fixed argument number must be assigned.

Different sets of primitives yield different predicate languages.

Examples. 1) Our "language for people" is based on: a) object variables x, y, z, ...; b) object constants: John, Britney, ...; c) function constants: none; d) predicate constants: *Male(x), Female(x), Mother(x, y), Father(x, y), Married(x, y), x=y.*

2) The language of first order arithmetic is based on: a) object variables x, y, z, ...; b) object constants: 0, 1; c) function constants: x+y, x*y; d) predicate constant: x=y.

3) The language of set theory is based on: a) object variables x, y, z, ...; b) object constants: none; c) function constants: none; d) predicate constants: x in y, x=y.

The remaining part of the language definition is common for all predicate languages.

Terms and formulas

By using the language primitives, we can build terms, atomic formulas and (compound) formulas.

Terms are expressions used to denote objects and functions:

a) Object variables and object constants (if any), are terms.

b) If f is a k-argument function constant, and t_1, ..., t_k are terms, then the string $f(t_1, ..., t_k)$ is a term.

c) There are no other terms.

Examples. 1) In our "language for people" only variables x, y, z, ..., and object constants *John, Britney, ...* are terms.

2) In the language of first order arithmetic, for addition and multiplication the "infix" notation is used: if t_1, t_2 are terms, then (t_1+t_2) and (t_1*t_2) are terms. Of course, the object constants 0, 1 and variables x, y, z, ... are terms. Examples of more complicated terms: (x+y), ((1+1)*(1+1)), (((1+1)*x)+1).

3) In the language of set theory, variables x, y, z, ... are the only kind of terms.

If a term does not contain variable names, then it denotes an "object" of our "domain" (for example, ((1+1)+1) denotes a specific natural number – the number 3). If a term contains variables, then it denotes a function. For example, (((x*x)+(y*y))+1) denotes the function x^2+y^2+1. (**Warning!** Note that the language of first order arithmetic **does not** contain a function constant denoting the **exponentiation** x^y, thus, for example, we must write x*x instead of x^2.)

Of course, the key element of our efforts in describing "objects", their properties and relations, will be assertions, for example, the commutative law in arithmetic: ((x+y)=(y+x)). In predicate languages, assertions are called **formulas** (or, sometimes, well formed formulas – wff-s, or sentences).

Atomic formulas (in some other textbooks: elementary formulas, prime

formulas) are defined as follows:

a) If p is a k-argument predicate constant, and t_1, ..., t_k are terms, then the string $p(t_1, ..., t_k)$ is an atomic formula.

b) There are no other atomic formulas.

For the equality symbol, the "infix" notation is used: if t_1, t_2 are terms, then $(t_1=t_2)$ is an atomic formula.

Examples. 1) In our "language for people", the following are examples of atomic formulas: *Male(x), Female(Britney), Male(Britney)* (not all formulas that are well formed, must be true!), *Father(x, Britney), Mother(Britney, John), Married(x, y).*

2) Summary of the atomic formulas of the language of first order arithmetic: a) constants 0 and 1, and all variables are terms; b) if t_1 and t_2 are terms, then (t_1+t_2) and (t_1*t_2) also are terms; c) atomic formulas are built only as $(t_1=t_2)$, where t_1 and t_2 are terms.

3) In the language of set theory, there are only two kinds of atomic formulas: $x \in y$, and x=y (where x and y are arbitrary variables).

In the language of first order arithmetic, even by using the only available predicate constant "=" atomic formulas can express a lot of clever things:

((x+0)=x); ((x+y)=(y+x)); ((x+(y+z))=((x+y)+z));
((x*0)=0); ((x*1)=x); ((x*y)=(y*x)); ((x*(y*z))=((x*y)*z));
(((x+y)*z)=((x*z)+(y*z))).

Exercise 1.2.1. As the next step, translate the following assertions into the language of first order arithmetic (do not use abbreviations!): 2*2=4, 2*2=5, $(x+y)^2 = x^2+2xy+y^2$.

(Compound) Formulas

The following definition is common for all predicate languages. Each language is specific only by its set of language primitives.

To write more complicated assertions, we will need compound formulas, built of atomic formulas by using a fixed set of **propositional connectives** and **quantifiers** (an invention due to G. Frege and C. S. Peirce). In this book, we will use the following set:

Implication symbol: B→C means "if B, then C", or "B implies C", or "C follows from B".

Conjunction symbol, $B \wedge C$ means "B and C".

Disjunction symbol, $B \vee C$ means "B, or C, or both", i.e. the so-called non-exclusive "or".

Negation symbol, $\neg B$ means "not B.

Universal quantifier, $\forall x\, B$ means "for all x: B".

Existential quantifier, $\exists x\, B$ means "there is x such that B".

The widely used equivalence connective ↔ can be derived in the following way: B↔C stands for $((B \rightarrow C) \wedge (C \rightarrow B))$. If you like using the so-called exclusive "or" ("B, or C, but not both"), you could define B *xor* C as $\neg(B \leftrightarrow C)$.

Warning! For programmers, conjunction, disjunction and negation are familiar "logical operations" – unlike the implication that is not used in "normal" programming languages. In programming, the so-called IF-statements, when compared to logic, mean a different thing: in the statement IF x=y THEN z:=17, the condition, x=y is, indeed, a formula, but the "consequence" z:=17 is not a formula – it is an executable statement. In logic, in B→C ("if B, then C"), B and C both are formulas.

We define the notion of **formula** of our predicate language as follows:

a) Atomic formulas are formulas.

b) If B and C are formulas, then $(B \rightarrow C), (B \wedge C), (B \vee C)$, and $(\neg B)$ also are formulas (B and C are called sub-formulas).

c) If B is a formula, and x is an object variable, then (∀xB) and (∃xB) also are formulas (B is called a sub-formula).

d) (If you like so,) there are no other formulas.

See also:

Notes on Logic Notation on the Web by **Peter Suber**.

Knowledge Representation by Means of Predicate Languages

Examples. 1) In our "language for people", the following are examples of compound formulas:

$((Father(x, y)) \vee (Mother(x, y)))$	"x is a parent of y"
$(\forall x(\forall y((Father(x, y)) \rightarrow (Male(x)))))$	"fathers are males" – could serve as an **axiom**!
$(\forall x(\forall y((Mother(x, y)) \rightarrow (\neg Male(x)))))$	"mothers are not males" – could be **derived** from the axioms!
$(\forall x(\exists y(Mother(y, x))))$	"each x has some y as a mother" – could serve as an **axiom**!
$(\forall x(Male(x) \vee Female(x)))$	What does it mean? It could serve as an **axiom**!

$\forall x(\forall y(\forall z((Mother(x,z)\wedge Mother(y,z))\rightarrow(x=y))))$ $\forall x(\forall y(\forall z((Father(x,z)\wedge Father(y,z))\rightarrow(x=y))))$ What does it mean? It could serve as an **axiom**!

2) Some simple examples of compound formulas in the language of first order arithmetic:

Warning! Speaking strictly, predicate symbols "<", ">", "≤", "≥", "≠" etc. do not belong to the language of first order arithmetic.

(∃u(x=(u+u)))	"x is an even number"
(∃u(((x+u)+1)=y))	"x is less than y", or, x<y
$(0<y\wedge\exists\, u(x=(y*u)))$	"x is divisible by y", speaking strictly, x<y must be replaced by ∃u(((x+u)+1)=y)).
$((1<x)\wedge(\neg(\exists\, y(\exists\, z(((y<x)\wedge(z<x))\wedge(x=(y*z)))))))$ formula *prime(x)*, "x is a prime number", speaking strictly, the 3 subformulas of the kind x<y should be replaced by their full version of the kind ∃u(((x+u)+1)=y)).	
$(\forall w(\exists x((w<x)\wedge(prime(x)))))$	"There are infinitely many prime numbers" (one of the first mathematical theorems, VI century BC), speaking strictly, w<x must be replaced by ∃u(((w+u)+1)=x)), and *prime(x)* must be replaced by the above formula.
$\forall\, x\forall\, y(0<y\rightarrow\exists\, z\exists\, u(u<y\wedge x=y*z+u))$	What does it mean?

3) Some simple examples of compound formulas in the language of set theory:

$(\exists\, y(y\in x))$	"x is a non-empty set"
$(\forall\, z((z\in x)\rightarrow(z\in y)))$	"x is a subset of y", or x≤y
$((\forall\, z((z\in x)\leftrightarrow(z\in y)))\rightarrow(x=y))$	What does it mean? Will serve as an **axiom**!
$(\forall\, y(\forall\, z((y\in x)\wedge(z\in x))\rightarrow y=z))$	"x contains zero or one member"
$(\forall\, u((u\in x)\leftrightarrow((u\in y)\vee(u\in z))))$	"x is union of y and z", or x=yUz

Of course, **having a predicate language is not enough** for expressing all of our knowledge formally, i.e. for communicating it to computers. Computers do not know in advance, for example, how to handle sexes. We must tell them

how to handle these notions by introducing **axioms**. Thus, the above-mentioned formulas like as

$$(\forall x(Male(x) \vee Female(x))) \quad \text{, or}$$
$$(\forall\ x(\forall\ y((Father(x, y)) \rightarrow (Male(x)))))$$

will be absolutely necessary as **axioms**. As we will see later,

language + axioms + logic = theory,

i.e. in fact, **to formulate all of our knowledge formally, we must create theories.**

Exercise 1.2.2. Translate the following assertions into our "language for people":

"x is child of y";
"x is grand-father of y";
"x is brother of x"; "x is sister of y";
"x is cousin of y"; "x is nephew of y"; "x is uncle of y".

Exercise 1.2.3. Translate the following assertions into the language of first order arithmetic:

"x and y do not have common divisors" (note: 1 is not a divisor!);
" $\sqrt{2}$ is not a rational number".

(Warning! $\neg\exists\ p \exists\ q(\sqrt{2}=\frac{p}{q})$, and $\exists\ x(x*x=2)$ are not correct solutions. Why?)

Exercise 1.2.3A. Imagine an alternative language of arithmetic that does not contain function constants + and *, but contains predicates *sum(x, y, z)* and *prod(x, y, z)* instead (meaning x+y=z and x*y=z correspondingly). Translate the following assertions into this language:

$$x+0=x\,;\, x+y=y+x\,;\, x+(y+1)=(x+y)+1\,;\,(x+y)*z=(x*z)+(y*z) \quad .$$

Warning! Some typical errors!

1a) Trying to say "for all x>2, F(x)", do not write $\forall x(x>2 \wedge F(x))$. This formula would imply that $\forall x(x>2)$ – a silly conclusion! Indeed, how about x=1? The correct version: $\forall x(x>2 \rightarrow F(x))$.

1b) Trying to say "there is x>2, such that F(x)", do not write $\exists x(x>2 \rightarrow F(x))$. The formula under the quantifier is true for x=1, hence, the entire formula cannot guarantee that "there is x>2, such that F(x)". The correct version: $\exists(x>2 \wedge F(x))$.

2) Some computer programmers do not like using the implication connective →, trying to write formulas as conditions of IF- or WHILE-statements, i.e. by using conjunction, disjunction and negation only. This "approach" makes most logical tasks much harder than they really are! More than that − some people

try saying, for example, "Persons are not Departments", as follows: $\forall x(Person(x) \wedge \neg Department(x))$ − instead of the correct version:

$$\forall x(Person(x) \rightarrow \neg Department(x)) \quad .$$

3) Do not use abbreviations at this early stage of your studies. For example, do not write *(∀x>2)∃yF(x, y)* to say that "for all x that are >2, there is y such that F(x, y)". Instead, you should write *∀x(x>2→∃yF(x, y))*. Similarly, instead of *(∃a>2)∃bG(a, b)*, you should write $\exists a(a > 2 \wedge \exists b\, G(a, b))$.

To say "there is one and only one x such that F(x)", you shouldn't write *∃!x F(x)* (at this stage of your studies), you should write, instead,

$$((\exists x F(x)) \wedge (\forall x1(\forall x2((F(x1) \wedge F(x2)) \rightarrow (x1 = x2))))) \quad .$$

4) Predicates cannot be substituted for object variables. For example, having 3 predicate constants *working(x, y), Person(x), Department(y)*, do not try writing *working(Person(x), Department(y))* to say that "only persons are working, and only in departments". The correct verssion:

$$\forall x \forall y(working(x, y) \rightarrow Person(x) \wedge Department(y)) \quad .$$

5) Trying to say “each person is working in some department”, do not write

$$\forall x \exists y(Person(x) \wedge Department(y) \rightarrow working(x, y)) \quad .$$

The correct version:

$$\forall x(Person(x) \rightarrow \exists y(Department(y) \wedge working(x, y))) \quad .$$

What is the difference?

Exercise 1.2.4. Try inventing your own predicate language. Prepare and do your own Exercise 1.2.2 for it.

Exercise 1.2.5 (optional). In computer science, currently, one the most popular means of knowledge representation are the so-called **UML class diagrams** and **OCL** (UML − Unified Modeling Language, OCL − Object Constraint Language). The above diagram representing our “domain of people” is an example. In our “language of people”, put down as many **axioms** of the domain you can notice in the diagram. For example, “every person is either male, or female”, “all fathers are males”, “every person has exactly one mother”, “a person can marry no more than one person” etc.

Many-sorted Languages

Maybe, you have to describe two or more kinds of "objects" that you do not wish to reduce to "sub-kinds" of one kind of "objects" (for example, integer numbers and character strings). Then you may need introducing for each of your "domains" a separate kind ("sort") of object variables. In this way you arrive to the so-called **many-sorted predicate languages**. In such languages: a) each object constant must be assigned to some sort; b) for each function constant, each argument must be assigned to some some sort, and function values must be

assigned to a (single) sort; c) for each predicate constant, each argument must be assigned to some sort. In many-sorted predicate languages, the term and atomic formula definitons are somewhat more complicated: building of the term $f(t_1, ..., t_k)$ or the formula $p(t_1, ..., t_k)$ is allowed only, if the sort of the term t_i values coincides with the sort of the i-th argument of f or p respectively. And the "meaning" of quantifiers depends on the sort of the variable used with them. For example, ∀x means "for all values of x from the domain of the sort of x".

Theoretically, many-sorted languages can be reduced to one-sorted languages by introduding the corresponding predicates $Sort_i(x)$ ("the value of x belongs to the sort i"). Still, in applications of logic (for example, in computer science) the many-sorted approach is usually more natural and more convenient. (See *Chapter 10. Many-Sorted First Order Logic*, by Jean Gallier.)

Warning about second order languages!

In our definition of predicate languages only the following kinds of primitives were used: object variables, object constants, function constants and predicate constants. You may ask: how about **function variables** and **predicate variables**? For, you may wish to denote by r "an arbitrary property" of your "objects". Then, r(x) would mean "x possess the property r", and you would be able to say something about "all properties", for example, ∀r∀x∀y(x=y→(r(x)↔r(y)). In this way you would have arrived at a **second order language**! In such languages, function and predicate variables are allowed. But properties lead to sets of objects, for example, {x | r(x)} would mean the set of all objects that possess the property r. But, why should we stop at the properties of objects? How about "properties of sets of objects" etc.? As it was detected long ago, all kinds of sets can be fully treated only in **set theory**! Thus, instead of building your own second order language, you should better try applying your favorite ("first order") set theory. An unpleasant consequence: the existence of the (much less significant) notion of second order languages forces many people to call predicate languages "**first order languages**" − to emphasize that, in these languages, the only kind of variables allowed are object variables.

On the other hand, when trying to implement realistic **formal reasoning software**, then using of some second order constructs is, as a rule, more efficient than implementing of a pure first order reasoning. See, for example, *Notices of the AMS*, Special Issue on Formal Proof, Vol. 55, N 11, 2008 (available online).

For details, see: Second-order-logic. About **second order arithmetic** see Reverse Mathematics. About an almost (but not 100%) successful attempt to create a set theory "as simple as logic" (by Georg Cantor and Gottlob Frege) – see Podnieks [1997], Section 2.2.

Omitting Parentheses

Our formal definitions of terms and formulas lead to expressions containing many parentheses. Let us remind, for example, our formula expressing that "x is a prime number":

$$((1<x)\wedge(\neg(\exists\, y(\exists\, z(((y<x)\wedge(z<x))\wedge(x=(y*z)))))))\quad .$$

Such formulas are an easy reading for computers, yet inconvenient for human reading (and even more inconvenient – for putting them correctly). In the

usual mathematical practice (and in programming languages) we are allowed to improve the look of our formulas by omitting some of the parentheses − according to (some of) the following rules:

a) Omit the outermost parentheses, for example, we may write A→(B→C) instead of the formally correct (A→(B→C)). In this way we may improve the final look of our formulas. Still, if we wish to use such formulas as parts of more complicated formulas, we must restore the outermost parentheses, for example: (A→(B→C))→D.

b) We may write, for example, simply:

$$x+y+z+u,\ x*y*z*u,\ A\wedge B\wedge C\wedge D,\ A\vee B\vee C\vee D,\ \exists x\,\forall y\,\exists z\,\forall u\,F$$

instead of the more formal

((x+y)+z)+u, ((x*y)*z)*u, $((A\wedge B)\wedge C)\wedge D$, $((A\vee B)\vee C)\vee D$,
∃x(∀y(∃z(∀u(F)))).

In this way we can simplify the above expression "x is a prime number" as follows:

$$(1<x)\wedge(\neg(\exists y\,\exists z((y<x)\wedge(z<x)\wedge(x=(y*z)))))\ .$$

c) We can apply the so-called **priority rules**. For example, the priority rank of multiplications is supposed to be higher than the priority rank of additions. This rule allows writing x+y*z instead of the more formal x+(y*z) − because of its higher priority rank, multiplication must be "performed first". The most popular priority rules are the following:

c1) The priority rank of **function constants** is higher than the priority rank of **predicate constants**. This allows, for example, writing x*y = y*x instead of (x*y)=(y*x), or $x\in y\cup z$ − instead of $x\in(y\cup z)$.

c2) The priority rank of **predicate constants** is higher than the priority rank of **propositional connectives and quantifiers**. This allows, for example, writing $y<x\wedge z<x$ instead of $(y<x)\wedge(z<x)$.

c3) The priority rank of **quantifiers** is higher than the priority rank of **propositional connectives**. This allows, for example, writing $\exists x\,F\wedge\forall y\,G$ instead of $(\exists x(F))\wedge(\forall y(G))$, or writing $\neg\exists x\,F$ instead of $\neg(\exists x(F))$.

c4) The priority rank of **negations** is higher than the priority rank of **conjunctions and disjunctions**. This allows, for example, writing $\neg A\wedge\neg B$ instead of $(\neg A)\wedge(\neg B)$.

c5) The priority rank of **conjunctions and disjunctions** is higher than the priority rank of **implications**. This allows, for example, writing $A\rightarrow A\vee B$ instead of $A\rightarrow(A\vee B)$.

In the usual mathematical practice some additional priority rules are used, but some of them are not allowed in the common programming languages. To avoid confusions do not use too many priority rules simultaneously!

According to the above priority rules, we can simplify the above expression "x is a prime number" obtaining a form that is much easier for human reading (but is somewhat complicated for computers to process it):

$$1<x\wedge\neg\exists\, y\,\exists\, z\,(y<x\wedge z<x\wedge x=y*z)\quad .$$

As you see, all the above rules are mere abbreviations. In principle, you could use any other set of abbreviation rules accepted by your audience. If computers would do logic themselves, they would not need such rules at all (except, maybe, for displaying some of their results to humans, but why?).

Exercise 1.2.6. "Translate" the following assertions into our "language for people":

"x and y are siblings";
"x and y are brothers"; "x and y are sisters";

"x is cousin of y";
"parents of x and y are married";
construct formulas expressing as much well-known relationships between people as you can.

But how about the predicate **Ancestor(x, y)** − "x is an ancestor of y"? Could it be expressed as a formula of our "language for people"? The first idea − let us "define" this predicate recursively:

$$\forall\, x\,\forall\, y\,(Father(x,y)\vee Mother(x,y)\rightarrow Ancestor(x,y))\quad ;$$
$$\forall\, x\,\forall\, y\,\forall\, z\,(Ancestor(x,y)\wedge Ancestor(y,z)\rightarrow Ancestor(x,z))\quad .$$

The second rule declares the transitivity property of the predicate. The above two formulas are **axioms**, allowing to derive the essential properties of the predicate *Ancestor(x, y)*. But how about a single formula F(x, y) in the "language for people", expressing that "x is an ancestor of y"? Such a formula should be a tricky combination of formulas *Father(x, y), Mother(x, y)* and *x=y*. And such a formula is **impossible**! See **Carlos Areces**. Ph.D. Thesis, 2000, (a non-trivial!) Theorem 1.2.

Exercise 1.2.7. "Translate" the following assertions into the language of first order arithmetic:

"x and y are twin primes" (examples of twin pairs: 3,5; 5,7; 11,13; 17,19;...),
"There are infinitely many pairs of twin primes" (the famous Twin Prime Conjecture),
"x is a power of 2" (Warning! $\exists n(x=2^n)$ is not a correct solution. Why? Because exponentiation does not belong to the language of first order

arithmetic.),
"Each positive even integer ≥4 can be expressed as a sum of two primes"
(the famous Goldbach Conjecture).

Free Variables and Bound Variables

The above expression "x is a prime number":

$$1<x\wedge\neg\exists\, y\exists\, z(y<x\wedge z<x\wedge x=y*z)$$

contains 3 variables: x − occurs 4 times in terms, y − 2 times in terms and 1 time in quantifiers, z − occurs 2 times in terms and 1 time in quantifiers. Of course, x is here a "free" variable – in the sense that the "truth value" of the formula depends on particular "values" taken by x. On the contrary, the "truth value" of the formula does not depend on the particular "values" taken by the two "bound" variables y and z − the quantifiers ∃y, ∃z force these variables to "run across their entire range".

More precisely, first, we will count **only** the occurrences of variables in terms, not in quantifiers. And second, we will define a particular occurrence o_x of a variable x in (a term of) a formula F as a **free occurrence** or a **bound occurrence** according to the following rules:

a) If F does not contain quantifiers ∃x, ∀x, then o_x is free in F.

b) If F is ∃xG or ∀xG, then o_x is bound in F.

c_1) If F is $G\wedge H, G\vee H$, or G→H, and o_x is free in G (or in H), then o_x is free in F.

c_2) If F is ¬G, ∃yG, or ∀yG, where y **is not** x, and o_x is free in G, then o_x is free in F.

d_1) If F is $G\wedge H, G\vee H$, or G→H, and o_x is bound in G (or in H), then o_x is bound in F.

d_2) If F is ¬G, ∃yG, or ∀yG (where y is any variable, x included), and o_x is bound in G, then o_x is bound in F.

Thus, the above formula $1<x\wedge\neg\exists\, y\exists\, z(y<x\wedge z<x\wedge x=y*z)$ contains 4 free occurrences of x, 2 bound occurrences of y, and 2 bound occurrences of z.

Exercise 1.2.8. Verify that an occurrence of x in F cannot be free and bound simultaneously. (Hint: assume that it is not the case, and consider the sequence of all sub-formulas of F containing this particular occurrence of x.)

Formally, we can use formulas containing free and bound occurrences of a

single variable simultaneously, for example, $x>1\rightarrow\exists\, x(x>1)$. Or, many bound occurrences of a single variable, for example,

$$(\forall\, xF(x)\wedge\exists\, xG(x))\vee\forall\, xH(x)$$

means the same as

$$(\forall\, xF(x)\wedge\exists\, yG(y))\vee\forall\, zH(z)\quad .$$

Still, we do not recommend using a single variable in many different roles in a single formula. Such formulas do not cause problems for computers, but they may become inconvenient for human reading.

Let us say, that x is a **free variable** of the formula F, if and only if F contains at least one free occurrence of x, or F does not contain occurrences of x at all.

If a formula contains free variables, i.e. variables that are not bound by quantifiers (for example: $x=0\vee x=1$), then the "truth value" of such formulas may depend on particular values assigned to free variables. For example, the latter formula is "true" for x=1, yet it is "false" for x=2. Formulas that do not contain free occurrences of variables, are called **closed formulas**, for example:

$$\forall\, w\exists\, x(w<x\wedge prime(x))\quad .$$

Closed formulas represent "definite assertions about objects of theory", they are expected to be (but not always really are) either "true", or "false".

Term Substitution

To say that x is a free variable of the formula F, we may wish to write F(x) instead of simply F. Replacing all free occurrences of x by a term t yields an "instance" of the formula F. It would be natural to denote this "instance" by F(t).

For example, if F(x) is ∃y(y+y=x) and t is z*z+z, then F(t), or F(z*z+z) will denote ∃y(y+y=z*z+z).

However, if t would be y*y+y, then F(t), or F(y*y+y) would be ∃y(y+y=y*y+y). Is this really F(y*y+y)?

Thus, sometimes, substitutions can lead to crazy results. Another example: in our expression "x is a prime number", let us replace x by y. Will the resulting formula mean "y is a prime number"? Let's see:

$$1<y\wedge\neg\exists\, y\exists\, z(y<y\wedge z<y\wedge y=y*z)\quad .$$

Since y<y is always false, the second part ¬∃y∃z(...) is true, hence, the latter formula means simply that "1 is less than y", and not that "y is a prime number".

Of course, we failed because we replaced a free variable x by a variable y in such a way that some **free** occurrence of x became **bound** by a quantifiers for y (∃y). In this way we **deformed the initial meaning** of our formula.

The following simple rule allows to avoid such situations. Suppose, x is a free variable of the formula F. We will say that the **substitution F(x/t)** (i.e. the substitution of the term t for x in the formula F) **is admissible**, if and only if no free occurrences of x in F are located under quantifiers that bind variables contained in t. If the substitution F(x/t) is admissible, then, by replacing all free occurrences of x in F by t, of course, we do not change the initial meaning of the formula F(x), and hence, we may safely denote the result of this substitution by F(t).

Exercise 1.2.9. Is x/y an admissible substitution in the following formulas? Why?

$$x=0\vee\exists\, y(y>z) \quad ;$$
$$x=0\vee\exists\, y(y>x) \quad .$$

Exercise 1.2.10 (optional). a) Mathematicians: think over the analogy between bound variables in logic and bound variables in sum expressions and integrals. b) Programmers: think over the analogy between bound variables in logic and loop counters in programs.

1.3. Axioms of Logic: Minimal System, Constructive System and Classical System

The Problem of Reasoning

Now we go on to the second phase of formalization: after having defined a *formal language* (predicate language) allowing to put down propositions about our domain of interest, and having formulated some of the propositions as *axioms*, and must introduce some *means of reasoning* allowing to derive other statements that are "true" of our domain.

Indeed, having formulated some fragment of our knowledge as a set of axioms A_1, ..., A_n in some predicate language L, we do not think that A_1, ..., A_n represent **all** statements that are "true" of the objects we are trying to investigate. Many other statements will **follow** from A_1, ..., A_n as consequences.

Example. Assume, we have formulated the following axioms in our "language for people": $\neg\exists x(Male(x)\wedge Female(x))$; $\forall\, x(Male(x)\vee Female(x))$,

and the following facts: $Male(John); Female(Britney)$. Then we do not need to formulate $\neg Female(John); \neg Male(Britney)$ as separate facts. These facts can be derived from the already registered facts.

The problem of reasoning: "formula F follows from A_1, ..., A_n", what does it mean? The answer must be absolutely precise, if we wish to **teach reasoning to computers**.

Solution of the Problem

First of all, let us notice that there are axioms and rules of inference that are applicable to any predicate languages, independently of the specific features of their domains. Such axioms and rules could be called “generally valid”.

For example, assume, some formula F has the following form:

$$(B \to D) \to ((C \to D) \to (B \vee C \to D)) \quad ,$$

where B, C, D are some formulas. Then F is “true” independently of the specific facts represented in the formulas B, C, D.

Similarly, the following rule of inference MP is applicable independently of the facts represented in the formulas B, C:

Having derived the formulas B, B→C, derive the formula C.

If we have B→D and B→D already derived, then – by applying the rule MP to the above formula F – we derive that $B \vee C \to D$.

Now, we will try formulating a complete set of "generally valid" principles (axioms and rules of inference) of "logically correct reasoning". "Generally valid" means that these principles will be applicable to any predicate language. I.e. for any fixed predicate language L, we wish to formulate a uniform list of **logical axioms and inference rules** that would allow formalization of principles of reasoning that are "valid” for all languages. Such principles are called sometimes "pure logical" principles. The existence of such general principles (and even, in a sense, a complete system of them) is the result of a 2500 year long history of great discoveries (or inventions? − see below).

Aristotle (384-322 BC),

Gottlob Frege (1848-1925),

Charles Sanders Peirce (1839-1914).

Bertrand Russell (1872-1970) − "The **Principia Mathematica** is a three-volume work on the foundations of mathematics, written by Bertrand Russell and Alfred North Whitehead and published in 1910-1913. It is an attempt to derive all mathematical truths from a well-defined set of axioms and inference rules in symbolic logic. The main inspiration and motivation for the Principia was Frege's earlier work on logic, which had led to some

contradictions discovered by Russell." (according to Wikipedia).

David Hilbert (1862-1943),

Wilhelm Ackermann (1896-1962).

D.Hilbert, W.Ackermann. Grundzüge der theoretischen Logik. Berlin (Springer), 1928 (see also: Hilbert and Ackermann's 1928 Logic Book by Stanley N. Burris).

The first version of logical axioms was introduced in 1879 by G. Frege in his above-mentioned *Begriffsschrift*. The next important version was proposed in 1910-1913 by B. Russell and A. Whitehead in their famous book *Principia Mathematica*. And finally, in 1928 D. Hilbert and W. Ackermann published in their above-mentioned book, in a sense, the final version of logical axioms. Modifications of this version are now used in all textbooks of mathematical logic.

In our version, the axioms will be represented by means of the so-called **axiom schemas** (programmers might call them *templates*). Each schema (template) represents an infinite, yet easily recognizable collection of single axioms. For example, schema L_3: $B \wedge C \to B$ may represent the following axioms ("instances of the schema") in the language of first order arithmetic:

$$x = y \wedge x = x \to x = y \quad ,$$

$$1 * 1 = 1 \wedge 1 + 1 = 1 + 1 \to 1 * 1 = 1 \quad ,$$

and many other axioms: take any formulas B, C, and you will obtain an axiom $B \wedge C \to B$.

We will not specify properties of the **equivalence** connective in axioms. We will regard this connective as a derived one: B↔C will be used as an abbreviation of $(B \to C) \wedge (C \to B)$.

Axioms of Logic

Suppose, we have specified some predicate language L. We adopt the following 15 axiom schemas as **the logical axioms for the language L.**

In the axiom schemas L_1-L_{11} below, B, C and D are any formulas in the language L.

The first two axiom schemas L_1, L_2 represent the "definition" of the implication connective:

L_1: $B \to (C \to B)$ (what does it mean?),

L_2: $(B \to (C \to D)) \to ((B \to C) \to (B \to D))$ (what does it mean?).

The following axiom schemas L_3–L_5 represent the "definition" of the AND-connective (conjunction):

L_3: $B \wedge C \to B$ (what does it mean?),

L_4: $B \wedge C \to C$ (what does it mean?),

L_5: $B \to (C \to B \wedge C)$ (what does it mean?).

The following axiom schemas L_6–L_8 represent the "definition" of the (non-exclusive) OR-connective (disjunction):

L_6: $B \to B \vee C$ (what does it mean?),

L_7: $C \to B \vee C$ (what does it mean?),

L_8: $(B \to D) \to ((C \to D) \to (B \vee C \to D))$ (what does it mean?).

The next xiom schema L_9 represents the "definition" of the NO-connective. In fact, it is a formal version of a proof method well-known in mathematics − refutation by deriving a contradiction (*Reductio ad absurdum*):

L_9: $(B \to C) \to ((B \to \neg C) \to \neg B)$ (what does it mean?).

The next axiom schema L_{10} represents the famous principle "Contradiction Implies Anything" (*Ex contradictione sequitur quodlibet*, or *Ex falso sequitur quodlibet*):

L_{10}: $\neg B \to (B \to C)$ (what does it mean?).

The following axiom schema L_{11} represents the famous **Law of Excluded Middle** (*Tertium non datur)*:

L_{11}: $B \vee \neg B$ (what does it mean?).

The above 11 schemas (plus the *Modus Ponens* rule of inference, see below) represent the **classical propositional logic** in the language L.

Now, the "definitions" of the universal and existential quantifiers follow.

In the following axiom schemas L_{12}, L_{13}, F is any formula, and t is a term such that the substitution F(x/t) is admissible (in particular, t may be x itself):

L_{12}: $\forall x F(x) \to F(t)$ (in particular, $\forall x F(x) \to F(x)$, what does it mean?),

L_{13}: $F(t) \to \exists x F(x)$ (in particular, $F(x) \to \exists x F(x)$, what does it mean?).

In the following schemas L_{14}, L_{15}, F is any formula, and G is a formula that does not contain x as a free variable:

L_{14}: $\forall x(G \to F(x)) \to (G \to \forall x F(x))$ (what does it mean?),

L_{15}: $\forall x(F(x) \to G) \to (\exists x F(x) \to G)$ (what does it mean?).

Rules of Inference

In the following rules of inference, B, C and F are any formulas.

Modus Ponens: B→C; B ⊢C (what does it mean?).

Generalization: F(x) ⊢ $\forall x F(x)$ (what does it mean?).

This list of logical axioms and rules of inference represents the so-called **classical predicate logic** in the predicate language L (or, simply − the **classical logic** in the language L).

Some of the logical axioms are "wrong, but useful"!

The axioms L_1, L_2 represent the (currently) most popular version of "defining" the implication connective. About other (equivalent) versions − containing 3 or 4 axioms − see Hilbert, Bernays [1934] (Chapter III) and Exercise 1.5.2.

The axiom L_9 represents the (currently) most popular version of "defining" the negation connective. About other (equivalent) versions − see Hilbert, Bernays [1934] (Chapter III), Exercise 2.4.2.

Three of the above axiom schemas seem to be (at least partly) problematic.

For example, how do you find the funny axiom L_{10}: $\neg B \to (B \to C)$? If ¬B and B were true simultaneously, then anything were true? *Ex contradictione sequitur quodlibet?* Is this a really "true" axiom? Of course, it is not. Still, this does not matter: we do not need to know, were C "true" or not, if ¬B and B were "true" simultaneously. By assuming that "if ¬B and B were true simultaneously, then anything were true" **we greatly simplify our logical apparatus**. For example, we will prove in Section 2.6 that, in the classical logic, ¬¬B→B. This simple formula can't be proved without the "crazy" axiom L_{10} (see Section 2.8).

In fact, the first axiom L_1: B→(C→B) also is funny. If B is (unconditionally) true, then B follows from C, even if C has nothing in common with B? Moreover, in Exercise 1.4.2(d) we will see that the axioms L_1, L_9 allow proving that ¬B, B ⊢ ¬C, i.e. if ¬B and B were true simultaneously, then

anything were false (thus, in a sense, L_1 contains already 50% of L_{10}!). After this, could we think of L_1 as a really "true" axiom? Of course, we can't. Still, this does not matter: if B is (unconditionally) true, then we do not need to know, follows B from C or not. By assuming that "if B is true, then B follows from anything" **we greatly simplify our logical apparatus**.

The above two phenomena are called **paradoxes of the material implication**, see Paradoxes of Material Implication by Peter Suber, and Falsity Implies Anything by Alexander Bogomolny.

May our decision to "greatly simplify" the logical apparatus have also some undesirable consequences? Let us consider the following formula F(x): ∀y(child(x, y)→Female(y)). It seems, F(x) is intended to mean: "All the children of x are female". However, in our system of logic, F(x) is regarded as true also, if x does not have children at all! If you do not have children at all, then all your children are female! Or male? Or smart? Etc. Seems funny, but is, in fact, harmless...

Constructive Logic

Still, the most serious problem is caused by the axiom L_{11}: B∨¬B − the Law of Excluded Middle. How can we think of L_{11} as a "true" axiom, if (according to Gödel's Incompleteness Theorem) each sufficiently strong consistent theory contains undecidable propositions? I.e. we postulate that either B, or ¬B "must be true", yet for some B we cannot prove neither B, nor ¬B! Knowing that B∨¬B is "true" may inspire us to work on the problem, but it may appear useless, if we do not succeed... Should we retain L_{11} as an axiom after this?

Some other **strange consequences** of L_{11} also should be mentioned (see Exercise 2.6.4):

$B \vee (B \to C)$,
$(B \to C) \vee (C \to B)$,
$((B \to C) \to B) \to B$ (the so-called **Peirce's Law**).

For these (and some other) reasons some people reject L_{11} as a "valid" logical axiom.

The above list of 15 axiom schemas as it stands is called the **classical logic**.

By excluding L_{11} from the list the so-called **constructive** (historically, and in most textbooks − **intuitionistic**) **logic** is obtained. As a concept, it was introduced by Luitzen Egbertus Jan Brouwer in 1908:

L. E. J. Brouwer. De onbetrouwbaarheid der logische principes (The unreliability of the

logical principles), *Tijdschrift voor Wijsbegeerte*, 2 (1908), pp.152-158.

Brouwer's main objection was against non-constructive proofs which are enabled mainly by "improper" use of the Law of Excluded Middle.

For elegant examples of non-constructive proofs see Constructive Mathematics by Douglas Bridges in Stanford Encyclopedia of Philosophy.

Note. A similar kind of non-constructive reasoning is represented by the so-called Double Negation Law: ¬¬B→B, see Section 2.6.

As a formal system, the intuitionistic logic was formulated by Arend Heyting in 1930:

A. Heyting. Die formalen Regeln der intuitionistischen Mathematik. *Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-mathematische Klasse*, 1930, pp.42-56.

The constructive concept of logic differs from the classical one mainly in its interpretation of disjunction and existence assertions:

− To prove BvC **constructively**, you must prove B, or prove C. To prove $B \vee C$ by using the classical logic, you are allowed to assume $\neg(B \vee C)$ as a hypothesis to derive a contradiction. Then, by the Law of Excluded Middle $(BvC) \vee \neg(BvC)$ you obtain $B \vee C$. Having only such a "negative" proof, you may be unable to determine, which part of the disjunction BvC is true − B, or C, or both. Knowing that $B \vee C$ is "true" may inspire you to work on the problem, but it may appear useless, if you do not succeed...

− To prove ∃xB(x) **constructively**, you must provide a particular value of x such that B(x) is true. To prove ∃xB(x) by using the classical logic, you are allowed to assume ∀x¬B(x) as a hypothesis to derive a contradiction. Then, by the Law of Excluded Middle $\exists\, xB(x) \vee \neg \exists\, xB(x)$ you obtain ∃xB(x). Having only such a "negative" proof, you may be unable to find a particular x for which B(x) is true. Knowing that ∃xB(x) is "true" may inspire you to work on the problem, but it may appear useless, if you do not succeed...

Note. Informally, we may regard existence assertions as "huge disjunctions". For example, in the language of first order arithmetic, ∃xB(x) could be "thought" as $B(0) \vee B(1) \vee B(2) \vee \ldots$, i.e. as an infinite "formula". Thus, the above two theses are, in a sense, "equivalent".

The constructive (intuitionist) logic is one of the great discoveries in mathematical logic − surprisingly, a complete system of constructive reasoning (as we will see later, in Section 4.4) can be obtained simply by dropping the Law of Excluded Middle from the list of valid logical principles.

See also Intuitionistic Logic by Joan Moschovakis in Stanford Encyclopedia of Philosophy.

Luitzen Egbertus Jan Brouwer (1881-1966): "He rejected in mathematical proofs the Principle

of the Excluded Middle, which states that any mathematical statement is either true or false. In 1918 he published a set theory, in 1919 a measure theory and in 1923 a theory of functions all developed without using the Principle of the Excluded Middle." (according to MacTutor History of Mathematics archive). "Como Heinrich Scholz solia decir en sus cursos: no son ni Heidegger ni Sartre los verdaderos renovadores de la filosofia, sino Brouwer porque sólo él ha atacado el bastión dos veces milenario del platonismo: la concepción de los entes matematicos como cosas en si." (quoted after Andrés R. Raggio, Escritos Completos, *Prometeo Libros*, 2002).

Minimal Logic

By excluding both L_{10} and L_{11} the so-called **minimal logic** is obtained. It was introduced by Ingebrigt Johansson in 1936:

I.Johansson. Der Minimalkalkül, ein reduzierter intuitionistischer Formalismus. *Compositio Mathematica*, 1936, Vol. 4, N1, pp.119-136.

As a separate concept, the minimal logic is much less significant than the constructive logic. Indeed, since it allows proving of $\neg B, B \vdash \neg C$ (in a sense, 50% of L_{10}!), dropping of L_{10} is not a very big step.

First Order Theories

Having defined our predicate language L, and having formulated for L all the logical axioms and rules of inference, do we need more?

To complete the formalization of our informal vision of our domain of interest, we must formulate at least some specific axioms describing the *specific features of the domain*. Logical axioms and rules of inference are valid for any domains, i.e. they are "content-free" in the sense that, by using them only, one cannot derive specific information about the domain.

For example, one cannot derive from (any) logic, that

$$\forall\, x(Male(x) \vee Female(x)) \quad .$$

To communicate this fact to the computer, we must formulate it as a *specific axiom*.

And, as we will prove in Section 4.3, we will never need introducing of *specific rules of inference*. All we need are the two logical rules of inference – *Modus Ponens* and *Generalization*.

Thus, as the result of the formalization process, we will obtain the so-called **first order theories**.

Each first order theory T includes:

a) a specific predicate **language** L(T);

b) **logical axioms** and rules of inference for this language (classical or constructive version may be adopted, see below);

c) a set of **specific (non-logical) axioms** of T.

As the first example, let's use our "language for people" to build a “theory for people”.

Examples of instances of logical axioms for the “language for people”:

L_1: $\forall\, x(Male(x)\rightarrow(Female(x)\rightarrow Male(x)))$;

L_6: $\forall\, x\forall\, y(Mother(x,y)\rightarrow Mother(x,y)\vee Father(x,y))$;

L_{11}: $Male(John)\vee\neg Male(John)$;

L_{12}: $\forall x(Female(x))\rightarrow Female(Britney)$;

etc.

And let's introduce the following non-logical axioms:

$\forall\, x(Male(x)\vee Female(x))$;
$\neg\exists x(Male(x)\wedge Female(x))$;
$\forall\, x\forall\, y(Father(x,y)\rightarrow Male(x))$;
$\forall\, x(\forall\, y(\forall\, z((Father(x,z)\wedge Father(y,z))\rightarrow(x=y))))$ ...

Exercise 1.3.1. Extend this list of axioms as far as you can. Is your list complete? What do you mean by “complete”?

Another example of a first order theory − the so-called **first order arithmetic PA** (also called Peano arithmetic):

The language of PA:

a) The constants 0 and 1, and all variables are terms.

b) If t_1 and t_2 are terms, then (t_1+t_2) and (t_1*t_2) also are terms.

c) Atomic formulas are built as $(t_1=t_2)$, where t_1 and t_2 are terms.

Since we can use, for example, the expression $2x^2-3y^2-1=0$ as a shortcut for (1+1)*x*x=(1+1+1)*y*y+1, we can say simply that, in first order arithmetic, atomic formulas of are arbitrary Diophantine equations.

Instances of logical axioms for the language of first order arithmetic:
L_1: $x=0\rightarrow(y=1\rightarrow x=0)$;

L_6: $x=y\rightarrow x=y\vee z=1$;

L_{11}: $0=1\vee\neg(0=1)$;

L_{12}: $\forall x(x=1)\rightarrow x=1$;

etc.

The specific (non-logical) axioms of first order arithmetic:

x=x,
x=y→y=x,
x=y→(y=z→x=z),
x=y→x+1=y+1,

¬(0=x+1),
x+1=y+1→x=y,
x+0=x,
x+(y+1)=(x+y)+1,
x*0=0,
x*(y+1)=(x*y)+x,
$B(0)\wedge\forall x(B(x)\rightarrow B(x+1))\rightarrow\forall xB(x)$, where B is any formula.

The axioms 7-10 represent recursive definitions of addition and multiplication. As the last the so-called **induction schema** is listed.

For the most popular **axiom system of set theory** – see Zermelo-Fraenkel's set theory.

Proofs and Theorems

In general, any sequence of formulas F_1, F_2, ..., F_m could be regarded as a (correct or incorrect) **formal proof** (or simply, a proof) of its last formula F_m. In a **correct proof**, formulas can play only the following roles:

a) *Axioms*. Some formulas may be instances of logical or non-logical axioms.

b) *Consequences* of earlier formulas, obtained by using rules of inference. For example, if F_{25} is A, and F_{34} is A→B, and F_{51} is B, then we can say that F_{51} has been obtained from F_{25} and F_{34} by using the *Modus Ponens* rule. Or, if F_{62} is C(x), and F_{63} is ∀xC(x), then we can say that F_{63} has been obtained from F_{62} by using the *Generalization* rule.

c) *Hypotheses*. Some formulas may appear in the proof without any formal justification, simply by assuming that they are "true".

Thus, the following notation can describe the actual status of a formal proof:

$$[T]: A_1, A_2, ..., A_n \vdash B,$$

where T is a first order theory (it determines which formulas are axioms and which are not), A_1, A_2, ..., A_n are **all** the hypotheses used in the proof, and B is the formula proved by the proof. Each formula in such a proof must be either an axiom, or a hypothesis from the set A_1, A_2, ..., A_n, or it must be obtained from earlier formulas (in this proof) by using a rule of inference. You may read the above notation as "in theory T, by using formulas A_1, A_2, ..., A_n as hypotheses, the formula B is proved".

As the first example, let us consider the following proof:

$$[L_5, MP]: \quad B, C \vdash B\wedge C \quad .$$

(1)	B	Hypothesis.
(2)	C	Hypothesis.
(3)	$B \rightarrow (C \rightarrow B \wedge C)$	It's the axiom schema $\mathbf{L_5}$.
(4)	$C \rightarrow B \wedge C$	It follows from (1) and (3) by *Modus Ponens*.
(5)	$B \wedge C$	It follows from (2) and (4) by *Modus Ponens*.

For more serious examples of formal proofs see the next Section 1.4 (Theorem 1.4.1 and Theorem 1.4.2).

In the real mathematical practice, when proving [T]: $A_1, A_2, ..., A_n \vdash B$, we may apply some theorem Q that already has been proved earlier. If we would simply insert Q into our formal proof, then, formally, this would yield only a proof of [T]: $A_1, A_2, ..., A_n, Q \vdash B$, i.e. Q would be qualified as a hypothesis. To obtain the desired formal proof of [T]: $A_1, A_2, ..., A_n \vdash B$, we must insert not only Q itself, but **the entire proof** of Q! In this way we obtain the following

Theorem 1.3.1. If there is a proof [T]: $A_1, A_2, ..., A_n, Q \vdash B$, and a proof [T]: $A_1, A_2, ..., A_n \vdash Q$, then there is a proof of [T]: $A_1, A_2, ..., A_n \vdash B$.

Proof. The proof of [T]: $A_1, A_2, ..., A_n, Q \vdash B$ is a sequence of formulas $F_1, F_2, ..., Q, ..., F_m, B$, and the proof of [T]: $A_1, A_2, ..., A_n \vdash Q$ is some sequence of formulas $G_1, G_2, ..., G_p, Q$. Let us replace Q by $G_1, G_2, ..., G_p, Q$:

$$F_1, F_2, ..., G_1, G_2, ..., G_p, Q, ..., F_m, B,$$

and eliminate the duplicate formulas. This sequence is a proof of [T]: $A_1, A_2, ..., A_n \vdash B$. Q.E.D.

If, in some proof, hypotheses are not used at all, then we may write simply [T]: $\vdash B$, or even $T \vdash B$, and say that B is a **theorem** of theory T. Of course, by using axioms directly one almost never can prove really complicated theorems. Still, we can retain our simple formal definition of the notion of theorem because of the following

Corollary 1.3.1. If there is a proof of [T]: $A_1, A_2, ..., A_n \vdash B$, and proofs of [T]: $\vdash A_1$, [T]: $\vdash A_2$, ..., [T]: $\vdash A_n$, then there is a proof of [T]: $\vdash B$.

Proof. Immediately, by Theorem 1.3.1.

Consistency

Sometimes, a seemingly plausible set of axioms allows deriving of contradictions (the most striking example − Russell's paradox in the "naive" set theory). A formula F is called a **contradiction** in the theory T, if and only if [T]: ⊢ F and [T]: ⊢ ¬F, i.e. if T proves and disproves F simultaneously. Theories allowinf to derive contradictions are called **inconsistent theories**. Thus, T is called a **consistent theory**; if and only if T does not allow deriving of contradictions.

Normally, for a first order theory, the set of all theorems is infinite, and, therefore, **consistency cannot be verified empirically**. We may only hope to establish this desirable property by means of some **theoretical proof** (see Podnieks [1997], Section 5.4 for a more detailed discussion of this problem).

For theories adopting the above logical axioms, inconsistency is, in a sense, "the worst possible property". Indeed, the axiom L_{10}: $\neg B \rightarrow (B \rightarrow C)$ says that if a theory allows deriving a contradiction, then, in this theory, anything is provable. In Section 2.4 we will − without L_{10} − prove 50% of it: $\neg B \rightarrow (B \rightarrow \neg C)$. Thus, even without L_{10} (but with L_1): if a theory allows deriving a contradiction, then, in this theory, anything is disprovable.

Is consistency enough for a theory to be "perfect"? In Section 4.3 we will prove the so-called Model Existence Theorem: if a first order theory is consistent, then there is a "model" (a kind of a "mathematical reality") where all its axioms and theorems are "true".

Completeness

If a formula contains free variables, i.e. variables that are not bound by quantifiers (for example: $x=0 \vee x=1$), then the "truth value" of such formulas may depend on particular values assigned to free variables. For example, the latter formula is "true" for x=1, yet it is "false" for x=2. Formulas that do not contain free occurrences of variables, are called **closed formulas**, for example:

$$\forall\, w \exists\, x(w < x \wedge prime(x)) \quad .$$

Closed formulas represent "definite assertions about the objects of our theory", and they are expected to be either "true", or "false". Or, in a first order **theory**, they are expected to be either provable, or disprovable (refutable). The above closed formula (stating that "there are infinitely many prime numbers") is

provable − if our theory is first order arithmetic.

T is called a **complete theory**, if and only if for each closed formula F in the language of T: [T]: ⊢F or [T]: ⊢¬F, i.e. if and only if T proves or disproves any closed formula of its language. In other words: a complete theory can solve any problem from the domain of its competence.

In an **incomplete theory**, some closed formulas ("definite assertions about the objects of theory") can be neither proved, not disproved. Thus, an incomplete theory can't solve some of the problems from the domain of its competence.

Formally, according to this definition, an inconsistent theory is complete. Indeed, the axiom L_{10}: ¬B→(B→C) says that if a theory allows deriving a contradiction, then, in this theory, anything is provable, i.e. it is a complete theory.

Of course, if T would be both consistent and complete, then we could call it "absolutely perfect". Unfortunately, Gödel's incompleteness theorem says thata all **fundamental mathematical theories are either inconsistent or incomplete**, i.e. none of them is "absolutely perfect" (see Mendelson [1997] or Podnieks [1997], Section 6.1).

Exercise 1.3.2 (optional). Re-formulate the above axiom system for a **many-sorted predicate language** (or, see Chapter 10. Many-Sorted First Order Logic, by Jean Gallier.)

1.4. The Flavor of Proving Directly

Theorem 1.4.1. [L_1, L_2, MP]: ⊢A→A for any formula A. What does it mean? It's the so-called **reflexivity property of implication**.

The following sequence of formulas represents a proof of the formula A→A:

(1)	(A→((C→A)→A))→((A→(C→A))→(A→A))	It's the axiom schema $\mathbf{L_2}$: (B→(C→D))→((B→C)→(B→D)), with B = A, C = C→A, D = A.
(2)	A→((C→A)→A)	It's the axiom schema $\mathbf{L_1}$: B→(C→B), with B = A, C = C→A.

(3) $(A\to(C\to A))\to(A\to A)$	It follows from (1) and (2) by *Modus Ponens*.
(4) $A\to(C\to A)$	It's the axiom schema $\mathbf{L_1}$: $B\to(C\to B)$, with B = A, C = C.
(5) $A\to A$	It follows from (3) and (4) by *Modus Ponens*.

As you can see, the proof is easy to verify, but it could be hard to build it from scratch. "Why" should we take "the axiom L_2 with B = A, C = C→A, D = A" for (1)?

How could one invent a proof like the above one? One of the versions could be as follows. First, let's try to find an axiom, from which we could get A→A as a consequence. By trying L_1, i.e. B→(C→B), and setting B=C=A, we could obtain A→(A→A), a dead end, perhaps. So, let's try L_2, i.e. (B→(C→D))→((B→C)→(B→D)). By setting B=D=A we obtain (A→(C→A))→((A→C)→(A→A)). It seems to be a good decision − because the first premise A→(C→A) is, in fact, L_1. Hence, by applying the MP rule, we obtain (A→C)→(A→A). Now, how to make A→C "provable"? Since C is, in fact, an arbitrary formula, we can replace C by C→A, obtaining (A→(C→A))→(A→A). The premise is here, again, L_1, hence, by applying the MP rule, we obtain A→A. Q.E.D. By performing all our replacements at the very beginning, we obtain the above proof of the formula A→A. [BTW, the above two smart "operations" − obtaining A→A within L_2, and making L_1 of A→C, are applications of the so-called **unification**, a very general and very important method used in intellectual computer programs, for details, see Section 5.7.]

Theorem 1.4.2. [L_1, L_2, MP]: A→B, B→C ⊢A→C, for any formulas A, B, C.

What does it mean? It's the so-called **Law of Syllogism** (by Aristotle), or the **transitivity property of implication**.

The following sequence of formulas represents a proof of the formula A→C from the hypotheses A→B and B→C:

(1) A→B	Hypothesis.
(2) B→C	Hypothesis.
(3) $(A\to(B\to C))\to((A\to B)\to(A\to C))$	It's the axiom schema $\mathbf{L_2}$: $(B\to(C\to D))\to((B\to C)\to(B\to D))$, with B = A, C = B, D = C.
(4) $(B\to C)\to(A\to(B\to C))$	It's the axiom schema $\mathbf{L_1}$: $B\to(C\to B)$, with B = B→C, C = A.

(5) $A\rightarrow(B\rightarrow C)$	It follows from (2) and (4) by *Modus Ponens*.
(6) $(A\rightarrow B)\rightarrow(A\rightarrow C)$	It follows from (3) and (5) by *Modus Ponens*.
(7) $A\rightarrow C$	It follows from (1) and (6) by *Modus Ponens*.

Note. Only axiom schemas L_1 and L_2 , and inference rule *Modus Ponens* are used for proving the Theorems 1.4.1 and 1.4.2. Hence, **these theorems will remain valid for any logical system containing L_1, L_2 and *Modus Ponens*.**

Exercise 1.4.1. Build sequences of formulas representing the following proofs (only the axiom schemas L_1 and L_2 and *Modus Ponens* are necessary):

a) [L_1, MP]: $A\vdash B\rightarrow A$ (a sequence of 3 formulas). What does it mean?

b) [L_2, MP]: $A\rightarrow B$, $A\rightarrow(B\rightarrow C)\vdash A\rightarrow C$ (a sequence of 5 formulas). What does it mean?

c) [L_1, L_2, MP]: $A\rightarrow(B\rightarrow C)\vdash B\rightarrow(A\rightarrow C)$ (a sequence of 9 formulas − thanks to Pavel Andreyev for the idea). What does it mean? It's the so-called **Premise Permutation Law**.

d) [L_1, L_2, MP]: $A\rightarrow(A\rightarrow B)\vdash A\rightarrow B$ (easy solution - a sequence of 9 formulas, smart solution by Arnold Ostrovsky – 8 formulas). What does it mean?

Theorem 1.4.3. [L_{14}, MP, Gen] If F is any formula, and G is any formula that does not contain x as a free variable, then

$G\rightarrow F(x)\vdash G\rightarrow\forall xF(x)$.

The following sequence of formulas represents a proof of the formula $G\rightarrow\forall xF(x)$ from the hypothesis $G\rightarrow F(x)$:

(1) $G\rightarrow F(x)$	Hypothesis.
(2) $\forall x(G\rightarrow F(x))$	It follows from (1) by Generalization.
(3) $\forall x(G\rightarrow F(x))\rightarrow(G\rightarrow\forall xF(x))$	It's the axiom schema $\mathbf{L_{14}}$ (G does not contain x as a free variable).

(4) G→∀xF(x)	It follows from (2) and (3) by *Modus Ponens*.

Exercise 1.4.2. Build sequences of formulas representing the following proofs (F is any formula, and G is a formula that does not contain x as a free variable):

a) [L_{15}, MP, Gen]: F(x)→G ⊢ ∃xF(x)→G (a sequence of 4 formulas). What does it mean?

b) [L_3-L_5, MP]: $A \wedge B \vdash B \wedge A$ (a sequence of 8 formulas). What does it mean?

c) [L_6-L_8, MP]: $\vdash A \vee B \to B \vee A$ (a sequence of 5 formulas). What does it mean?

d) [L_1, L_9, MP]: B, ¬B ⊢¬C (a sequence of 9 formulas). What does it mean? It's 50% of the axiom L_{10}!

e) [L_3, L_4, L_9, MP]: $\vdash \neg(A \wedge \neg A)$ (a sequence of 5 formulas). What does it mean? It's the so-called **Law of Non-Contradiction**.

f) [L_1, L_8, L_{10}, MP]: $\vdash \neg A \vee B \to (A \to B)$ (a sequence of 5 formulas). What does it mean?

g) [L_8, L_{11}, MP]: A→B, ¬A→B ⊢B (a sequence of 7 formulas). What does it mean?

h) [L_1-L_8, MP]: $A \to B \vdash A \vee C \to B \vee C$ (a sequence of 11 formulas). What does it mean?

i) [L_1-L_{11}, MP]: $\vdash A \vee (A \to B)$ (a sequence of 15 formulas). What does it mean? Does it mean anything at all?

Exercise 1.4.3 (optional, for smart students). Could you build shorter sequences proving the formulas of Exercise 1.4.1 c, d) and Exercise 1.4.2 b, d)? Evgeny Vihrov verified in 2011 that any proof of the formula of Exercise 1.4.1 d) will be longer than 5 formulas.

1.5. Deduction Theorems

If, by assuming B as a hypothesis, we have proved C, then we have proved that B implies C. This natural way of reasoning is formalized in the so-called

deduction theorems (introduced by Jacques Herbrand and Alfred Tarski):

J. Herbrand. Recherches sur la théorie de la démonstration. PhD Thesis, University of Paris, 1930 (approved in April 1929).

A. Tarski. Ueber einige fundamentale Begriffe der Metamathematik. "Comptes Rendus de Séances de la Société des Sciences et des Lettres de Varsovie, Classe III", 1930, Vol.23, pp. 22-29.

We will prove two such theorems – Deduction Theorem 1 (for propositional logic) and Deduction Theorem 2 (for predicate logic).

Theorem 1.5.1 (Deduction Theorem 1). If T is a first order theory, and there is a proof of

$$[T, MP]: A_1, A_2, ..., A_n, B \vdash C,$$

then there is a proof of

$$[L_1, L_2, T, MP]: A_1, A_2, ..., A_n \vdash B \rightarrow C.$$

I.e. having a *Modus Ponens* proof of C from the hypotheses $A_1, A_2, ..., A_n, B$, we can build a *Modus Ponens* proof of $B \rightarrow C$ from the hypotheses $A_1, A_2, ..., A_n$.

It appears that, usually, proving of [T, MP]: ... $B \vdash C$ is easier (technically simpler) than proving of [T, MP]: ... $\vdash B \rightarrow C$.

Exercise 1.5.1 (optional, for smart students). Do not read the proof below. Try proving yourself.

Proof (thanks to Sergey Kozlovich for the idea, see also Kleene [1967], Exercise 10C). We must define a procedure allowing to convert any proof of $[T, MP]: A_1, A_2, ..., A_n, B \vdash C$ into a proof of $[L_1, L_2, T, MP]: A_1, A_2, ..., A_n \vdash B \rightarrow C$.

The easy way to do this would be using an induction by the number of formulas in the proof of $[T, MP]: A_1, A_2, ..., A_n, B \vdash C$. But we will use a more elegant idea. Any proof of $[T, MP]: A_1, A_2, ..., A_n, B \vdash C$ is a sequence of formulas $F_1, F_2, ...F_m$. We will replace each formula F_i by 3 or 5 formulas, the last of these being the formula $B \rightarrow F_i$, retaining our sequence as a valid proof.

We must consider the following cases:

1) F is an axiom (i.e. an instance of a logical axiom or a non-logical axiom of T). Replace F by 3 formulas: F, $F \rightarrow (B \rightarrow F)$, $B \rightarrow F$. The second formula is an instance of L_1, the third formula is obtained from the first two ones by using *Modus Ponens*.

2) F is one of the hypotheses A_i. Replace F by 3 formulas: F, $F \to (B \to F)$, $B \to F$. The second formula is an instance of L_1, the third formula is obtained from the first two ones by using *Modus Ponens*.

3) F is B. Replace F by the 5 formulas from the proof of Theorem 1.4.1, where D can be any formula:

$(B \to ((D \to B) \to B)) \to ((B \to (D \to B)) \to (B \to B))$ (an instance of L_2),

$B \to ((D \to B) \to B)$ (an instance of L_1),

$B \to (D \to B)) \to (B \to B)$ (by *Modus Ponens*),

$B \to (D \to B)$ (an instance of L_1),

$B \to B$ (by *Modus Ponens*).

The last formula is here, of course, $B \to F$.

4) F is derived from some previous formulas F_i and F_j by *Modus Ponens*, F_i having the form $F_j \to F$ (i.e. $F_j \to F$ and F_j yield F by *Modus Ponens*). Then, the formulas

$B \to F_j$,

$B \to (F_j \to F)$

are already present in the converted proof (they appeared during the replacement operations applied to the formulas F_j and $F_j \to F$). So, replace F by 3 formulas:

$(B \to (F_j \to F)) \to ((B \to F_j) \to (B \to F))$ (an instance of L_2),

$(B \to F_j) \to (B \to F)$ (by *Modus Ponens*),

$B \to F$ (by *Modus Ponens*).

Thus, what we have now, is a correct proof in $[L_1, L_2, MP]$ that is using the hypotheses $A_1, A_2, ..., A_n$, but not B! The last formula of this proof is $B \to C$ (because C is the last formula of our initial proof of $[L_1, L_2, MP]$: $A_1, A_2, ..., A_n, B \vdash C$). Thus, we have a proof of $[L_1, L_2, MP]$: $A_1, A_2, ..., A_n \vdash B \to C$. Q.E.D.

The above proof of Deduction Theorem 1 includes, in fact, an **algorithm** allowing to obtain a proof of $[L_1, L_2, MP]$: $A_1, A_2, ..., A_n \vdash B \to C$ from a given proof of $[L_1, L_2, MP]$: $A_1, A_2, ..., A_n, B \vdash C$. The resulting proof is longer than the given one: if the given proof consists of *m* formulas, then the resulting proof consists of *3m* or 3*m*+2 formulas).

Corollaries 1.5.1. 1) If there is a proof of

$$[T, MP]: A_1, A_2, ..., A_n, B_1, B_2, ..., B_k \vdash C,$$

then there is a proof of

$$[L_1, L_2, T, MP]: A_1, A_2, ..., A_n \vdash (B_1 \to (B_2 \to (... \to (B_k \to C)...))).$$

2) If T includes (or proves) schemas L_1, L_2, then, if there is a proof of [T, MP]: $A_1, A_2, ..., A_n, B \vdash C$ then there is a proof of [T, MP]: $A_1, A_2, ..., A_n \vdash B \to C$.

In particular, if [T, MP]: $B \vdash C$, then [T, MP]: $\vdash B \to C$.

And, if [T, MP]: $B, C \vdash D$, then [T, MP]: $\vdash B \to (C \to D)$.

Proof. 1) By iterating Deduction Theorem 1.

2) If T is a theory which includes or proves the schemas L_1, L_2, then [L_1, L_2, T, MP] is equivalent to [T, MP]. Q.E.D.

Exercise 1.5.2 (optional, for smart students). In earlier versions of logical axioms, instead of the axiom L_2, in some texts, the following 3 axioms were in use:

L_{21}: $(A \to (A \to B)) \to (A \to B)$,

L_{22}: $(A \to (B \to C)) \to (B \to (A \to C))$ (i.e. the Premise Permutation Law),

L_{23}: $(A \to B) \to ((B \to C) \to (A \to C))$ (the Law of Syllogism, or the transitivity property of implication).

Verify that both versions, i.e. [L_1, L_2, MP] and [L_1, L_{21}, L_{23}, L_{23}, MP] are equivalent. (Hint: a) See Section 2.1 to verify that [L_1, L_2, MP] proves L_{21}, L_{23}, L_{23}. b) Verify that [L_1, L_{21}, L_{23}, L_{23}, MP] proves L_2 either directly, or by proving the Deduction Theorem 1 for [L_1, L_{21}, L_{23}, L_{23}, MP].)

Exercise 1.5.3 (optional, thanks to Sergey Kozlovich for the idea).

a) Prove the following "generalization" of the *Modus Ponens* rule:

[L_1, L_2, MP]: $(D_1 \to (D_2 \to ... (D_k \to B)...))$, $(D_1 \to (D_2 \to ... (D_k \to (B \to C))...)) \vdash (D_1 \to (D_2 \to ... (D_k \to C)...))$.

b) Prove the following "generalization" of the axiom L_{14} (formulas D_1, D_2, ..., D_k do not contain x as a free variable):

[L_1, L_2, L_{14}, MP]: $\vdash \forall x(D_1 \to (D_2 \to ... (D_k \to F(x))...) \to (D_1 \to (D_2 \to ...$

$(D_k \to \forall x F(x))...)$.

Exercise 1.5.4 (optional, for smart students). Investigate the size (the number of formulas) of the proof of $[L_1, L_2, MP]: A_1, A_2, ..., A_n, \vdash B \to C$ as a function f(m) of the size m of the proof of $[L_1, L_2, MP]: A_1, A_2, ..., A_n, B \vdash C$. You may wish to report your result. We will publish your report on the web as an appendix to this book. The current record holder is Sergey Kozlovich, 2004: $f(m) \leq 3m+2$. Improve this result, or prove that it is the best one possible.

Exercise 1.5.5 (optional, for smart students). Investigate the size (the number of instances of **atomic** formulas) of the proof of $[L_1, L_2, MP]: A_1, A_2, ..., A_n, \vdash B \to C$ as a function g(m) of the size m of the proof of $[L_1, L_2, MP]: A_1, A_2, ..., A_n, B \vdash C$. You may wish to report your result. We will publish your report on the web as an appendix to this book. The current record holder is Kirils Solovjovs, 2008: $g(m, n) \leq 7m+24n-2$, where n is the number of instances of atomic formulas in the formula B. Improve this result, or prove that it is the best one possible.

Warning! *Generalization* involved...

Now, what, if in the proof of $A_1, A_2, ..., A_n, B \vdash C$ not only *Modus Ponens*, yet also *Generalization* is used?

We must be careful, because, trying "simply" to apply Deduction Theorem 1, we can obtain crazy results. Indeed, having a formula F(x), by *Generalization*, we obtain the formula $\forall x F(x)$. Thus, $F(x) \vdash \forall x F(x)$. If Deduction Theorem 1 could be extended to Gen without any restrictions, then we could conclude that $\vdash F(x) \to \forall x F(x)$. If this is true for any x, it is true also for x=2, hence, $\vdash F(2) \to \forall x F(x)$. Thus, if the number 2 is prime, then all numbers are prime?

So, let us try deriving a restricted formulation of the Deduction Theorem − it seems, we should **prohibit application of Gen to the free variables of B** − the hypothesis "to be moved".

Theorem 1.5.2 (Deduction Theorem 2). If T is a first order theory, and there is a proof of

$$[T, MP, Gen]: A_1, A_2, ..., A_n, B \vdash C,$$

where *Generalization* is not applied to the free variables of B, then there is a proof of

$$[L_1, L_2, L_{14}, T, MP, Gen]: A_1, A_2, ..., A_n \vdash B \to C.$$

Proof. We must extend the above proof of the Deduction Theorem 1 that consisted of 4 cases. First, we must extend the first case:

1') F is an axiom (i.e. an instance of a logical axiom or a non-logical axiom of T). Replace F by 3 formulas: F, F→(B→F), B→F. The second formula is an instance of L_1, the third formula is obtained from the first two ones by using *Modus Ponens*.

And we must add the following case:

5) F is derived from some previous formula F_i by *Generalization*, thus, F having the form $\forall xF_i$, where x is not free in the formula B. Replace F by the following 3 formulas:

$\forall x(B\rightarrow F_i)\rightarrow(B\rightarrow\forall xF_i)$,

$\forall x(B\rightarrow F_i)$,

$B\rightarrow\forall xF_i$.

Since x is not free in B, the first formula is an instance of L_{14}. The second formula is obtained by *Generalization* from the formula $B\rightarrow F_i$ that is already present in the converted proof (it appeared during the replacement operation applied to the formula F_i). The third formula is obtained from the first two ones by using *Modus Ponens*.

Thus, what we have now, is a correct proof in [L_1, L_2, L_{14}, MP, Gen] that is using the hypotheses $A_1, A_2, ..., A_n$, but not B! The last formula of this proof is B→C (because C is the last formula our initial proof of [L_1, L_2, L_{14}, MP, Gen]: $A_1, A_2, ..., A_n, B \vdash C$). Thus, we have a proof of [L_1, L_2, L_{14}, MP, Gen]: $A_1, A_2, ..., A_n \vdash B\rightarrow C$. Q.E.D.

Corollary 1.5.2. 1) If there is a proof of

$$[T, MP, Gen]: A_1, A_2, ..., A_n, B_1, B_2, ..., B_k \vdash C,$$

where *Generalization* is not applied to the the free variables of the formulas $B_1, B_2, ..., B_k$, then there is a proof of

$$[L_1, L_2, L_{14}, T, MP, Gen]: A_1, A_2, ..., A_n \vdash (B_1\rightarrow(B_2\rightarrow(...\rightarrow(B_k\rightarrow C)...))).$$

2) If B is a **closed** formula, and there is a proof of

$$[T, MP, Gen]: A_1, A_2, ..., A_n, B \vdash C,$$

then there is a proof of

$$[L_1, L_2, L_{14}, T, MP, Gen]: A_1, A_2, ..., A_n \vdash B\rightarrow C.$$

3) If T is a theory whose axioms include schemas L_1, L_2, L_{14}, then, if there is a

proof of

$$[T, MP, Gen]: A_1, A_2, ..., A_n, B \vdash C,$$

where *Generalization* is not applied to the the free variables of B, then there is a proof of

$$[T, MP, Gen]: A_1, A_2, ..., A_n \vdash B \rightarrow C.$$

In particular, if [T, MP, Gen]: $B \vdash C$, where *Generalization* is not applied to the free variables of B, then [T, MP, Gen]: $\vdash B \rightarrow C$.

Proof. Similar to the proof of the above Corollaries of Deduction Theorem 1.

Warning! Previously proved theorems involved...

In the real mathematical practice, when proving [T, MP, Gen]: $A_1, A_2, ..., A_n \vdash C$, we may wish to apply some theorem Q that we have already proved earlier. If we would simply insert Q into our formal proof, then, formally, this would yield only that [T, MP, Gen]: $A_1, A_2, ..., A_n, Q \vdash C$. To obtain the desired formal proof of [T, MP, Gen]: $A_1, A_2, ..., A_n \vdash C$, we must insert not only Q itself, but **the entire proof** of Q!

Still, with the Deduction Theorem 2 this may be problematic. If we are proving [T, MP, Gen]: $A_1, A_2, ..., A_n, B \vdash C$ with the intention to apply Deduction Theorem 2 (to obtain [T, MP, Gen]: $A_1, A_2, ..., A_n \vdash B \rightarrow C$), then, before inserting the proof of Q, we must ensure that, in this proof, *Generalization* is not applied to the free variables of B. But, of course, the original proof of Q **could** contain such *Generalization*s! To solve this problem, we could try, in the proof of Q, before inserting it, rename simultaneously all the variables to which *Generalization* is applied and which are free variables in B. But this simultaneous renaming may affect the bound variables of Q, and thus − destroy the intended use of Q.

The problem is solved completely by the following extension of the Deduction Theorem 2:

Theorem 1.5.3 (Deduction Theorem 2A). If there is a proof of

$$[T, MP, Gen]: A_1, A_2, ..., A_n, B \vdash C,$$

where, **after B appears in the proof**, *Generalization* is not applied to the free variables of B, then there is a proof of

$$[L_1, L_2, L_{14}, T, MP, Gen]: A_1, A_2, ..., A_n \vdash B \rightarrow C.$$

Indeed, having such a theorem, we obtain the necessary

Corollary 1.5.3. If there is a proof of

$$[T, MP, Gen]: A_1, A_2, ..., A_n, B, Q \vdash C,$$

where, after B appears in the proof, *Generalization* is not applied to the free variables of B, and there is a proof of

$$[T, MP, Gen]: A_1, A_2, ..., A_n \vdash Q,$$

then there is a proof of

$$[T, MP, Gen]: A_1, A_2, ..., A_n \vdash B \rightarrow C.$$

Proof of the Corollary. In the proof of $[T, MP, Gen]: A_1, A_2, ..., A_n, B, Q \vdash C$, first, move all the hypotheses $A_1, A_2, ..., A_n$ to the beginning. Then, immediately after them, insert the proof of $[T, MP, Gen]: A_1, A_2, ..., A_n \vdash Q$. Now we have a proof of $[T, MP, Gen]: A_1, A_2, ..., A_n, B \vdash C$, where, after B appears in the proof, *Generalization* is not applied to the free variables of B. By Deduction Theorem 2A, then there is a proof of $[T, MP, Gen]: A_1, A_2, ..., A_n \vdash B \rightarrow C$. Q.E.D.

Proof of the Deduction Theorem 2A. Let us modify the above proof of the Deduction Theorem 2.

We must define a procedure allowing to convert any allowed proof of $[T, MP, Gen]: A_1, A_2, ..., A_n, B \vdash C$ into a proof of $[L_1, L_2, T, MP, Gen]: A_1, A_2, ..., A_n \vdash B \rightarrow C$.

Unlike the above proof, let us leave unchanged all the formulas of the proof of $[T, MP]: A_1, A_2, ..., A_n, B \vdash C$ **before B appears in the proof**. After this, **starting with B**, we will replace each formula F by 3 or 5 formulas, one of them being the formula $B \rightarrow F$.

We must consider the following cases:

1), 2), 3) − as in the proof of the Deduction Theorem 1.

4) F is derived from some previous formulas F_i and F_j by *Modus Ponens*, F_i having the form $F_j \rightarrow F$ (i.e. $F_j \rightarrow F$ and F_j yield F by *Modus Ponens*). Then, 4 subcases are possible.

4a) F_j and $F_j \rightarrow F$ both appear before B, i.e. they remain unchanged in the converted proof. Let us replace F by the following 3 formulas: F, $F \rightarrow (B \rightarrow F)$, $B \rightarrow F$. The second formula is an instance of L_1, the third formula is obtained by using *Modus Ponens* from the first two ones.

4b) F_j appears before B, and $F_j \to F$ is B or appears after B. Then, the formulas F_j and $B \to (F_j \to F)$ are already present in the converted proof. Let us replace F by the following 5 formulas:

$(B \to (F_j \to F)) \to ((B \to F_j) \to (B \to F))$ (an instance of L_2),

$(B \to F_j) \to (B \to F)$ (by *Modus Ponens*),

$Fj \to (B \to F_j)$ (an instance of L_1),

$B \to F_j$ (by *Modus Ponens*),

$B \to F$ (by *Modus Ponens*).

4c) F_j is B or appears after B, and $F_j \to F$ appears before B. Then, the formulas $B \to F_j$ and $F_j \to F$ are already present in the converted proof. Let us replace F by the following 5 formulas from the proof of Theorem 1.4.2:

$(B \to (F_j \to F)) \to ((B \to F_j) \to (B \to F))$ (an instance of L_2),

$(F_j \to F) \to (B \to (F_j \to F))$ (an instance of L_1),

$B \to (F_j \to F)$ (by *Modus Ponens*),

$(B \to F_j) \to (B \to F)$ (by *Modus Ponens*),

$B \to F$ (by *Modus Ponens*).

4d) F_j and $F_j \to F$ both are B or appear after B. Then, the formulas $B \to F_j$ and $B \to (F_j \to F)$ are already present in the converted proof (they appeared during the replacement operations applied to the formulas F_j and $F_j \to F$). Let us replace F by the following 3 formulas:

$(B \to (F_j \to F)) \to ((B \to F_j) \to (B \to F))$ (an instance of L_2),

$(B \to F_j) \to (B \to F)$ (by *Modus Ponens*),

$B \to F$ (by *Modus Ponens*).

5) F is derived from some previous formula F_i by *Generalization*, thus, F is in the form $\forall x F_i$. Then, 2 subcases are possible.

5a) F_i appears before B. Then x is not free in B. Let us replace F by the following 3 formulas:

F (by *Generalization*, x is not free in B),

$F \to (B \to F)$ (an instance of L_1),

$B \to F$

5b) F_i is B or appears after B. Then x is not free in B, and the formula $B \to F_i$ that is already present in the converted proof (it appeared during the replacement operation applied to the formula F_i). Let us replace F by the following 3 formulas:

$\forall x(B \to F_i)$ (by *Generalization*, x is not free in B),

$\forall x(B \to F_i) \to (B \to \forall x F_i)$ (an instance of L_{14}, since x is not free in B),

$B \to \forall x F_i$ (by *Modus Ponens*).

Thus, what we have now, is a correct proof in $[L_1, L_2, L_{14}, T, MP, Gen]$ that is using the hypotheses $A_1, A_2, ..., A_n$, but not B! The last formula of this proof is $B \to C$ (because C is the last formula our initial proof of [T, MP, Gen]: $A_1, A_2, ..., A_n, B \vdash C$). Thus, we have a proof of $[L_1, L_2, L_{14}, T, MP, Gen]$: $A_1, A_2, ..., A_n \vdash B \to C$. Q.E.D.

Exercise 1.5.6 (optional, for smart students). In some other textbooks, a somewhat different system of logical axioms is used: instead of the axioms L_{14}, L_{15} and the *Generalization* rule the following two rules of inference are used:

$G \to F(x) \vdash G \to \forall x F(x)$ ($\forall$-Introduction);

$F(x) \to G \vdash \exists x F(x) \to G$ ($\exists$-Elimination).

Of course, here, G is a formula that does not contain x as a free variable. Verify that both systems are equivalent in all of their versions (minimal, constructive, and classical).

2. Propositional Logic

George Boole (1815-1864): "In 1854 he published *An Investigation into the Laws of Thought, on Which are founded the Mathematical Theories of Logic and Probabilities.* Boole approached logic in a new way reducing it to a simple algebra, incorporating logic into mathematics. He pointed out the analogy between algebraic symbols and those that represent logical forms. It began the algebra of logic called Boolean algebra which now finds application in computer construction, switching circuits etc." (according to MacTutor History of Mathematics archive).

See also:

G.Boole. The Calculus of Logic. *The Cambridge and Dublin Mathematical Journal*, vol. 3 (1848) (available online at http://www.maths.tcd.ie/pub/HistMath/People/Boole/CalcLogic/, published by David R. Wilkins).

2.1. Proving Formulas Containing Implication only

Let us return to the Exercise 1.4.1(d), where you produced a sequence of 9 formulas proving the following:

d) $[L_1, L_2, MP]$: $A\to(A\to B) \vdash A\to B$.

Did you try the next step – proving of

d') $[L_1, L_2, MP]$: $\vdash(A\to(A\to B))\to(A\to B)$?

For proving directly – almost an impossible task!

Now, having deduction theorems, we can simplify the task of proving d), and make the task of proving d') feasible. **More precisely – the task of proving that d) and d') are provable.** Indeed,

(1)	$A\to(A\to B)$	Hypothesis.
(2)	A	Hypothesis.
(3)	$A\to B$	By MP, from (1), (2).
(4)	B	By MP, from (2), (3).

Thus, we have established that $A\to(A\to B), A \vdash B$. Now, by Deduction Theorem 1,

$$[L_1, L_2, MP]: A\to(A\to B) \vdash A\to B.$$

And let us apply this theorem once more,

$$[L_1, L_2, MP]: \vdash (A\to(A\to B))\to(A\to B).$$

Note. In fact, we proved here only $A\to(A\to B), A \vdash B$, but **we did not prove d) and d'), i.e. we did not produce the corresponding sequences of formulas. We just proved that these sequences do exist!** To produce them really, we must apply the algorithm described in the proof of Deduction Theorem 1.

Exercise 2.1.1. Imagine applying the algorithm described in the proof of Deduction Theorem 1: a) to the above 4 formula sequence – producing a sequence of 44 formulas proving $[L_1, L_2, MP]: \vdash (A\to(A\to B))\to(A\to B)$; b) to your 9 formula proof of (d) – producing a sequence of 29 formulas proving the same.

Warning! Always be careful when selecting hypotheses. For example, to prove the strange formula (the so-called Peirce's Law) $\vdash ((A\to B)\to A)\to A$ (it **is** provable in the classical logic, **not** in the constructive logic!), you can try proving that $(A\to B)\to A \vdash A$, but not $A\to B, A \vdash A$. Why? Because, by Deduction Theorem 1, from $A\to B, A \vdash A$ it follows that $A\to B \vdash A\to A$ and $\vdash (A\to B)\to(A\to A)$, or $A \vdash (A\to B)\to A$ and $\vdash A\to((A\to B)\to A)$. Where do you see $\vdash((A\to B)\to A)\to A$ here?

Exercise 2.1.2. Prove the following $[L_1, L_2, MP]$:

a) $\vdash((A\to B)\to(A\to C))\to(A\to(B\to C))$. What does this formula mean?

b) $\vdash (A\to B)\to((B\to C)\to(A\to C))$. What does this formula mean? It's another version of the so-called **Law of Syllogism** (by Aristotle), or the **transitivity property of implication**. Explain the difference between this formula and Theorem 1.4.2: $A\to B, B\to C \vdash A\to C$.

c) $\vdash (A\to(B\to C))\to(B\to(A\to C))$. What does this formula mean? It's another version of the so-called **Premise Permutation Law.** Explain the difference between this formula and Exercise 1.4.1(c): $A\to(B\to C) \vdash B\to(A\to C)$.

2.2. Proving Formulas Containing Conjunction

Theorem 2.2.1. a) $[L_5, MP]$ $A, B \vdash A\wedge B$.

b) $[L_3, L_4, MP]$: $A\wedge B \vdash A$, $A\wedge B \vdash B$.

Let us prove (a).

(1) A	Hypothesis.
(2) B	Hypothesis.
(3) $A \to (B \to A \wedge B)$	Axiom $\mathbf{L_5}$: $B \to (C \to B \wedge C)$ with B = A, C = B.
(4) $B \to A \wedge B$	By MP, from (1) and (3).
(5) $A \wedge B$	By MP, from (2) and (4).

Now, let us prove (b).

(1) $A \wedge B$	Hypothesis.
(2) $A \wedge B \to A$	Axiom $\mathbf{L_3}$: $B \wedge C \to B$ with B = A, C = B.
(3) A	By MP, from (1) and (2).

Thus, $A \wedge B \vdash$A.

(1) $A \wedge B$	Hypothesis.
(2) $A \wedge B \to B$	Axiom $\mathbf{L_4}$: $B \wedge C \to C$ with B = A, C = B.
(3) B	By MP, from (1) and (2).

Thus, $A \wedge B \vdash$B.

Theorem 2.2.1 allows easy proving of **equivalences**. Let us remind that B↔C is defined as a shortcut for $(B \to C) \wedge (C \to B)$. Of course, we will call B and C equivalent formulas, if and only if ⊢B↔C. For example, by Theorem 1.4.1, [L_1, L_2, MP] ⊢A→A, hence, [L_1, L_2, L_5, MP] ⊢ $(A \to A) \wedge (A \to A)$, i.e.

$$[L_1, L_2, L_5, MP] \vdash A \leftrightarrow A.$$

Of course, (a) of the Exercise 2.1.2 is the reverse formula of the axiom L_2. Hence, by Theorem 2.2.1:

$$[L_1, L_2, L_5, MP] \vdash (A \to (B \to C)) \leftrightarrow ((A \to B) \to (A \to C)).$$

By (c) of the Exercise 2.1.2, and Theorem 2.2.1:

$$[L_1, L_2, L_5, MP] \vdash (A \to (B \to C)) \leftrightarrow (B \to (A \to C))$$

Now, let us prove another form of the Law of Syllogism, or Theorem 1.4.2 [L_1, L_2, MP]: A→B, B→C ⊢A→C:

$$[L_1\text{-}L_4, MP] \vdash (A \to B) \wedge (B \to C) \to (A \to C) \quad .$$

(1)	$(A \to B) \wedge (B \to C)$	Hypothesis.
(2)	$(A \to B) \wedge (B \to C) \to (A \to B)$	Axiom $\mathbf{L_3}$: $B \wedge C \to B$ with B = A→B, C = B→C.
(3)	$(A \to B) \wedge (B \to C) \to (B \to C)$	Axiom $\mathbf{L_4}$: $B \wedge C \to C$ with B = A→B, C = B→C.
(4)	A→B	By MP, from (1) and (2).
(5)	B→C	By MP, from (1) and (3).
(6)	A→C	By by the transitivity property of implication (Theorem 1.4.2).

Thus, we have established that [L_1-L_4, MP]: $(A \to B) \wedge (B \to C)$ ⊢A→C. By Deduction Theorem 1, [L_1-L_4, MP] ⊢ $((A \to B) \wedge (B \to C)) \to (A \to C)$.

Exercise 2.2.1. Prove the following [L_1- L_5, MP]:

a) A→B, A→C ⊢ $A \to B \wedge C$. What does it mean?

b) ⊢ $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$. What does it mean?

c) $A \to B \wedge C$ ⊢A→B. What does it mean?

d) $A \to B \wedge C$ ⊢A→C. What does it mean?

e) ⊢ $(A \to B \wedge C) \to (A \to B) \wedge (A \to C)$. What does it mean?

Hence,

$$[L_1\text{-}L_5, MP]: \vdash (A \to B \wedge C) \leftrightarrow (A \to B) \wedge (A \to C) \quad .$$

Exercise 2.2.2. Prove the following, [L_1- L_5, MP]:

a) ⊢ $A \wedge B \leftrightarrow B \wedge A$. What does it mean? That **conjunction is commutative**.

b) ⊢ $A \wedge (B \wedge C) \leftrightarrow (A \wedge B) \wedge C$. What does it mean? That **conjunction is associative**.

c) ⊢ $A \wedge A \leftrightarrow A$. What does it mean? That **conjunction is idempotent**.

Exercise 2.2.3. Prove the following, [L_1- L_5, MP]:

a) $\vdash (A \to (B \to C)) \leftrightarrow (A \wedge B \to C)$. What does it mean?

b) $\vdash (A \to B) \to (A \wedge C \to B \wedge C)$. What does it mean? The converse formula $(A \wedge C \to B \wedge C) \to (A \to B)$ cannot be true. Explain, why.

c) A $\vdash B \leftrightarrow B \wedge A$. What does it mean?

Exercise 2.2.4. Let us remind that the equivalence connective A↔B is defined as a shortcut for $(A \to B) \wedge (B \to A)$. Prove the following properties of this connective [L_1- L_5, MP]:

(a) ⊢A↔A (reflexivity),

(b) ⊢(A↔B)→(B↔A) (symmetricity),

(c) $\vdash (A \leftrightarrow B) \wedge (B \leftrightarrow C) \to (A \leftrightarrow C)$ (transitivity).

2.3. Proving Formulas Containing Disjunction

Exercise 2.3.1. Prove the following [L_1, L_2, L_6-L_8, MP]:

a) [L_8, MP]: A→C, B→C ⊢ $A \vee B \to C$. What does it mean?

b) [L_5, L_6-L_8, MP]: ⊢ $A \vee B \leftrightarrow B \vee A$. What does it mean? That **disjunction is commutative**.

c) [L_1, L_2, L_5, L_6-L_8, MP]: ⊢ $A \vee A \leftrightarrow A$. What does it mean? That **disjunction is idempotent**.

Theorem 2.3.0. [L_1, L_2, L_8, MP]: If there is a proof of

$$A_1, A_2, ..., A_n, B \vdash D,$$

and a proof of

$$A_1, A_2, ..., A_n, C \vdash D,$$

then there is a proof of

$$A_1, A_2, ..., A_n, B \vee C \vdash D.$$

Exercise 2.3.2. Prove Theorem 2.3.0.

By using Theorem 2.3.0, we can prove that **disjunction is associative:**

$$[L_1, L_2, L_5, L_6\text{-}L_8, MP]: \vdash A \vee (B \vee C) \leftrightarrow (A \vee B) \vee C .$$

Indeed, to prove, for example,

$$\vdash A \vee (B \vee C) \rightarrow (A \vee B) \vee C \qquad (*)$$

we can first prove $\vdash A \rightarrow (A \vee B) \vee C$ and $\vdash B \vee C \rightarrow (A \vee B) \vee C$, and after that – apply Theorem 2.3.0. Proving of the second formula can be reduced in the same way. Thus, (*) would be proved, if we could prove that

$$\vdash A \rightarrow (A \vee B) \vee C \;, \vdash B \rightarrow (A \vee B) \vee C \;, \vdash C \rightarrow (A \vee B) \vee C \;.$$

(1) $C \rightarrow (A \vee B) \vee C$ Axiom $\mathbf{L_7}$.

Now, let us prove that

$\vdash B \rightarrow (A \vee B) \vee C$.

(2) $B \rightarrow A \vee B$ Axiom $\mathbf{L_7}$.

(3) $A \vee B \rightarrow (A \vee B) \vee C$ Axiom $\mathbf{L_6}$.

(4) $B \rightarrow (A \vee B) \vee C$ From (2) and (3), by the transitivity property of implication (Theorem 1.4.2).

Now, let us prove that

$\vdash A \rightarrow (A \vee B) \vee C$.

(5) $A \rightarrow A \vee B$ Axiom $\mathbf{L_6}$.

(6) $A \vee B \rightarrow (A \vee B) \vee C$ Axiom $\mathbf{L_6}$.

(7) $A \rightarrow (A \vee B) \vee C$ From (5) and (6), by the transitivity property of implication (Theorem 1.4.2).

Exercise 2.3.3. a) Prove the converse:

$[L_1, L_2, L_6\text{-}L_8, MP]$: $\vdash (A \vee B) \vee C \rightarrow A \vee (B \vee C)$.

b) Prove (use Deduction Theorem 1) that $[L_1, L_2, L_6\text{-}L_8, MP]$: $\vdash (A \rightarrow B) \rightarrow (A \vee C \rightarrow B \vee C)$. What does it mean? The converse formula $(A \vee C \rightarrow B \vee C) \rightarrow (A \rightarrow B)$ cannot be true. Explain, why.

c) Prove that $[L_1, L_2, L_6\text{-}L_8, MP]$: $\vdash A \rightarrow B, C \rightarrow D \vdash A \vee C \rightarrow B \vee D$. What does it mean?

The following theorem corresponds to the well-known **distributive property** of (number) addition to multiplication: (a+b)c = ac+bc. Of course, the "dual" distributive property (i.e. – of multiplication to addition) does not hold for numbers: ab+c=(a+c)(b+c) would imply ab+c=ab+ac+bc+cc, c=ac+bc+cc, and, if c<>0, then 1=a+b+c. Still, surprisingly, in logic,

Theorem 2.3.1. Conjunction is distributive to disjunction, and disjunction is distributive to conjunction:

$$[L_1\text{-}L_8, MP]: \vdash (A \wedge B) \vee C \leftrightarrow (A \vee C) \wedge (B \vee C) \quad .$$
$$[L_1\text{-}L_8, MP]: \vdash (A \vee B) \wedge C \leftrightarrow (A \wedge C) \vee (B \wedge C) \quad .$$

First, let us prove that $\vdash (A \wedge B) \vee C \rightarrow (A \vee C) \wedge (B \vee C)$.

(1) Prove $\vdash A \wedge B \rightarrow (A \vee C) \wedge (B \vee C)$

(2) Prove $\vdash C \rightarrow (A \vee C) \wedge (B \vee C)$

(3) $\vdash (A \wedge B) \vee C \rightarrow (A \vee C) \wedge (B \vee C)$ From (1) and (2), by Exercise 2.3.1(a).

Exercise 2.3.4. a) Prove (1) and (2). b) (optional) Do not read the following proof. Try proving yourself.

Now, let us prove the converse: $\vdash (A \vee C) \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$.

Note. The proof below starts with C as a hypothesis. Why not with $(A \vee C) \wedge (B \vee C)$? Because, we will use Deduction Theorem 1 to prove the intermediate formula (6) $C \rightarrow (B \vee C \rightarrow (A \wedge B) \vee C)$, not the final result!

(1)	C	Hypothesis.
(2)	B→C	From (1).
(3)	$C \rightarrow (A \wedge B) \vee C$	Axiom L_7.
(4)	$B \rightarrow (A \wedge B) \vee C$	From (2) and (3).
(5)	$B \vee C \rightarrow (A \wedge B) \vee C$	From (4) and (3).
(6)	$C \rightarrow (B \vee C \rightarrow (A \wedge B) \vee C)$	From (1)-(5), by Deduction Theorem 1.
(7)	$(B \rightarrow A \wedge B) \rightarrow (B \vee C \rightarrow (A \wedge B) \vee C)$	Exercise 2.3.3(b).
(8)	$A \rightarrow (B \rightarrow A \wedge B)$	Axiom L_3.
(9)	$A \rightarrow (B \vee C \rightarrow (A \wedge B) \vee C)$	From (8) and (7).
(10)	$A \vee C \rightarrow (B \vee C \rightarrow (A \wedge B) \vee C)$	From (9) and (6).
(11)	$(A \vee C) \wedge (B \vee C) \rightarrow (A \wedge B) \vee C$	From (10), by Exercise 2.2.3(a).

Now, we must prove that $\vdash \ (A \vee B) \wedge C \rightarrow (A \wedge C) \vee (B \wedge C)$.

(1) Prove $\vdash \ A \rightarrow (C \rightarrow (A \wedge C) \vee (B \wedge C))$

(2) Prove $\vdash \ B \rightarrow (C \rightarrow (A \wedge C) \vee (B \wedge C))$

(3) Prove $\vdash$
$(A \vee B) \wedge C \rightarrow (A \wedge C) \vee (B \wedge C)$

Exercise 2.3.5. Prove the above (1), (2) and (3).

Finally, we must prove that $\vdash \ (A \wedge C) \vee (B \wedge C) \rightarrow (A \vee B) \wedge C$.

Exercise 2.3.6. Prove that.

2.4. Formulas Containing Negation – Minimal Logic

Theorem 2.4.1. a) If

$$[L_1, L_2, L_9, MP]: A_1, A_2, ..., A_n, B \vdash C,$$

and

$$[L_1, L_2, L_9, MP]: A_1, A_2, ..., A_n, B \vdash \neg C,$$

then

$$[L_1, L_2, L_9, MP]: A_1, A_2, ..., A_n \vdash \neg B.$$

What does this mean?

b) $[L_3, L_4, L_9, MP]: \vdash \ \neg(A \wedge \neg A)$. What does it mean? It's the so-called **Law of Non-Contradiction.**

Proof. a) By Deduction Theorem 1, $A_1, A_2, ..., A_n \vdash B \rightarrow C$, and $A_1, A_2, ..., A_n \vdash B \rightarrow \neg C$. Let us continue this proof by adding the axiom $\mathbf{L_9}$: $(B \rightarrow C) \rightarrow ((B \rightarrow \neg C) \rightarrow \neg B)$ as the next step. After this, by applying MP twice we obtain $\neg B$. Q.E.D.

b) See Exercise 1.4.2 (e).

Exercise 2.4.1. a) (optional, for smart students) Investigate the size (the number of formulas) of the proof of $[L_1, L_2, L_9, MP]: A_1, A_2, ..., A_n, \vdash \neg B$ as a function f(k, m) of the sizes k, m of the proofs of $[L_1, L_2, L_9, MP]: A_1, A_2, ..., A_n, B \vdash C$ and $_1, A_2, ..., A_n, B \vdash \neg C$. You may wish to report your result. We will publish your report on the web as an appendix to this book. The current

record holder is Aiga Romane, 2008: f(k, m) ≤ 3(k+m)+7. Improve this result, or prove that it is the best possible one.

b) $[L_1, L_2, L_9, MP]$: A, ¬B ⊢ ¬(A→B). Or, $[L_1\text{-}L_4, L_9, MP]$: ⊢ $A \wedge \neg B \to \neg(A \to B)$. What does it mean?

c) ⊢$[L_1, L_2, L_9, MP]$: (A→¬A)→¬A. What does it mean?

Attention: non-constructive reasoning! In Section 2.6, we will use the classical logic $[L_1\text{-}L_{11}, MP]$ to prove the converse formula of (c): $\neg(A \to B) \to A \wedge \neg B$, i.e. the equivalence $\neg(A \to B) \leftrightarrow A \wedge \neg B$. This formula cannot be proved in the constructive logic $[L_1\text{-}L_{10}, MP]$ (see Section 2.8).

Theorem 2.4.2. $[L_1, L_2, L_9, MP]$: ⊢(A→B)→(¬B→¬A).

What does it mean? It's the so-called **Contraposition Law**.

Note. The following form of Theorem 2.4.2 is called **Modus Tollens**:

$$[L_1, L_2, L_9, MP]: \vdash A \to B, \neg B \vdash \neg A.$$

Attention: non-constructive reasoning! In Section 2.6, we will use the classical logic $[L_1\text{-}L_{11}, MP]$ to prove the converse formula (¬B→¬A)→(A→B), i.e. the equivalence (A→B)↔(¬B→¬A). We will see also that these formulas cannot be proved in the constructive logic $[L_1\text{-}L_{10}, MP]$ (see Section 2.8).

Exercise 2.4.2. a) Prove Theorem 2.4.2.

b) (optional) Verify that, in our axiom system, the Law of Non-Contradiction and the Contraposition Law could be used instead of the axiom L_9. More precisely: prove L_9 in the logic $[L_1\text{-}L_5$, Law of Non-Contradiction, Contraposition Law, MP]. Be careful: do not use theorems depending on the axiom L_9.

Theorem 2.4.3. $[L_1\text{-}L_9, MP]$: ⊢(A→¬B)↔(B→¬A). What does it mean?

First we prove that ⊢(A→¬B)→(B→¬A).

(1)	A→¬B	Hypothesis.
(2)	B	Hypothesis.
(3)	(A→B)→((A→¬B)→¬A)	Axiom $\mathbf{L_9}$: (B→C)→((B→¬C)→¬B) with B = A, C = B.

(4)	$A\to B$	From (2) by Axiom L_1 and MP.
(5)	$(A\to\neg B)\to\neg A$	From (3) and (4).
(6)	$\neg A$	From (1) and (5).

Thus, by Deduction Theorem 1, $\vdash(A\to\neg B)\to(B\to\neg A)$. By swapping A and B we obtain the converse formula: $\vdash(B\to\neg A)\to(A\to\neg B)$. Q.E.D.

Attention: non-constructive reasoning! Warning! The (very similar to Theorem 2.4.3) formula $(\neg A\to B)\leftrightarrow(\neg B\to A)$ can be proved only in the classical logic!

Theorem 2.4.4. $[L_1, L_2, L_9, MP]$: $\vdash A\to\neg\neg A$. What does it mean?

(1)	A	Hypothesis.
(2)	$(\neg A\to A)\to((\neg A\to\neg A)\to\neg\neg A)$	Axiom $\mathbf{L_9}$.
(3)	$A\to(\neg A\to A)$	Axiom $\mathbf{L_1}$.
(4)	$\neg A\to A$	From (1) and (3) by MP.
(5)	$(\neg A\to\neg A)\to\neg\neg A$	From (2) and (4) by MP.
(6)	$\neg\neg A$	From (5) and Theorem 1.4.1 by MP.

Attention: non-constructive reasoning! In Section 2.6, we will use the classical logic $[L_1\text{-}L_{11}, MP]$ to prove the converse formula $\vdash\neg\neg A\to A$, i.e. the equivalence $\vdash\neg\neg A\leftrightarrow A$ (the so-called **Double Negation Law**). We will see also (Section 2.8) that these formulas cannot be proved in the constructive logic $[L_1\text{-}L_{10}, MP]$.

Still, in the minimal logic we can prove (Brouwer, 1923?):

Theorem 2.4.5. $[L_1, L_2, L_9, MP]$: $\vdash\neg\neg\neg A\leftrightarrow\neg A$. What does it mean?

Indeed, by Theorem 2.4.4, $\vdash\neg A\to\neg\neg\neg A$. By the Contraposition Law (Theorem 2.4.2), $\vdash(A\to\neg\neg A)\to(\neg\neg\neg A\to\neg A)$. Hence, by Theorem 2.4.4, $\vdash\neg\neg\neg A\to\neg A$. Q.E.D.

Theorem 2.4.5 (and some of the following formulas in this and in the next section containing double negations) may seem uninteresting to people believing unconditionally in the equivalence $\neg\neg A\leftrightarrow A$. Still, it seems interesting (at least – for a mathematician) to obtain a general characterization of logical formulas that do not depend on the Law of Excluded Middle. In Section 2.7 we will use these formulas to prove the elegant and non-trivial

Glivenko's theorem: a) A is provable in the classical propositional logic (i.e. in [L_1-L_{11}, MP]), if and only if ¬¬A is provable in the constructive propositional logic (i.e. in [L_1-L_{10}, MP]), b) ¬A is provable in the classical propositional logic, if and only if ¬A is provable in the constructive propositional logic.

Theorem 2.4.6. a) [L_1, L_2, L_9, MP]: ⊢(¬A→A)→¬¬A. What does it mean?

b) [L_1, L_2, L_6, L_7, L_9, MP]: ⊢¬¬(Av¬A). What does it mean?

In this weak form, the Law of Excluded Middle can be "proved constructively". The formula $\neg\neg(A\vee\neg A)$ can be proved in the constructive logic, but $A\vee\neg A$ can't – as we will see in Section 2.8.

Exercise 2.4.3. Prove (a) and (b) of Theorem 2.4.6. The axiom L_{11} can't be used in these proofs! (Hint for (b): derive a contradiction from ¬(Av¬A).)

Theorem 2.4.7. [L_1-L_9, MP]: a) ⊢ (A→B)→(¬¬A→¬¬B). What does it mean?

b) ⊢¬¬(A→B)→(¬¬A→¬¬B). What does it mean?

c) ⊢(A→(B→C))→(¬¬A→(¬¬B→¬¬C)). What does it mean?

d) ¬¬(A→B), ¬¬(B→C) ⊢¬¬(A→C). What does it mean?

e) ¬¬A, ¬¬(A→B) ⊢¬¬B. What does it mean?

The converse of (a): (¬¬A→¬¬B)→(A→B) cannot be proved in the constructive logic (see Section 2.8).

To prove (a), we must simply apply twice the Contraposition Law: (A→B)→(¬B→¬A)→(¬¬A→¬¬B). And, of course, (e) is an easy consequence of (b).

Now, let us prove (b).

(1)	¬¬(A→B)	Hypothesis.
(2)	¬¬A	Hypothesis.
(3)	¬¬A→((A→B)→¬¬B)	From (a), by transposing A→B and ¬¬A, by the Premise Permutation Law.
(4)	(A→B)→¬¬B	From (2) and (3).
(5)	((A→B)→¬¬B)→(¬¬¬B→¬(A→B))	By the Contraposition Law.

(6)	$\neg\neg\neg B\rightarrow\neg(A\rightarrow B)$	From (4) and (5).
(7)	$(\neg\neg\neg B\rightarrow\neg(A\rightarrow B))\rightarrow(\neg\neg(A\rightarrow B)\rightarrow\neg\neg\neg\neg B)$	By the Contraposition Law.
(8)	$\neg\neg(A\rightarrow B)\rightarrow\neg\neg\neg\neg B$	From (6) and (7).
(9)	$\neg\neg\neg\neg B$	From (1) and (8).
(10)	$\neg\neg\neg\neg B\rightarrow\neg\neg B$	By Theorem 2.4.5.
(11)	$\neg\neg B$	From (9) and (10).

Thus, by Deduction Theorem 1, $\vdash\neg\neg(A\rightarrow B)\rightarrow(\neg\neg A\rightarrow\neg\neg B)$.

Let us prove (c).

(1)	$A\rightarrow(B\rightarrow C)$	Hypothesis.
(2)	$\neg\neg A$	Hypothesis.
(3)	$\neg\neg B$	Hypothesis.
(4)	$\neg\neg A\rightarrow\neg\neg(B\rightarrow C)$	From (1), by (a).
(5)	$\neg\neg(B\rightarrow C)$	From (2) and (4).
(6)	$\neg\neg B\rightarrow\neg\neg C$	From (5), by (b).
(7)	$\neg\neg C$	From (3) and (6).

Thus, by Deduction Theorem 1, $\vdash(A\rightarrow(B\rightarrow C))\rightarrow(\neg\neg A\rightarrow(\neg\neg B\rightarrow\neg\neg C))$.

Now we can prove (d). First, let us take (c) with A = A→B, B = B→C, C = A→C:

(1)$\vdash((A\rightarrow B)\rightarrow((B\rightarrow C)\rightarrow(A\rightarrow C)))\rightarrow(\neg\neg(A\rightarrow B)\rightarrow(\neg\neg(B\rightarrow C)\rightarrow\neg\neg(A\rightarrow C)))$.

(2)	$\vdash(A\rightarrow B)\rightarrow((B\rightarrow C)\rightarrow(A\rightarrow C)$	By transitivity of implication and Deduction Theorem 1.
(3)	$\neg\neg(A\rightarrow B)$	Hypothesis.
(4)	$\neg\neg(B\rightarrow C)$	Hypothesis.
(5)	$\neg\neg(A\rightarrow C)$	From (1), (3) and (4).

Theorem 2.4.8. [L_1-L_9, MP]: a) $\vdash\ \neg\neg(A\wedge B)\leftrightarrow(\neg\neg A\wedge\neg\neg B)$. What does it mean?

b) $\vdash \neg\neg A \vee \neg\neg B \rightarrow \neg\neg(A \vee B)$. What does it mean?

Attention: non-constructive reasoning! The converse of (b): $\neg\neg(A \vee B) \rightarrow \neg\neg A \vee \neg\neg B$ cannot be proved in the constructive logic (see Section 2.8). What does it mean? If we simply succeed in deriving a contradiction from $\neg(A \vee B)$, then, perhaps, we do not have a method allowing to decide, which part of $\neg\neg A \vee \neg\neg B$ is true – ¬¬A, or ¬¬B?

Exercise 2.4.4. Prove Theorem 2.4.8. (Hint: use the result of Exercise 2.2.3(a), if needed.)

Theorem 2.4.9. [L_1, L_2, L_9, MP] $\vdash \neg A \rightarrow (A \rightarrow \neg B)$ (compare with Exercise 1.4.2(d)). What does it mean?

It's a weak form of the "crazy" axiom $\mathbf{L_{10}}$: $\neg A \rightarrow (A \rightarrow B)$. This axiom says: "Contradiction implies anything". In the minimal logic we can prove 50% of L_{10}: "Contradiction implies that all is wrong". Of course, this 50%-provability of L_{10} decreases the significance of the minimal logic accordingly.

Proof. See Exercise 2.4.5.

Theorem 2.4.10. [L_1-L_9, MP]:

a) $\vdash \neg A \vee \neg B \rightarrow \neg(A \wedge B)$. It's a **half of** the so-called **First de Morgan Law**. What does it mean?

b) $\vdash \neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$. It's the so-called **Second de Morgan Law**. What does it mean?

Attention: non-constructive reasoning! The second half of (a) – the converse implication, i.e. the equivalence $\neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$ can be proved in the classical logic, yet not in the constructive logic (see Section 2.8). Explain, why.

Augustus de Morgan (1806-1871): "He recognised the purely symbolic nature of algebra and he was aware of the existence of algebras other than ordinary algebra. He introduced de Morgan's laws and his greatest contribution is as a reformer of mathematical logic." (according to MacTutor History of Mathematics archive).

Use Contraposition Law to prove (a) and (b→) in Exercise 2.4.5.

Let us prove (b←).

(0)	$\neg A \wedge \neg B$	Hypothesis.
(1)	¬A	From (0), by Axiom L_3.
(2)	¬B	From (0), by Axiom L_4.

(3)	A→¬C	From (1), by Theorem 2.4.9: ¬A→(A→¬C). C is any formula.
(4)	B→¬C	From (2), by Theorem 2.4.9: ¬B→(B→¬C). C is any formula.
(5)	$A \vee B \to \neg C$	From (3) and (4), by Axiom L_8: $(A \to \neg C) \to ((B \to \neg C) \to (A \vee B \to \neg C))$.
(6)	$A \vee B \to \neg\neg C$	Repeat (3)-(5) with ¬¬C instead of ¬C.
(7)	$\neg(A \vee B)$	From (5) and (6), by Axiom L_9: $(A \vee B \to \neg C) \to ((A \vee B \to \neg\neg C) \to \neg(A \vee B))$

Thus, by [L_1, L_2] Deduction Theorem 1,

$$[L_1\text{-}L_9, \text{MP}] \vdash \neg A \wedge \neg B \to \neg(A \vee B) \ .$$

Exercise 2.4.5. Prove:

a) Theorem 2.4.9.

b) (a) and (b→) of Theorem 2.4.10. (Hint: use Contraposition Law).

c) [L_1-L_9, MP]: $\vdash (A \to B) \to \neg(A \wedge \neg B)$. What does it mean? Compare with Exercise 2.4.1.

d) [L_1-L_8, MP]: $\vdash A \vee B \to ((A \to B) \to B)$. What does it mean?

Attention: non-constructive reasoning! The converse implication of (a), $\neg(A \wedge \neg B) \to (A \to B)$ cannot be proved in the constructive logic (see Section 2.8). Explain, why. Still, we will prove this formula in the classical logic.

The converse of (b): $((A \to B) \to B) \to A \vee B$ cannot be proved in the constructive logic (see Section 2.8). Explain, why. Still, we will prove this formula in the classical logic.

2.5. Formulas Containing Negation – Constructive Logic

In this book, **constructive logic** is used as a synonym of **intuitionistic logic**!

Constructive logic includes the "crazy" axiom $\mathbf{L_{10}}$: ¬B→(B→C), but rejects the Law of Excluded Middle $\mathbf{L_{11}}$: Bv¬B as a general logical principle.

Theorem 2.5.1. a) [L_{10}, MP]: A, $\neg$A $\vdash$B. What does it mean?

b) [L_1, L_2, L_8, L_{10}, MP]: $\vdash \quad A \vee B \rightarrow (\neg A \rightarrow B)$. What does it mean?

c) [L_1, L_8, L_{10}, MP]: $\vdash \quad \neg A \vee B \rightarrow (A \rightarrow B)$. What does it mean?

Of course, (a) follows directly from L_{10}, by MP.

Exercise 2.5.1. Prove (b) and (c) of Theorem 2.5.1. Note: when proving (c), you cannot use Deduction Theorem 1 (because of the missing axiom L_2). So, simply build a sequence of 5 formulas representing the proof of (c).

Attention: non-constructive reasoning! The converse of (b), i.e. $(\neg A \rightarrow B) \rightarrow A \vee B$ cannot be proved in the constructive logic (see Section 2.8). Explain, why. The converse of (c), i.e. $(A \rightarrow B) \rightarrow \neg A \vee B$ cannot be proved in constructive logic (see Section 2.8). Explain, why.

Surprisingly, (b), i.e. the rule $A \vee B, \neg A \vdash$B seems to be a quite a "natural" logical principle, yet it cannot be proved without the "crazy" axiom L_{10}! Why not? Because it implies L_{10}! Indeed,

(1)	$A \vee B \rightarrow (\neg A \rightarrow B)$	Hypothesis.
(2)	$\neg$A	Hypothesis.
(3)	A	Hypothesis.
(4)	$A \rightarrow A \vee B$	Axiom L_6.
(5)	$A \vee B$	By MP, from (3) and (4).
(6)	B	By MP, from (1), (5) and (2).

Hence, by Deduction Theorem 1, [L_1, L_2, L_6, MP]: $A \vee B \rightarrow (\neg A \rightarrow B) \vdash \neg$A$\rightarrow$(A$\rightarrow$B).

In Section 2.8 we will prove that L_{10} cannot be derived from L_1-L_9, hence, (b) also cannot be derived from L_1-L_9 (i.e. without L_{10}).

Theorem 2.5.2. [L_1-L_{10}, MP]:

a) $\vdash (\neg\neg$A$\rightarrow\neg\neg$B$)\rightarrow\neg\neg$(A$\rightarrow$B). It's the converse of Theorem 2.4.7(b). Hence, [L_1-L_{10}, MP]: $\vdash\neg\neg$(A$\rightarrow$B$)\leftrightarrow(\neg\neg$A$\rightarrow\neg\neg$B).

b) $\vdash\neg\neg$A$\rightarrow(\neg$A$\rightarrow$A). It's the converse of Theorem 2.4.6(a). Hence,

[L_1-L_{10}, MP]: $\vdash\neg\neg$A$\leftrightarrow(\neg$A$\rightarrow$A).

c) $\vdash \; A \vee \neg A \rightarrow (\neg \neg A \rightarrow A)$. What does it mean?

d) $\vdash \neg\neg(\neg\neg A \rightarrow A)$. What does it mean?

Of course, (b) is an instance of the **axiom L_{10}.**

To prove (a), let us prove that [L_1-L_{10}, MP]: $\neg\neg A \rightarrow \neg\neg B, \neg(A \rightarrow B) \vdash \neg B, \neg\neg B$. Then, by Theorem 2.4.1, (a) $\vdash (\neg\neg A \rightarrow \neg\neg B) \rightarrow \neg\neg(A \rightarrow B)$.

Exercise 2.5.2. a) Prove that [L_1-L_{10}, MP]: $\neg\neg A \rightarrow \neg\neg B, \neg(A \rightarrow B) \vdash \neg B, \neg\neg B$.

b) Prove (c) and (d) of Theorem 2.5.2.

Exercise 2.5.3. Prove that in [L_1-L_{10}, MP]:

a) $A \vdash \; B \leftrightarrow B \vee \neg A$. What does it mean?

b) $\vdash \; B \vee (A \wedge \neg A) \leftrightarrow B$. What does it mean?

c) $\vdash \; ((A \wedge \neg A) \wedge B) \vee C \leftrightarrow C$. What does it mean?

2.6. Formulas Containing Negation – Classical Logic

If you agree to adopt the formula $B \vee \neg B$, i.e. the Law of Excluded Middle (Axiom L_{11} in the list of Section 1.3), you can prove, first of all, the so called **Double Negation Law**:

Theorem 2.6.1. [L_1-L_{11}, MP]: $\vdash \neg\neg A \rightarrow A$. Hence, [$L_1$-$L_{11}$, MP]: $\vdash \neg\neg A \leftrightarrow A$.

Indeed, by Theorem 2.5.2, [L_1-L_{10}, MP]: $\vdash \; A \vee \neg A \rightarrow (\neg \neg A \rightarrow A)$, hence, [L_1-L_{11}, MP]: $\vdash \neg\neg A \rightarrow A$. Q.E.D.

In the minimal logic we proved Theorem 2.4.4: [L_1, L_2, L_9, MP]: $\vdash A \rightarrow \neg\neg A$. Hence, [$L_1$-$L_{11}$, MP]: $\vdash \neg\neg A \leftrightarrow A$.

Attention: non-constructive reasoning! The formula $\neg\neg A \rightarrow A$ cannot be proved in the constructive logic, see Section 2.8. Why? Because it represents a kind of non-constructive reasoning. Indeed, imagine, you wish to prove that $\exists x B(x)$. Assume the contrary, $\neg\exists x B(x)$, and derive a contradiction. Thus you have proved... the negation of $\neg\exists x B(x)$, i.e. $\neg\neg\exists x B(x)$. To conclude $\exists x B(x)$ from $\neg\neg\exists x B(x)$, you need the Double Negation Law. Hence, by adopting this law as a logical principle, you would allow non-constructive existence proofs – if you prove $\exists x B(x)$ by assuming $\neg\exists x B(x)$, and deriving a contradiction, then

you may not obtain a method allowing to find a particular x satisfying B(x).

Exercise 2.6.1. Prove that [L_8, L_{11}, MP]: A→B, ¬A→B ⊢ B. Or, by Deduction Theorem 1, [L_1, L_2, L_8, L_{11}, MP]: (A→B)→((¬A→B)→B). What does it mean? This formula cannot be proved in the constructive logic (see Section 2.8). Explain, why.

In the classical logic, you can prove also the full form of the **Contraposition Law:**

Theorem 2.6.2. [L_1-L_{11}, MP]: ⊢(A→B) ↔ (¬B→¬A).

We proved a half of this Law in the minimal logic as Theorem 2.4.2: [L_1, L_2, L_9, MP]: ⊢ (A→B)→(¬B→¬A). Let us prove the remaining half: [L_1-L_{11}, MP] ⊢(¬B→¬A) → (A→B).

(1)	¬B→¬A	Hypothesis.
(2)	A	Hypothesis.
(3)	¬¬A→¬¬B	From (1), by the first half.
(4)	A→¬¬A	Double Negation Law.
(5)	¬¬B→B	Double Negation Law.
(6)	B	From (4), (3) ans (5).

By Deduction Theorem 1, [L_1-L_{11}, MP] ⊢(¬B→¬A) → (A→B).

Attention: non-constructive reasoning! The formula (¬B→¬A) → (A→B) cannot be proved in the constructive logic, see Section 2.8. Explain, why.

Exercise 2.6.1A. Prove that in [L_1-L_{11}, MP]:

a) ⊢(¬A→B)↔(¬B→A) (compare with Theorem 2.4.3).

b) ⊢(A→B)→((¬A→¬B)→(B↔A)).

Attention: non-constructive reasoning! These two formulas cannot be proved in the constructive logic, see Section 2.8.

Theorem 2.6.3. [L_1-L_{11}, MP]: ⊢ $\neg(A\wedge B)\leftrightarrow\neg A\vee\neg B$. It's the so-called **First de Morgan Law.**

A half of this Law we proved in the minimal logic as Theorem 2.4.10(a): [L_1-L_9, MP] ⊢ $\neg A\vee\neg B\rightarrow\neg(A\wedge B)$. Let us prove the remaining half: [L_1-L_{11}, MP] ⊢ $\neg(A\wedge B)\rightarrow\neg A\vee\neg B$.

Attention: non-constructive reasoning! This formula cannot be proved in the constructive logic, see Section 2.8. Explain, why.

Let us start by proving $\neg(\neg A \vee \neg B) \rightarrow A \wedge B$.

(1) $\neg(\neg A \vee \neg B)$ Hypothesis.

(2) $\neg(\neg Av \neg B) \rightarrow \neg\neg A \wedge \neg\neg B$ By the Second de Morgan Law -Theorem 2.4.10(b).

(3) $\neg\neg A \wedge \neg\neg B \rightarrow \neg\neg(A \wedge B)$ Theorem 2.4.8(a). [L_1-L_9, MP]!

(4) $\neg\neg(A \wedge B)$ From (1), (2) and (3).

Thus, by Deduction Theorem 1, [L_1-L_9, MP] $\vdash \neg(\neg A \vee \neg B) \rightarrow \neg\neg(A \wedge B)$. By applying the first half of the Contraposition Law (provable in the minimal logic): [L_1-L_9, MP] $\vdash \neg\neg\neg(A \wedge B) \rightarrow \neg\neg(\neg A \vee \neg B)$. By Theorem 2.4.5: [L_1-L_9, MP] $\vdash \neg(A \wedge B) \rightarrow \neg\neg\neg(A \wedge B)$, hence,

[L_1-L_9, MP] $\vdash \neg(A \wedge B) \rightarrow \neg\neg(\neg A \vee \neg B)$. Now, by the Double Negation Law, [L_1-L_{11}, MP] $\vdash \neg\neg(\neg A \vee \neg B) \rightarrow \neg A \vee \neg B$, hence,

[L_1-L_{11}, MP] $\vdash \neg(A \wedge B) \rightarrow \neg A \vee \neg B$. Q.E.D.

In the classical logic, we can express implication by negation and disjunction. Indeed, we already know that [L_1-L_{10}, MP]: $\vdash \neg A \vee B \rightarrow (A \rightarrow B)$ (Theorem 2.5.1(c)).

Theorem 2.6.4. a) [L_1-L_8, MP]: $A \vee C \vdash (A \rightarrow B) \rightarrow B \vee C$. Hence, [L_1-L_8, MP]: $A \vee \neg A \vdash (A \rightarrow B) \rightarrow \neg A \vee B$.

b) [L_1-L_{11}, MP]: $\vdash (A \rightarrow B) \leftrightarrow \neg A \vee B$.

Of course, (b) follows from (a) and Theorem 2.5.1(c). Let us prove (a).

(1) A, A→B ⊢B

(2) A, A→B ⊢ $B \vee C$ By Axiom L_6.

(3) A ⊢ $(A \rightarrow B) \rightarrow B \vee C$ By Deduction Theorem 1.

(4) C, A→B ⊢C

(5) C, A→B ⊢ $B \vee C$ By Axiom L_7.

(6) C $\vdash$ $(A \to B) \to B \vee C$ By Deduction Theorem 1.

(7) AvC $\vdash$ $(A \to B) \to B \vee C$ By Axiom L_8.

Exercise 2.6.2. Prove that in [L_1-L_{11}, MP]:

a) $\vdash$ $B \wedge (A \vee \neg A) \leftrightarrow B$. What does it mean?

b) $\vdash$ $((A \vee \neg A) \vee B) \wedge C \leftrightarrow C$. What does it mean?

c) $\vdash$ $((A \to B) \to B) \to A \vee B$. What does it mean? Hence, by Exercise 2.4.5(d), [L_1-L_{11}, MP]: $\vdash$ $((A \to B) \to B) \leftrightarrow A \vee B$.

Exercise 2.6.3. Prove that in [L_1-L_{11}, MP]:

a) $\vdash$ $(A \to B) \leftrightarrow \neg(A \wedge \neg B)$. What does it mean?

b) $\vdash$ $\neg(A \to B) \leftrightarrow A \wedge \neg B$. What does it mean?

c) $\vdash$ $A \vee B \leftrightarrow (\neg A \to B)$. What does it mean?

d) $\vdash$ $A \wedge B \leftrightarrow \neg(A \to \neg B)$. What does it mean?

e) (optional, for smart students) Try detecting, which parts of these equivalences are provable: 1) in the minimal logic, 2) in the constructive logic.

Strange formulas

Exercise 2.6.4. Prove in [L_1-L_{11}, MP] the following strange formulas:

a) $\vdash$ $A \vee (A \to B)$. What does it mean? Does it mean anything at all?

b) $\vdash$ $(A \to B) \vee (B \to A)$. What does it mean? Does it mean anything at all? The most crazy theorem of the classical propositional logic?

c) $\vdash$ $((A \to B) \to A) \to A$. What does it mean? Does it mean anything at all? It is the so-called **Peirce's Law** from:

C. S. Peirce. On the algebra of logic: A contribution to the philosophy of notation. *American Journal of Mathematics*, 1885, vol.7, pp.180-202.

2.7. Constructive Embedding. Glivenko's Theorem

Let us remind some of the results of previous sections concerning double negations:

Theorem 2.4.4. [L_1, L_2, L_9, MP]: $\vdash A \to \neg\neg A$.

Theorem 2.4.5. [L_1-L_9, MP]: $\vdash \neg\neg\neg A \leftrightarrow \neg A$.

Theorem 2.4.6(b). [L_1-L_9, MP]: $\vdash \neg\neg(A \vee \neg A)$. In this weak form, the Law of Excluded Middle can be "proved constructively".

Theorem 2.4.7. [L_1-L_9, MP]: a) $\vdash (A \to B) \to (\neg\neg A \to \neg\neg B)$.

b) $\vdash \neg\neg(A \to B) \to (\neg\neg A \to \neg\neg B)$.

c) $\vdash (A \to (B \to C)) \to (\neg\neg A \to (\neg\neg B \to \neg\neg C))$.

d) $\neg\neg(A \to B), \neg\neg(B \to C) \vdash \neg\neg(A \to C)$.

e) $\neg\neg A, \neg\neg(A \to B) \vdash \neg\neg B$.

Theorem 2.4.8. [L_1-L_9, MP]: a) $\vdash \neg\neg(A \wedge B) \leftrightarrow (\neg\neg A \wedge \neg\neg B)$.

b) $\vdash \neg\neg A \vee \neg\neg B \to \neg\neg(A \vee B)$.

Theorem 2.5.2. [L_1-L_{10}, MP]: a) $\vdash (\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B)$. It's the converse of Theorem 2.4.7(b).

d) $\vdash \neg\neg(\neg\neg A \to A)$.

Theorem 2.6.1. [L_1-L_{11}, MP]: $\vdash \neg\neg A \leftrightarrow A$.

Does it mean that for **any** formula A: if [L_1-L_{11}, MP]: $\vdash A$, then [L_1-L_{10}, MP]: $\vdash \neg\neg A$? (The converse is obvious: if [L_1-L_{10}, MP]: $\vdash \neg\neg A$, then [L_1-L_{11}, MP]: $\vdash A$ by Theorem 2.6.1.)

Imagine, we have a proof of [L_1-L_{11}, MP]: $\vdash A$. It is a sequence of formulas $R_1, R_2, ..., R_n$, where $R_n = A$. If this sequence does not contain instances of the axiom L_{11}, then it is a proof of [L_1-L_{10}, MP]: $\vdash A$ as well. Hence, according to Theorem 2.4.4, [L_1-L_{10}, MP]: $\vdash \neg\neg A$

If the sequence $R_1, R_2, ..., R_n$ contains some instances of L_{11}, i.e. formulas having the form $B \vee \neg B$, then, according to Theorem 2.4.6(b)**,** we could try replacing each such formula by a sequence proving that [L_1-L_9, MP]: $\vdash \neg\neg(B \vee \neg B)$. It appears that **each** of the formulas $\neg\neg R_1, \neg\neg R_2, ..., \neg\neg R_n$ is provable in [L_1-L_{10}, MP].

a) If R_k is an instance of the axioms L_1-L_{10}, then [L_1-L_{10}, MP]: $\vdash \neg\neg R_k$ (Theorem 2.4.4).

b) If R_k is an instance of the axiom L_{11}, then [L_1-L_{10}, MP]: $\vdash \neg\neg R_k$ (Theorem 2.4.6(b)).

c) Now, let us assume that i, j < k, and R_i, $R_j \vdash R_k$ directly by MP, i.e. R_j is $R_i \to R_k$. We know already that $[L_1\text{-}L_{10}, MP]: \vdash \neg\neg R_i$ and $[L_1\text{-}L_{10}, MP]: \vdash \neg\neg(R_i \to R_k)$. By Theorem 2.4.7(b), $[L_1\text{-}L_9, MP]: \vdash \neg\neg(R_i \to R_k) \to (\neg\neg R_i \to \neg\neg R_k)$. Hence, $[L_1\text{-}L_{10}, MP]: \vdash \neg\neg R_k$.

Because $A = R_n$, we have proved the remarkable Glivenko's theorem from 1929:

V.Glivenko. Sur quelques points de la logique de M. Brouwer. *Academie Royale de Belgique, Bulletins de la classe des sciences*, 1929, ser.5, vol.15, pp.183-188.

Valery Ivanovich Glivenko (1896-1940, see http://www.math.ru/history/people/glivenko, in Russian) is best known by the so-called Glivenko-Cantelli theorem in probability theory.

Glivenko's Theorem. $[L_1\text{-}L_{11}, MP]: \vdash A$, if and only if $[L_1\text{-}L_{10}, MP]: \vdash \neg\neg A$. Or: a formula A is provable in the classical propositional logic, if and only if its double negation $\neg\neg A$ is provable in the constructive propositional logic.

This theorem provides a kind of a "constructive embedding" for the classical propositional logic: any classically provable formula can be "proved" in the constructive logic, if you simply put two negations before it.

Corollary. $[L_1\text{-}L_{11}, MP]: \vdash \neg A$, if and only if $[L_1\text{-}L_{10}, MP]: \vdash \neg A$. Or: a "negative" formula $\neg A$ is provable in the classical propositional logic, if and only if it is provable in the constructive propositional logic.

Indeed, if $[L_1\text{-}L_{11}, MP]: \vdash \neg A$, then by Glivenko's theorem, $[L_1\text{-}L_{10}, MP]: \vdash \neg\neg\neg A$, and by Theorem 2.4.5, $[L_1\text{-}L_{10}, MP]: \vdash \neg A$. Q.E.D.

Exercise 2.7.1. Prove the following version of Glivenko's theorem (see Kleene [1952]):

a) If $[L_1\text{-}L_{11}, MP]: A_1, A_2, ..., A_n \vdash C$, then

$[L_1\text{-}L_{10}, MP]: \neg\neg A_1, \neg\neg A_2, ..., \neg\neg A_n \vdash \neg\neg C$.

b) If $[L_1\text{-}L_{11}, MP]: \neg A_1, \neg A_2, ..., \neg A_n, B_1, B_2, ..., B_p \vdash \neg C$, then

$[L_1\text{-}L_{10}, MP]: \neg A_1, \neg A_2, ..., \neg A_n, \neg\neg B_1, \neg\neg B_2, ..., \neg\neg B_p \vdash \neg C$.

2.8. Axiom Independence. Using Computers in Mathematical Proofs

If one of our axioms L_i could be proved by using the remaining *n-1* axioms, then we could simplify our logical system by dropping L_i as an axiom. A striking example:

Theorem 2.8.1. The axiom $\mathbf{L_9}$: (A→B)→((A→¬B)→¬A) can be proved in [L_1, L_2, L_8, L_{10}, L_{11}, MP].

This fact was established by Augusts Kurmitis (on the web, also: A. A. Kurmit):

A. A. Kurmitis. On independence of a certain axiom system of the propositional calculus. *Proc. Latvian State University*, 1958, Vol. 20, N3, pp. 21-25 (in Russian).

The following proof of L_9 in [L_1, L_2, L_8, L_{10}, L_{11}, MP] is due to Janis Sedols (1939-2011).

First, let us establish that the formula (A→¬A)→¬A can be proved in [L_1, L_2, L_8, L_{10}, L_{11}, MP] (in Exercise 2.4.1 we established that [L_1, L_2, L_9, MP]: ⊢ (A→¬A)→¬A):

(1)	$(A \to \neg A) \to ((\neg A \to \neg A) \to (A \vee \neg A) \to \neg A)$	Axiom L_8.
(2)	A→¬A	Hypothesis.
(3)	¬A→¬A	This is provable in [L_1, L_2, MP] (Theorem 1.4.1).
(4)	$A v \neg A$	Axiom L_{11}.
(4)	¬A	From (1), (2), (3) and (4), by MP.
(6)	(A→¬A)→¬A	By Deduction Theorem 1 (which is valid for any logical system containing [L_1, L_2, MP]).

Now let us establish that in [L_1, L_2, L_{10}, MP]: A→B, A→¬B ⊢A→¬A.

(7)	$A \rightarrow B$	Hypothesis.
(8)	$A \rightarrow \neg B$	Hypothesis.
(9)	A	Hypothesis.
(9)	B	From (7), (9), by MP.
(10)	$\neg B$	From (8), (9), by MP.
(11)	$\neg B \rightarrow (B \rightarrow \neg A)$	Axiom L_{10}.
(12)	$\neg A$	From (9), (10) and (11) by MP.
(13)	$A \rightarrow B, A \rightarrow \neg B \vdash A \rightarrow \neg A$	By Deduction Theorem 1 (which is valid for any propositional system containing $[L_1, L_2, MP]$).

Finally, let us merge the proofs of (6) and (13), then by MP we obtain $\neg A$, i.e.

$[L_1, L_2, L_8, L_{10}, L_{11}, MP]: A \rightarrow B, A \rightarrow \neg B \vdash \neg A$.

Now, by Deduction Theorem 1 (which is valid for any propositional system containing $[L_1, L_2, MP]$) we obtain the axiom L_9:

$[L_1, L_2, L_8, L_{10}, L_{11}, MP]: (A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$.

Q.E.D.

What should we do after establishing that one of our axioms is "dependent"?

Do you think, we should drop L_9 as an axiom of our logical system?

First, let's note that **we have proved L_9 by using three problematic axioms**: L_1, L_{10}, L_{11}. But L_9 itself is not problematic!

Secondly, L_9 cannot be proved in $[L_1\text{-}L_8, L_{10}, MP]$ (see Theorem 2.8.2 below). Hence, if we would drop L_9, then, instead of a simple definition

$$\text{classical logic} = \text{constructive logic} + L_{11},$$

we would have a more complicated one:

$$\text{constructive logic} = \text{classical logic} - L_{11} + L_9.$$

Now, the question of questions:

Is the Law of Excluded Middle an independent logical principle?

I.e., could we prove the Law of Excluded Middle (the axiom L_{11}: $B \vee \neg B$) by using the other axioms (i.e. [L_1-L_{10}, MP]) as we proved L_9 in [L_1, L_2, L_8, L_{10}, L_{11}, MP]? If not, how could we demonstrate that this is impossible at all? How could we demonstrate that some logical principle is **independent**, i.e. that it cannot be derived from other principles?

Let us assume, we have an algorithm *q* computing for each formula A some its "property" q(A) such that:

a) q(L_1) is true, q(L_2) is true, ..., q(L_{10}) is true (i.e. the axioms L_1-L_{10} possess property *q*).

b) If q(A) is true and q(A→B) is true, then q(B) is true (i.e. MP "preserves" property q). Hence, q(F) is true for all the formulas F that are provable in [L_1-L_{10}, MP].

c) q(L_{11}) is false (L_{11} does not possess property *q*).

If we could obtain such a property *q*, then, of course, this would demonstrate that L_{11} cannot be proved in [L_1-L_{10}, MP], i.e. that the **Law of Excluded Middle is an independent logical principle**.

The most popular way of introducing such properties of formulas are the so-called "multi-valued logics" or "many-valued logics", introduced by Jan Lukasiewicz and Emil Post:

J.Lukasiewicz. O logice trojwartosciowej. *Ruch Filozoficzny (Lwow)*, 1920, vol. 5, pp. 169-171

E.Post. Introduction to a general theory of elementary propositions. *Amer. journ. math.*, 1921, vol. 21, pp.163-195

Read more: Many-Valued Logic by Siegfried Gottwald in Stanford Encyclopedia of Philosophy.

For example, let us consider a kind of "three-valued logic", where 0 means "false", 1 – "unknown" (or NULL – in terms of SQL), 2 – "true". Then it would be natural to define conjunction and disjunction as

$$A \wedge B = \min(A,B)$$
$$A \vee B = \max(A,B).$$

But how should we define implication and negation?

A	B	$A \wedge B$	$A \vee B$	A→B
0	0	0	0	i_1

0	1	0	1	i_2
0	2	0	2	i_3
1	0	0	1	i_4
1	1	1	1	i_5
1	2	1	2	i_6
2	0	0	2	i_7
2	1	1	2	i_8
2	2	2	2	i_9

A	¬A
0	i_{10}
1	i_{11}
2	i_{12}

Thus, theoretically, we have here to explore: 3^9 = 19683 variants of implication definitions and 3^3 = 27 negation definitions.

Do you think, it would be natural to set the values of ¬A as follows?

A	¬A
0	2
1	1
2	0

Yes, if we would try building a "natural" three-valued logic, in which "1" would mean, indeed, "unknown". To fill in the “natural” table of three-valued implication, we could use, for example, the classical equivalence (A→B)↔¬AvB. In this way we could obtain the “natural” three-valued logic used, for example, for handling of NULL-values in SQL.

However, our aim is here, in a sense, just the opposite of “natural”. We will

consider

"under the above truth tables, formula A always takes "true" values"

as a kind of the above-mentioned "property" q(A). Hence, we will try to define the tables for implication and negation in such a way that:

a) the axioms L_1, L_2, ..., L_{10} always take "true" values (i.e. 2),

b) Modus Ponens "preserves" taking always "true" values (i.e. if the formulas A and A→B are always 2, then B also is always 2),

c) the axiom L_{11} sometimes takes the values 0 or 1.

Because of "violating" L_{11}, the definitions of implication and negation, having these properties, cannot be 100% natural. So, we must explore (at least some of) the "unnatural" versions as well.

Exercise 2.8.1 (optional). Develop a simple (recursive) computer program receiving as input:
a) Any such "truth tables".

b) Any formula F consisting of letters A, B, C, and propositional connectives.

and printing out "truth values" of the formula F, for example, if F = B→(A→B):

A	B	B→(A→B)
0	0	2
0	1	2
0	2	2
1	0	2
1	1	2
1	2	2
2	0	2
2	1	2
2	2	2

In this example the axiom L_1 always takes "true" values. Perhaps, we should be interested only in those variants of our "truth tables" that "satisfy" at least the axioms L_1, L_2, ..., L_8 forcing them always to take "true" values.

Note. See my version of the program in C++: header file, implementation.

Thus, we consider

"under the above truth tables, formula A always takes "true" values"

as a kind of the "property" q(A).

Will Modus Ponens preserve this property? If A is "true", and A→B is "true", how could B be not? Let us consider the relevant part of the above truth tables (i.e. the part where A is "true"):

A	B	A→B
2	0	i_7
2	1	i_8
2	2	i_9

If we would consider only those variants of our "truth tables" where $i_7 = 0$ or 1, $i_8 = 0$ or 1, and $i_9 = 2$, then, if B would not be 2 for some values of its arguments, then A→B also would not be 2 for the same values of arguments.

Hence, if we restrict ourselves to "truth tables" with $i_7 = 0$ or 1, $i_8 = 0$ or 1, and $i_9 = 2$, then MP preserves the property of "being true". **I.e., from "true" formulas MP can derive only "true" formulas.**

The next idea: if we wish the axiom L_6: A→AvB always taking the value "true" (i.e. the value 2), then, **if A≤B, then A→B must be 2.**

Thus, of all the $3^9 = 19683$ possible implication definition variants only the following 3*2*2 = 12 variants are worth of exploring:

A	B	A→B
0	0	2
0	1	2
0	2	2
1	0	i_4=0,1,2
1	1	2
1	2	2

2	0	i_7=0,1
2	1	i_8=0,1
2	2	2

Exercise 2.8.2. a) Verify that under any of these 12 implication definitions the axioms L_3, L_4, L_6, L_7 always take the value 2, i.e. you do not need testing these axioms any more.

b) For each of the axioms, L_1, L_2, L_5 and L_8, determine all the possible combinations of the values of i_4, i_7, i_8 forcing it to take always the value 2.

Note. The "intersection" of b) consists of 5 combinations (see the results file #00).

Exercise 2.8.3 (optional) Extend your previous computer program by adding 6 nested loops: for i_4=0 to 2, for i_7=0 to 1, for i_8=0 to1, for i_aa=0 to 2, for i_b=0 to 2, for i_c=0 to 2. Let the program print out only those variants of "truth tables" that make "true" all the axioms L_1-L_8. (My program yields 135 such variants, see the results file #00).
Thus, now we have 135 variants of "truth tables" that make "true" all the axioms L_1-L_8. Let us search among them for the variants that allow proving of axiom independency results we are interested in.

Axiom L_9

In Theorem 2.8.1 we established that the axiom **L_9**: (A→B)→((A→¬B)→¬A) can be proved in [L_1-L_8, L_{10}, L_{11}, MP]. Still,

Theorem 2.8.2. The axiom L_9 **cannot** be proved in [L_1-L_8, L_{10}, MP].

Proof. Let your program print out only those variants of "truth tables" that make "true" all the axioms L_1-L_8, and make: L_9 – not "true", and L_{10} – "true". My program yields 66 such variants, see the results file #01. I like especially the (most natural?) variant #33:

Implication variant #3:
2 2 2 2 2 2 0 1 2 L1-L8 true.
Variant #33. Negation: 2 1 0 L9 not true. L10 true. L11 not true.

A	B	A→B
0	0	2

0	1	2
0	2	2
1	0	2
1	1	2
1	2	2
2	0	0
2	1	1
2	2	2

A	¬A
0	2
1	1
2	0

See the extended results file #1 for this variant.

Under this variant the axioms L_1-L_8 and L_{10} are "true". As we know, under this variant, by MP, from "true" formulas only "true" formulas can be derived. The axiom L_9 is not "true" under this variant:

A	B	(A→B)→((A→¬B)→¬A)
0	0	2
0	1	2
0	2	2
1	0	1
1	1	1
1	2	1
2	0	2
2	1	2

2	2	2

Hence, L_9 **cannot** be proved in [L_1-L_8, L_{10}, MP]. Q.E.D.

In a similar way, we can obtain other independence results.

Axiom L_{10}

Theorem 2.8.3. The "crazy" axiom $\mathbf{L_{10}}$: ¬B→(B→C) cannot be proved in the minimal logic [L_1-L_9, MP], and even not in [L_1-L_9, L_{11}, MP].

Proof. Let your program print out only those variants of "truth tables" that make "true" all the axioms L_1-L_8, and make: L_9 – "true", L_{10} – not "true", and L_{11} – "true". My program yields 6 such variants, see the results file #02. I like especially the (somewhat natural?) variant #1:

Implication variant #1:
2 2 2 0 2 2 0 1 2 L1-L8 true.
Variant #1. Negation: 2 2 1 L9 true. L10 not true. L11 true.

See the extended results file #2 for this variant.

Under this variant the axioms L_1-L_9 and L_{11} are "true". As we know, under this variant, by MP, from "true" formulas only "true" formulas can be derived. The axiom L_{10} is not "true" under this variant:

A	B	¬A→(A→B)
0	0	2
0	1	2
0	2	2
1	0	2
1	1	2
1	2	2
2	0	0
2	1	1
2	2	2

Hence, L_{10} **cannot** be proved in [L_1-L_9, L_{11}, MP]. Q.E.D.

Axiom L_{11}

Now, let us prove the main result of this section:

Theorem 2.8.4. The Law of Excluded Middle $\mathbf{L_{11}}$: $B \vee \neg B$ cannot be proved in the constructive propositional logic [L_1-L_{10}, MP]. I.e. the **Law of Excluded Middle is an independent logical principle**.

Proof. Let your program print out only those variants of "truth tables" that make "true" all the axioms L_1-L_8, and make: L_9 – "true", L_{10} – "true", L_{11} – not "true". My program yields only one such variant, see the results file #03:

Implication variant #1:
2 2 2 0 2 2 0 1 2 L1-L8 true.
Variant #1. Negation: 2 0 0 L9 true. L10 true. L11 not true.

See the extended results file #3 for this variant.

Under this variant the axioms L_1-L_{10} are "true". As we know, under this variant, by MP, from "true" formulas only "true" formulas can be derived. The axiom L_{11} is not "true" under this variant:

B	¬B	$B \vee \neg B$
0	2	2
1	0	1
2	0	2

Hence, L_{11} **cannot** be proved in [L_1-L_{10}, MP]. Q.E.D.

The results file #03 proves also the following

Theorem 2.8.5 (thanks to Pavels Mihailovs for a correction**).** The following (classically provable) formulas cannot be proved in the constructive propositional logic [L_1-L_{10}, MP]:

$$\neg\neg A \to A$$
$$(\neg B \to \neg A) \to (A \to B)$$
$$(\neg A \to B) \to (\neg B \to A)$$
$$(\neg\neg A \to \neg\neg B) \to (A \to B)$$
$$(A \to B) \to \neg A \vee B$$
$$((A \to B) \to B) \to A \vee B$$
$$((A \to B) \to A) \to A$$
$$\neg(A \wedge \neg B) \to (A \to B)$$
$$\neg(A \to B) \to A \wedge \neg B$$

$$A \vee (A \to B)$$
$$(A \to B) \to ((\neg A \to \neg B) \to (B \to A))$$

Indeed, all these formulas take non-"true" values under the "truth tables" from the proof of Theorem 2.8.4.

The following three formulas also cannot be proved in the constructive propositional logic, yet, unfortunately, the "truth tables" from our proof of Theorem 2.8.4 do not allow proving this:

$$\neg (A \wedge B) \to \neg A \vee \neg B$$
$$\neg \neg (A \vee B) \to \neg \neg A \vee \neg \neg B$$
$$(A \to B) \vee (B \to A)$$

Indeed, under the above "truth tables", these formulas always take "true" values (see results file #03). Another interesting conclusion: **add these three formulas as additional axioms to [L_1-L_{10}, MP] – and L_{11} will remain still unprovable!**

Thus, we did not succeed in building a three-valued logic that would allow showing that the latter three formulas cannot be proved in the constructive propositional logic. Is it possible at all to build a multi-valued logic that would separate constructively provable propositional formulas from unprovable ones? Kurt Gödel showed in 1932 that this is impossible: **none of the finitely-valued logics "matches" exactly the constructive propositional logic:**

K. Gödel. Zum intuitionistischen Aussagenkalkül, *Anzeiger Akademie der Wissenschaften Wien*, Math.-naturwiss. Klasse, 1932, Vol. 69, pp.65-66.

Exercise 2.8.4. a) (optional, for smart students) Verify that the latter three formulas cannot be proved in the constructive propositional logic [L_1-L_{10}, MP]. Or, see Section 4.4.

b) Verify that any of the following formulas could be used – instead of $B \vee \neg B$ – as the axiom L_{11} of the classical propositional logic: i) $(A \to B) \to \neg A \vee B$, ii) ¬¬B→B, iii) ¬(A→B)→A (Hint: since all these formulas are provable in [L_1-L_{11}, MP], it remains to prove L_{11} in [L_1-L_{10}, MP] + (i), in [L_1-L_{10}, MP] + (ii), and in [L_1-L_{10}, MP] + (iii)).

c) Verify that with ¬¬B→B instead of L_{11} the "crazy" axiom L_{10} becomes 100% derivable from the other axioms. Perhaps, this is why many textbooks prefer the combination L_1-L_9 + ¬¬B→B as the axiom list for the classical propositional logic. But, then, we are forced to define the constructive propositional logic not as a subset of the classical one, but as the classical logic with the axiom ¬¬B→B replaced by the "crazy" axiom L_{10}: ¬B→(B→C)!

Axiom L_{10} again...

Finally, let us check which of the main results of Sections 2.5 (constructive logic) and 2.6 (classical logic) depend on the "crazy" axiom L_{10}: ¬A→(A→B). Let your program print out only those variants of "truth tables" that make "true" all the axioms L_1-L_8, and make: L_9 – "true", L_{10} – not "true". My program yields 6 such variants, see the results file #04. Surprisingly, in all these variants L_{11} also is "true" (thus, the results file #04 equals the results file #02). As the most productive appears

Implication variant #1:
2 2 2 0 2 2 0 1 2 L1-L8 true.
Variant #1. Negation: 2 2 1 L9 true. L10 not true. L11 true.
Constructively provable formulas
Not true: $(A \vee B) \rightarrow ((\neg A) \rightarrow B)$
Not true: $((\neg A) \vee B) \rightarrow (A \rightarrow B)$
Not true: ((¬¬A)→(¬¬B))→(¬¬(A→B))
Not true: (¬¬A)→((¬A)→A)
Not true: $(A \vee (\neg A)) \rightarrow ((\neg \neg A) \rightarrow A)$
Not true: ¬¬((¬¬A)→A)

Classically provable formulas
True: $(\neg \neg (A \vee B)) \rightarrow ((\neg \neg A) \vee (\neg \neg B))$
True: $(\neg (A \wedge B)) \rightarrow ((\neg A) \vee (\neg B))$
Not true: (¬¬A)→A
Not true: ((¬B)→(¬A))→(A→B)
Not true: ((¬A)→B)→((¬B)→A)
Not true: ((¬¬A)→(¬¬B))→(A→B)
True: $(A \rightarrow B) \rightarrow ((\neg A) \vee B)$
Not true: $((A \rightarrow B) \rightarrow B) \rightarrow (A \vee B)$
Not true: ((A→B)→A)→A
Not true: $(\neg (A \wedge (\neg B))) \rightarrow (A \rightarrow B)$
True: (A→B)→(((¬A)→B)→B)
Not true: $(\neg (A \rightarrow B)) \rightarrow (A \wedge (\neg B))$
Not true: $A \vee (A \rightarrow B)$
True: $(A \rightarrow B) \vee (B \rightarrow A)$
Not true: (A→B)→(((¬A)→(¬B))→(B→A))

Thus, the following constructively provable formulas cannot be proved in the minimal logic [L_1-L_9, MP] (and even in [L_1-L_9, L_{11}, MP]), i.e. they cannot be proved without the "crazy" axiom L_{10}:

$$(A \vee B) \rightarrow (\neg A \rightarrow B)$$

$$\neg A \vee B \rightarrow (A \rightarrow B)$$
$$(\neg\neg A \rightarrow \neg\neg B) \rightarrow \neg\neg(A \rightarrow B)$$
$$\neg\neg A \rightarrow (\neg A \rightarrow A)$$
$$A \vee \neg A \rightarrow (\neg\neg A \rightarrow A)$$
$$\neg\neg(\neg\neg A \rightarrow A)$$

And the following classically provable formulas cannot be proved without the "crazy" axiom L_{10} **(**thanks to Pavels Mihailovs for a correction**)**:

$$\neg\neg A \rightarrow A$$
$$(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)$$
$$(\neg A \rightarrow B) \rightarrow (\neg B \rightarrow A)$$
$$(\neg\neg A \rightarrow \neg\neg B) \rightarrow (A \rightarrow B)$$
$$((A \rightarrow B) \rightarrow B) \rightarrow A \vee B$$
$$((A \rightarrow B) \rightarrow A) \rightarrow A$$
$$\neg(A \wedge \neg B) \rightarrow (A \rightarrow B)$$
$$\neg(A \rightarrow B) \rightarrow A \wedge \neg B$$
$$A \vee (A \rightarrow B)$$
$$(A \rightarrow B) \rightarrow ((\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A))$$

Exercise 2.8.5 (thanks to Stanislav Golubcov for the idea). But how about the remaining four (classically provable) formulas:

a) $(A \rightarrow B) \rightarrow \neg A \vee B$,
b) $\neg(A \wedge B) \rightarrow \neg A \vee \neg B$,
c) $\neg\neg(A \vee B) \rightarrow \neg\neg A \vee \neg\neg B$,
c_1) (A→B)→((¬A→B)→B),
d) $(A \rightarrow B) \vee (B \rightarrow A)$?

Show that the formulas (a, b, c) **can be** proved without the "crazy" axiom L_{10}, i.e prove them in [L_1-L_9, L_{11}, MP]. (Hint: use Theorem 2.6.4 (a) [L_1-L_8, MP]: $A \vee \neg A \vdash (A \rightarrow B) \rightarrow \neg A \vee B$.). For smart students: how about the remaining formulas (c_1, d)?

Using computers in mathematical proofs

Do you trust the above proofs? Personally, I trust much more my ability to write (relatively) error-free computer programs than my ability to carry out error-free mathematical proofs. But how about you? Of course, you do not need trusting my (or your own) program generating the results files #00, #01, #02, #03 and #04. We used these files only to select the "truth table" variants allowing to prove our independence results. The critical points to be trusted are (see my implementation file) : a) the recursive program

```
int MyFormula::ValueAt(int A, int B, int C)
```

and b) the character string analyzer

```
int MyFormula::Analyze(int *pOperation, AnsiString *pSubFormula1, AnsiString
*pSubFormula2)
```

You may wish to remove your worries by verifying directly that under all the 3 truth table variants used above: a) the axioms L_1-L_8 are true, and b) the axioms L_9, L_{10}, L_{11} and other formulas are true or not true according to the goal of each particular proof. Before you have performed this 100%, you can feel **the flavor of using computers in mathematical proofs** (I trust this proof, I do not trust it, I trust this proof, I do not trust it, I trust this proof, I do not trust it, I trust this proof, I do not trust it, I trust this proof, I do not trust it, I trust this proof, I do not trust it, I trust this proof, I do not trust it, I trust this proof, I do not trust it, I trust this proof, I do not trust it, I trust this proof, I do not trust it, I trust this proof, I do not trust it,...)

Unfortunately, in more complicated cases the situation does not allow the above simple exit (i.e. manual verification of the solution found by a computer):

"The Four Colour Theorem was the first major theorem to be proved using a computer, having a proof that could not be verified directly by other mathematicians. Despite some worries about this initially, independent verification soon convinced everyone that the Four Colour Theorem had finally been proved. Details of the proof appeared in two articles in 1977. Recent work has led to improvements in the algorithm." (According to the article: The Four Colour Theorem in MacTutor History of Mathematics archive).

The proof of the Four Colour Theorem was completed in 1976 by Kenneth Appel and Wolfgang Haken, see their photographs published in European Mathematical Society, Newsletter No. 46, December 2002, pp. 15-19.

"The best-known, and most debated, instance is the use of computer analysis by Kenneth Appel and Wolfgang Haken of the University of Illinois in their 1976 proof of the four-colour conjecture (that four colours suffice to colour in any map drawn upon a plane in such a way that countries which share a border are given different colours). First put forward in 1852, the conjecture had become perhaps the most famous unsolved problem in mathematics, resisting a multitude of efforts at proof for over a century. Appel and Haken's demonstration rested upon computerized analysis, occupying 1,200 hours of computer time, of over 1,400 graphs. The analysis of even one of those graphs typically went beyond what an unaided human being could plausibly do: the ensemble of their demonstration certainly could not be checked in detail by human beings. In consequence, whether that demonstration constituted "proof" was deeply controversial..." (according to

Donald MacKenzie. Computers and the Sociology of Mathematical Proof. In: *Trends in the History and Philosophy of Mathematics*, Odense: University of Southern Denmark, 2004, pp.67-86).

See also

Ken Appel on the 4CT proof, December 1998

Robin Wilson. Four Colours Suffice. European Mathematical Society, Newsletter No. 46, December 2002, pp. 15-19 (online copy).

The Four Color Theorem, November 13, 1995, by Robin Thomas.

A computer-checked proof of the Four Colour Theorem, 2004, by Georges Gonthier.

Doron Zeilberger. Opinion 54: It is Important to Keep Looking for Non-Computer Proofs of the Four-Color Theorem, But Not For the "Usual" Reasons (available online at http://www.math.rutgers.edu/~zeilberg/Opinion54.html).

Two other famous computer assisted mathematical proofs:

- In 1989, by using Cray super-computers, Clement W. H. Lam finished his proof that finite projective plane of order 10 is impossible (for details see Projective plane in Wikipedia).

- In 1998, Thomas C. Hales finished his proof of Kepler conjecture about the densest arrangement of spheres in space (Johannes Kepler conjectured it in 1611, for details see Kepler conjecture in Wikipedia).

See logical software links selected by Peter Suber.

Visit the Mizar Project.

3. Predicate Logic

3.1. Proving Formulas Containing Quantifiers and Implication only

Theorem 3.1.0. $[L_1, L_2, L_{12}, L_{13}, MP] \vdash \forall xB(x) \rightarrow \exists xB(x)$. What does it mean? It prohibits "empty domains".

Indeed,

(1)	$\forall xB(x)$	Hypothesis.
(2)	$\forall xB(x) \rightarrow B(x)$	Axiom L_{12}.
(3)	$B(x)$	By MP.
(4)	$B(x) \rightarrow \exists xB(x)$	Axiom L_{13}.
(5)	$\exists xB(x)$	By MP.

Thus, by $[L_1, L_2, MP]$ Deduction Theorem 1, there is a proof of $[L_1, L_2, L_{12}, L_{13}, MP] \vdash \forall xB(x) \rightarrow \exists xB(x)$.

Theorem 3.1.1. a) $[L_1, L_2, L_{12}, L_{14}, MP, Gen] \vdash \forall x(B \rightarrow C) \rightarrow (\forall xB \rightarrow \forall xC)$. What does it mean?

b) $[L_1, L_2, L_{12}\text{-}L_{15}, MP, Gen] \vdash \forall x(B \rightarrow C) \rightarrow (\exists xB \rightarrow \exists xC)$. What does it mean?

Let us prove (a).

(1)	$\forall x(B \rightarrow C)$	Hypothesis.
(2)	$\forall xB$	Hypothesis.
(3)	$\forall x(B \rightarrow C) \rightarrow (B \rightarrow C)$	Axiom L_{12}: $\forall xF(x) \rightarrow F(x)$.
(4)	$B \rightarrow C$	From (1) and (3), by MP.
(5)	$\forall xB \rightarrow B$	Axiom L_{12}: $\forall xF(x) \rightarrow F(x)$.
(6)	B	From (2) and (5), by MP.

(7)	C	From (4) and (6), by MP.
(8)	∀xC	From (7), by Gen.

In this proof, Gen is applied only to x, which is not a free variable in ∀x(B→C) and ∀xB. Thus, by [L_1, L_2, L_{14}, MP, Gen] Deduction Theorem 2, there is a proof of [L_1, L_2, L_{12}, L_{14}, MP, Gen] ⊢ ∀x(B→C) → (∀xB→∀xC).

Let us prove (b).

(1)	∀x(B→C)	Hypothesis.
(2)	∀x(B→C)→(B→C)	Axiom L_{12}: ∀xF(x)→F(x).
(3)	B→C	From (1) and (2), by MP.
(4)	C→∃xC	Axiom L_{13}: F(x)→∃xF(x).
(5)	B→∃xC	From (3) and (4), by transitivity of implication [L_1, L_2, MP].
(6)	∀x(B→∃xC)	From (5), by Gen.
(7)	∀x(B→∃xC)→(∃xB→∃xC)	Axiom L_{15}: ∀x(F(x)→G)→(∃xF(x)→G) (∃xC does not contain x as a free variable).
(8)	∃xB→∃xC	From (6) and (7), by MP.

In this proof, Gen is applied only to x, which is not a free variable in ∀x(B→C). Thus, by [L_1, L_2, L_{14}, MP, Gen] Deduction Theorem 2, there is a proof of [L_1, L_2, L_{12}-L_{15}, MP, Gen] ⊢ ∀x(B→C) → (∃xB→∃xC).

Q.∃.D.

Theorem 3.1.2. a) [L_1, L_2, L_5, L_{12}, L_{14}, MP, Gen] ⊢ ∀x∀yB(x, y)↔∀y∀xB(x, y). What does it mean?

b) [L_1, L_2, L_5, L_{13}, L_{15}, MP, Gen] ⊢ ∃x∃yB(x, y)↔∃y∃xB(x, y). What does it mean?

c) [L_1, L_2, L_{12}-L_{15}, MP, Gen] ⊢ ∃x∀yB(x, y)→∀y∃xB(x, y). What does it mean? The converse implication ∀x∃yB(x, y)→∃y∀xB(x, y) cannot be true. Explain, why.

Let us prove (b).

(1)	$B(x, y)\rightarrow\exists xB(x, y)$	Axiom L_{13} with $F(x) = B(x, y)$.
(2)	$\exists xB(x, y)\rightarrow\exists y\exists xB(x, y)$	Axiom L_{13} with $F(y) = \exists xB(x, y)$.
(3)	$B(x, y)\rightarrow\exists y\exists xB(x, y)$	From (1) and (2), by transitivity of implication [L_1, L_2, MP].
(4)	$F(x)\rightarrow G \vdash \exists xF(x)\rightarrow G$	Exercise 1.4.3(a): [L_{15}, MP, Gen], x not free in G.
(5)	$\exists yB(x, y)\rightarrow\exists y\exists xB(x, y)$	From (3), by (4), with $F(y) = B(x, y)$, $G = \exists x\exists yB(x, y)$.
(6)	$\exists x\exists yB(x, y)\rightarrow\exists y\exists xB(x, y)$	From (5), by (4), with $F(x) = \exists yB(x, y)$, $G = \exists x\exists yB(x, y)$.

The proof of the converse implication $[L_1, L_2, L_{13}, L_{15}, MP, Gen] \vdash \exists y\exists xB(x, y)\rightarrow\exists x\exists yB(x, y)$ is identical.

Now, by Axiom L_5 we obtain the equivalence (b). Q.E.D.

Exercise 3.1.1. Prove (a) and (c) of Theorem 3.1.2.

Exercise 3.1.2. Prove in the constructive logic,

$[L_1\text{-}L_{10}, L_{12}\text{-}L_{15}, MP, Gen] \vdash \exists x(B(x)\rightarrow C(x)) \rightarrow (\forall xB(x)\rightarrow\exists xC(x))$.

3.2. Formulas Containing Negations and a Single Quantifier

Attention: non-constructive reasoning! $\neg\forall xB \rightarrow \exists x\neg B$. This formula is accepted in the classical logic: if no x can possess the property B, then there is an x that does not possess B. It represents non-constructive reasoning in its ultimate form: let us assume, all x-s possess the property B, if we succeed in deriving a contradiction from this assumption, then – what? Is this a proof that there is a particular x that does not possess the property B? Does our proof contain a method allowing to build at least one such x? If not, do we have a "real" proof of $\exists x\neg B$?

How many formulas can be built of the formula B by using negations and a single quantifier?

$$\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg\forall x\neg\neg\neg\neg\neg\neg\neg\neg\neg\neg B$$

¬¬¬¬¬¬¬¬¬¬∃x¬¬¬¬¬¬¬¬¬¬B

Let us remind Theorem 2.4.5 [L_1-L_9, MP]: ⊢¬¬¬A↔¬A. I.e., any number of negations can be reduced to zero, one, or two, and thus we obtain 3*2*3 = 18 formulas to be investigated. The following table represents the results of this investigation from

A.Heyting. On weakened quantification. *Journal of Symbolic Logic*, 1936, vol.11, pp.119-121 (see also Kleene [1952], Section 3.5).

Table 3.2

I ∀xB ---------- ¬¬∀xB ========== ∀x¬¬B ¬¬∀x¬¬B ¬∃x¬B	**III** ∃x¬B ---------- ¬¬∃x¬B ¬∀x¬¬B ========== ¬∀xB
II ∃xB ---------- ∃x¬¬B ---------- ¬¬∃xB ¬¬∃x¬¬B ¬∀x¬B	**IV** ∀x¬B ¬¬∀x¬B ¬∃x¬¬B ¬∃xB

Legend. a) In the classical logic, within each of the 4 groups all formulas are equivalent, for example, in group III: ¬∀xB↔∃x¬B. Of course, formulas of different groups cannot be equivalent (explain, why).

b) Two formulas within a group are constructively equivalent, if and only if they have no separating lines between them. For example, in group II: constructively, ¬∀x¬B↔¬¬∃xB, but not ¬∀x¬B↔∃xB (explain, why). All the formulas of the group IV are constructively equivalent.

c) If two formulas F_1, F_2 within a group (F_1 – above, F_2 – below) are separated by a **single line**, then: constructively, $F_1 \to F_2$, and ¬¬($F_2 \to F_1$), but not $F_2 \to F_1$. For example, in group II: constructively, ∃xB→¬∀x¬B, and ¬¬(¬∀x¬B→∃xB), but not ¬∀x¬B→∃xB (explain, why).

d) If two formulas F_1, F_2 within a group (F_1 – above, F_2 – below) are separated by a **double line**, then: constructively, $F_1 \to F_2$, but not $F_2 \to F_1$, and even not $\neg\neg(F_2 \to F_1)$. For example, in group III: constructively, $\exists x \neg B \to \neg \forall x B$, but not $\neg \forall x B \to \exists x \neg B$, and even not $\neg\neg(\neg \forall x B \to \exists x \neg B)$ (try to explain, why). Thus, the implication $\neg \forall x B \to \exists x \neg B$ could be qualified as **super-non-constructive**.

Now, let us prove the implications necessary for the positive part of the above legend to be true.

Group I

I-1. Constructively, [L_1, L_2, L_9, MP]: $\vdash \forall x B \to \neg\neg \forall x B$

Immediately, by Theorem 2.4.4 [L_1, L_2, L_9, MP]: $\vdash A \to \neg\neg A$.

I-2. Constructively, [L_1-L_9, L_{12}, L_{14}, MP, Gen]: $\vdash \neg\neg \forall x B \to \forall x \neg\neg B$

(1)	$\forall x B \to B$	Axiom L_{12}: $\forall x F(x) \to F(x)$.
(2)	$\neg\neg \forall x B \to \neg\neg B$	From (1), by Theorem 2.4.7(a) [L_1-L_9, MP]: $\vdash (A \to B) \to (\neg\neg A \to \neg\neg B)$.
(3)	$\forall x(\neg\neg \forall x B \to \neg\neg B)$	From (2), by Gen.
(4)	$\neg\neg \forall x B \to \forall x \neg\neg B$	From (3), by Axiom L_{14}: $\forall x(G \to F(x)) \to (G \to \forall x F(x))$.

I-3. Constructively, [L_1, L_2, L_9, MP]: $\vdash \forall x \neg\neg B \to \neg\neg \forall x \neg\neg B$

Immediately, by Theorem 2.4.4 [L_1, L_2, L_9, MP]: $\vdash A \to \neg\neg A$.

I-4. Constructively, [L_1, L_2, L_9, L_{12}, L_{15}, MP, Gen] $\vdash \neg\neg \forall x \neg\neg B \to \neg \exists x \neg B$

(1)	$\forall x \neg\neg B \to \neg\neg B$	Axiom L_{12}: $\forall x F(x) \to F(x)$.
(2)	$\neg\neg\neg B \to \neg \forall x \neg\neg B$	From (1), by the Contraposition Law – Theorem 2.4.2. [L_1, L_2, L_9, MP]: $\vdash (A \to B) \to (\neg B \to \neg A)$.
(3)	$\neg B \to \neg\neg\neg B$	By Theorem 2.4.4 [L_1, L_2, L_9, MP]: $\vdash A \to \neg\neg A$.

(4)	$\neg B \rightarrow \neg \forall x \neg\neg B$	From (2) and (3), by transitivity of implication – Theorem 1.4.2 [L_1, L_2, MP].
(5)	$\forall x(\neg B \rightarrow \neg \forall x \neg\neg B)$	From (4), by Gen.
(6)	$\exists x \neg B \rightarrow \neg \forall x \neg\neg B$	From (5), by Axiom L_{15}: $\forall x(F(x) \rightarrow G) \rightarrow (\exists x F(x) \rightarrow G)$.
(7)	$\neg\neg \forall x \neg\neg B \rightarrow \neg \exists x \neg B$	From (6), by the Contraposition Law [L_1, L_2, L_9, MP].

I-5. In the classical logic, [L_1-L_{11}, L_{13}, L_{14}, MP, Gen]: $\vdash \neg \exists x \neg B \rightarrow \forall x B$

(1)	$\neg B \rightarrow \exists x \neg B$	Axiom L_{13}: $F(x) \rightarrow \exists x F(x)$.
(2)	$\neg \exists x \neg B \rightarrow \neg\neg B$	From (1), by the Contraposition Law [L_1, L_2, L_9, MP].
(3)	$\neg\neg B \rightarrow B$	Classical logic, Theorem 2.6.1 [L_1-L_{11}, MP]: $\vdash \neg\neg A \rightarrow A$
(4)	$\neg \exists x \neg B \rightarrow B$	From (2) and (3), by transitivity of implication [L_1, L_2, MP].
(5)	$\forall x(\neg \exists x \neg B \rightarrow B)$	From (4), By Gen.
(6)	$\neg \exists x \neg B \rightarrow \forall x B$	From (5), by Axiom L_{14}: $\forall x(G \rightarrow F(x)) \rightarrow (G \rightarrow \forall x F(x))$.

Thus, we have proved that in Group I, constructively, $F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow F_4 \rightarrow F_5$, and, in the classical logic, $F_5 \rightarrow F_1$. I.e. we have proved that in Group I: a) in the classical logic, all the formulas are equivalent, and b) constructively, upper formulas imply lower formulas.

I-6. Constructively, [L_1, L_2, L_9, L_{13}, L_{14}, MP, Gen]: $\vdash \neg \exists x \neg B \rightarrow \forall x \neg\neg B$

(1)	$\neg B \rightarrow \exists x \neg B$	Axiom L_{13}: $F(x) \rightarrow \exists x F(x)$.
(2)	$\neg \exists x \neg B \rightarrow \neg\neg B$	From (1), by the Contraposition Law [L_1, L_2, L_9, MP].

(3) $\forall x(\neg\exists x\neg B\rightarrow\neg\neg B)$	From (2), by Gen.
(4) $\neg\exists x\neg B\rightarrow\forall x\neg\neg B$	From (3), by Axiom L_{14}: $\forall x(G\rightarrow F(x))\rightarrow(G\rightarrow\forall xF(x))$.

Thus, we have proved that in Group I, constructively, [L_1, L_2, L_9, L_{12} -L_{15}, MP, Gen]: $F_3\rightarrow F_4\rightarrow F_5\rightarrow F_3$, i.e. that formulas F_3, F_4, F_5 are constructively equivalent.

For Group I, it remains to prove

I-7. Constructively, [L_1-L_{10}, MP] $\vdash\neg\neg(\neg\neg\forall xB\rightarrow\forall xB)$

Immediately, by Theorem 2.5.2(d) [L_1-L_{10}, MP] $\vdash\neg\neg(\neg\neg A\rightarrow A)$.

Group II

II-1. Constructively, [L_1, L_2, L_9, L_{12}-L_{15}, MP, Gen] $\vdash\exists xB\rightarrow\exists x\neg\neg B$

(1) $B\rightarrow\neg\neg B$	By Theorem 2.4.4 [L_1, L_2, L_9, MP]: $\vdash A\rightarrow\neg\neg A$.
(2) $\forall x(B\rightarrow\neg\neg B)$	From (1), by Gen.
(3) $\exists xB\rightarrow\exists x\neg\neg B$	From (2), by Theorem 3.1.1(b) [L_1, L_2, L_{12}-L_{15}, MP, Gen]

II-2. Constructively, [L_1-L_9, L_{12}-L_{15}, MP, Gen] $\vdash\exists x\neg\neg B\rightarrow\neg\neg\exists xB$

(1) $B\rightarrow\exists xB$	Axiom L_{13}: $F(x)\rightarrow\exists xF(x)$.
(2) $\neg\neg B\rightarrow\neg\neg\exists xB$	From (1), by Theorem 2.4.7(a) [L_1-L_9, MP]: $\vdash(A\rightarrow B)\rightarrow(\neg\neg A\rightarrow\neg\neg B)$.
(3) $\forall x(\neg\neg B\rightarrow\neg\neg\exists xB)$	From (2), by Gen.
(4) $\exists x\neg\neg B\rightarrow\neg\neg\exists xB$	From (3), by Theorem 3.1.1(b) [L_1, L_2, L_{12}-L_{15}, MP, Gen]

II-3. Constructively, [L_1-L_9, L_{12}-L_{15}, MP, Gen] $\vdash\neg\neg\exists xB\rightarrow\neg\neg\exists x\neg\neg B$

Immediately from II-1, by From (1), by Theorem 2.4.7(a) [L_1-L_9, MP]: $\vdash(A\rightarrow B)\rightarrow(\neg\neg A\rightarrow\neg\neg B)$.

II-4. Constructively, [L_1-L_9, L_{12}, L_{15}, MP, Gen] $\vdash\neg\neg\exists x\neg\neg B\rightarrow\neg\forall x\neg B$

(1) $\forall x\neg B\rightarrow\neg B$	Axiom L_{12}: $\forall xF(x)\rightarrow F(x)$.
(2) $\neg\neg B\rightarrow\neg\forall x\neg B$	From (1), by the Contraposition Law – Theorem 2.4.2. [L_1, L_2, L_9, MP]: $\vdash (A\rightarrow B)\rightarrow(\neg B\rightarrow\neg A)$.
(3) $\forall x(\neg\neg B\rightarrow\neg\forall x\neg B)$	From (2), by Gen.
(4) $\exists x\neg\neg B\rightarrow\neg\forall x\neg B$	From (3), by Axiom L_{15}: $\forall x(F(x)\rightarrow G)\rightarrow(\exists xF(x)\rightarrow G)$.
(5) $\neg\neg\exists x\neg\neg B\rightarrow\neg\neg\neg\forall x\neg B$	From (4), by Theorem 2.4.7(a) [L_1-L_9, MP]: $\vdash(A\rightarrow B)\rightarrow(\neg\neg A\rightarrow\neg\neg B)$.
(6) $\neg\neg\neg\forall x\neg B\rightarrow\neg\forall x\neg B$	Theorem 2.4.5 [L_1-L_9, MP]: $\vdash \neg\neg\neg A\leftrightarrow\neg A$
(7) $\neg\neg\exists x\neg\neg B\rightarrow\neg\forall x\neg B$	From (5) and (6), by transitivity of implication [L_1, L_2, MP].

II-5. In the classical logic, [L_1-L_{11}, L_{13}, L_{14}, MP, Gen]: $\vdash\neg\forall x\neg B\rightarrow\exists xB$

(1) $\neg\forall x\neg B\rightarrow\neg\neg\exists xB$	II-6 [L_1, L_2, L_9, L_{13}, L_{14}, MP, Gen], see below.
(2) $\neg\neg\exists xB\rightarrow\exists xB$	Classical logic, Theorem 2.6.1 [L_1-L_{11}, MP]: $\vdash\neg\neg A\rightarrow A$
(3) $\neg\forall x\neg B\rightarrow\exists xB$	From (1) and (2), by transitivity of implication [L_1, L_2, MP].

Thus, we have proved that in Group II, constructively, $F_1\rightarrow F_2\rightarrow F_3\rightarrow F_4\rightarrow F_5$, and, in the classical logic, $F_5\rightarrow F_1$. I.e. we have proved that in Group II: a) in the classical logic, all the formulas are equivalent, and b) constructively, upper formulas imply lower formulas.

II-6. Constructively, [L_1, L_2, L_9, L_{13}, L_{14}, MP, Gen] $\vdash\neg\forall x\neg B\rightarrow\neg\neg\exists xB$

(1) $B\rightarrow\exists xB$	Axiom L_{13}: $F(x)\rightarrow\exists xF(x)$.

(2) $\neg\exists xB\rightarrow\neg B$	From (1), by the Contraposition Law – Theorem 2.4.2. [L_1, L_2, L_9, MP]: $\vdash (A\rightarrow B)\rightarrow(\neg B\rightarrow\neg A)$.
(3) $\forall x(\neg\exists xB\rightarrow\neg B)$	From (2), by Gen.
(4) $\neg\exists xB\rightarrow\forall x\neg B$	From (3), by Axiom L_{14}: $\forall x(G\rightarrow F(x))\rightarrow(G\rightarrow\forall xF(x))$.
(5) $\neg\forall x\neg B\rightarrow\neg\neg\exists xB$	From (4), by the Contraposition Law – Theorem 2.4.2. [L_1, L_2, L_9, MP]: $\vdash (A\rightarrow B)\rightarrow(\neg B\rightarrow\neg A)$.

Thus, we have proved that in Group II, constructively, [L_1-L_9, L_{12}-L_{15}, MP, Gen]: $F_3\rightarrow F_4\rightarrow F_5\rightarrow F_3$, i.e. that formulas F_3, F_4, F_5 are constructively equivalent.

II-7. Constructively, [L_1-L_{10}, MP]: $\vdash\neg\neg(\neg\neg\exists xB\rightarrow\exists xB)$

Immediately, by Theorem 2.5.2 [L_1-L_{10}, MP]: $\vdash\neg\neg(\neg\neg A\rightarrow A)$.

Thus, constructively, $\neg\neg(F_3\rightarrow F_1)$, and $F_1\rightarrow F_2\rightarrow F_3\rightarrow F_4\rightarrow F_5\rightarrow F_3$. By Theorem 2.4.7(d), [$L_1$-$L_9$, MP] $\neg\neg(A\rightarrow B)$, $\neg\neg(B\rightarrow C)\vdash\neg\neg(A\rightarrow C)$. Thus, in fact, we have proved that in Group II, for all i, j, constructively, $\neg\neg(F_i\rightarrow F_j)$ (a kind of "weak equivalence").

Group III

III-1. Constructively, [L_1, L_2, L_9, MP]: $\vdash\exists x\neg B\rightarrow\neg\neg\exists x\neg B$

Immediately, by Theorem 2.4.4 [L_1, L_2, L_9, MP]: $\vdash A\rightarrow\neg\neg A$.

III-2. Constructively, [L_1, L_2, L_9, L_{12}, L_{15}, MP, Gen]: $\vdash\neg\neg\exists x\neg B\rightarrow\neg\forall x\neg\neg B$

(1) $\forall x\neg\neg B\rightarrow\neg\neg\forall x\neg\neg B$	I-3 [L_1, L_2, L_9, MP], see above.
(2) $\neg\neg\forall x\neg\neg B\rightarrow\neg\exists x\neg B$	I-4 [L_1, L_2, L_9, L_{12}, L_{15}, MP, Gen], see above.
(3) $\forall x\neg\neg B\rightarrow\neg\exists x\neg B$	From (1) and (2), by transitivity of implication [L_1, L_2, MP].

(4) $\neg\neg\exists x\neg B\rightarrow\neg\forall x\neg\neg B$ — From (3), by the Contraposition Law [L_1, L_2, L_9, MP].

III-3. Constructively, [L_1-L_9, L_{12}, L_{14}, MP, Gen]: $\vdash\neg\forall x\neg\neg B\rightarrow\neg\forall xB$

(1) $\forall xB\rightarrow\neg\neg\forall xB$ — I-1 [L_1, L_2, L_9, MP], see above.

(2) $\neg\neg\forall xB\rightarrow\forall x\neg\neg B$ — I-2 [L_1-L_9, L_{12}, L_{14}, MP, Gen]

(3) $\forall xB\rightarrow\forall x\neg\neg B$ — From (1) and (2), by transitivity of implication [L_1, L_2, MP].

(4) $\neg\forall x\neg\neg B\rightarrow\neg\forall xB$ — From (3), by the Contraposition Law [L_1, L_2, L_9, MP].

III-4. In the classical logic, [L_1-L_{11}, L_{13}, L_{14}, MP, Gen]: $\vdash\neg\forall xB\rightarrow\exists x\neg B$

(1) $\neg\exists x\neg B\rightarrow\forall xB$ — I-5: in the classical logic, [L_1-L_{11}, L_{13}, L_{14}, MP, Gen]

(2) $\neg\forall xB\rightarrow\neg\neg\exists x\neg B$ — From (1), by the Contraposition Law [L_1, L_2, L_9, MP].

(3) $\neg\neg\exists x\neg B\rightarrow\exists x\neg B$ — Classical logic, Theorem 2.6.1 [L_1-L_{11}, MP]: $\vdash\neg\neg A\rightarrow A$

(4) $\neg\forall xB\rightarrow\exists x\neg B$ — From (2) and (3), by transitivity of implication [L_1, L_2, MP].

Thus, we have proved that in Group III, constructively, $F_1\rightarrow F_2\rightarrow F_3\rightarrow F_4$, and, in the classical logic, $F_4\rightarrow F_1$. I.e. we have proved that in Group III: a) in the classical logic, all the formulas are equivalent, and b) constructively, upper formulas imply lower formulas.

III-4. Constructively, [L_1, L_2, L_9, L_{13}, L_{14}, MP, Gen]: $\vdash\neg\forall x\neg\neg B\rightarrow\neg\neg\exists x\neg B$

(1) $\neg\exists x\neg B\rightarrow\forall x\neg\neg B$ — I-6 [L_1, L_2, L_9, L_{13}, L_{14}, MP, Gen]

(2) $\neg\forall x\neg\neg B\rightarrow\neg\neg\exists x\neg B$ — From (1), by the Contraposition Law [L_1, L_2, L_9, MP].

Thus, we have proved that in Group III, constructively, $F_2\rightarrow F_3\rightarrow F_2$, i.e. that

formulas F_2, F_3 are constructively equivalent.

III-5. Constructively, [L_1-L_{10}, MP]: $\vdash\neg\neg(\neg\neg\exists x\neg B\rightarrow\exists x\neg B)$

Immediately, by Theorem 2.5.2 [L_1-L_{10}, MP]: $\vdash\neg\neg(\neg\neg A\rightarrow A)$.

Group IV

IV-1. Constructively, [L_1, L_2, L_9, MP]: $\vdash\forall x\neg B\rightarrow\neg\neg\forall x\neg B$

Immediately, by Theorem 2.4.4 [L_1, L_2, L_9, MP]: $\vdash A\rightarrow\neg\neg A$.

IV-2. Constructively, [L_1-L_9, L_{12}-L_{15}, MP, Gen]: $\vdash\neg\neg\forall x\neg B\rightarrow\neg\exists x\neg\neg B$

(1)	$\exists x\neg\neg B\rightarrow\neg\forall x\neg B$	From II-2, II-3, II-4 [L_1-L_9, L_{12}-L_{15}, MP, Gen], by transitivity of implication [L_1, L_2, MP].
(2)	$\neg\neg\forall x\neg B\rightarrow\neg\exists x\neg\neg B$	From (1), by the Contraposition Law [L_1, L_2, L_9, MP].

IV-3. Constructively, [L_1, L_2, L_9, L_{12}-L_{15}, MP, Gen]: $\vdash\neg\exists x\neg\neg B\rightarrow\neg\exists xB$

(1)	$\exists xB\rightarrow\exists x\neg\neg B$	II-1 [L_1, L_2, L_9, L_{12}-L_{15}, MP, Gen]
(2)	$\neg\exists x\neg\neg B\rightarrow\neg\exists xB$	From (1), by the Contraposition Law [L_1, L_2, L_9, MP].

IV-4. Constructively, [L_1, L_2, L_9, L_{13}, L_{14}, MP, Gen]: $\vdash\neg\exists xB\rightarrow\forall x\neg B$

(1)	$B\rightarrow\exists xB$	Axiom L_{13}: $F(x)\rightarrow\exists xF(x)$.
(2)	$\neg\exists xB\rightarrow\neg B$	From (1), by the Contraposition Law [L_1, L_2, L_9, MP].
(3)	$\forall x(\neg\exists xB\rightarrow\neg B)$	From (2), by Gen.
(4)	$\neg\exists xB\rightarrow\forall x\neg B$	From (3), by Axiom L_{14}: $\forall x(G\rightarrow F(x))\rightarrow(G\rightarrow\forall xF(x))$.

Thus, we have proved that in Group IV all the formulas are constructively equivalent.

And thus, we have proved the positive part of the legend of Table 3.2. The negative part of the legend asserts that the following (classically provable)

formulas cannot be proved constructively:

(1)	¬¬∀xB→∀xB	See Group I. Simply, an instance of (the non-constructive) ¬¬A→A.
(2)	∀x¬¬B→¬¬∀xB	See Group I. **Super-non-constructive**: even ¬¬(2) is non-constructive!
(3)	¬¬(∀x¬¬B→¬¬∀xB)	¬¬(2). See Group I.
(4)	∃x¬¬B→∃xB	See Group II. Nearly, an instance of (the non-constructive) ¬¬A→A.
(5)	¬¬∃xB→∃x¬¬B	See Group II. Stronger than simply non-constructivity of ¬¬A→A?
(6)	¬¬∃x¬B→∃x¬B	See Group III. Simply, an instance of (the non-constructive) ¬¬A→A.
(7)	¬∀xB→¬∀x¬¬B	See Group III. **Super-non-constructive**: even ¬¬(7) is non-constructive!
(8)	¬¬(¬∀xB→¬∀x¬¬B)	¬¬(7). See Group III.

We will prove these facts in Section 4.5 (see Exercise 4.5.1).

Still, the most striking (classically provable) non-constructive quantifier implications correspond to existence proofs via *reductio ad absurdum*:

(8)	¬∀x¬B→∃xB	¬¬(8) is constructively provable, but (8) is not, see Group II. If we know how to derive a contradiction from ∀x¬B, then may be, we do not know how to find a particular x such that B.
(9)	¬∀x¬B→¬¬∃x¬¬B	(9) is weaker than (8), but still non-constructive, see Group II. If we know how to derive a contradiction from ∀x¬B, then may be, we do not know how to derive a contradiction from ¬∃x¬¬B.
(10)	¬∀xB→∃x¬B	∃ven ¬¬(10) is non-constructive, see Group III. If we know how to derive a contradiction from ∀xB, then may be, we do not know how to find a particular x such that ¬B.

(11) $\neg\forall$xB$\rightarrow\neg\neg\exists$x$\neg$B	(11) is weaker than (10), but still super-non-constructive (i.e. even $\neg\neg$(11) is non-constructive), see Group III. If we know how to derive a contradiction from $\forall$xB, then may be, we do not know how to derive a contradiction from $\neg\exists$x$\neg$B.

3.3. Proving Formulas Containing Conjunction and Disjunction

Theorem 3.3.1.

a) [L_1-L_5, L_{12}, L_{14}, MP, Gen]: $\vdash \forall x(B\wedge C)\leftrightarrow\forall xB\wedge\forall xC$.

b) [L_1, L_2, L_6-L_8, L_{14}, MP, Gen]: $\vdash \forall xB\vee\forall xC\rightarrow\forall x(B\vee C)$. The converse formula $\forall x(B\vee C)\rightarrow\forall xB\vee\forall xC$ cannot be true. Explain, why.

Exercise 3.3.1.

Prove [L_3-L_5, L_{12}, MP, Gen]: $\forall x(B\wedge C)\vdash\forall xB\wedge\forall xC$ and

[L_3-L_5, L_{12}, MP, Gen]: $\forall xB\wedge\forall xC\vdash\forall x(B\wedge C)$.

Since, in your first proof, Gen has been applied only to x, which is not a free variable in $\forall$x(B&C), then, by Deduction theorem 2 [L_1, L_2, L_{14}, MP, Gen] we obtain that

$$[L_1\text{- }L_5, L_{12}, L_{14}, \text{MP, Gen}]: \vdash \forall x(B\wedge C)\rightarrow\forall xB\wedge\forall xC \quad .$$

Similarly, in your second proof, Gen has been applied only to x, which is not a free variable in $\forall xB\wedge\forall xC$, then, by Deduction theorem 2 [L_1, L_2, L_{14}, MP, Gen] we obtain that

$$[L_1\text{- }L_5, L_{12}, L_{14}, \text{MP, Gen}]: \vdash \forall xB\wedge\forall xC\rightarrow\forall x(B\wedge C) \quad .$$

Now, by Theorem 2.2.1(a) [L_5]: A, B $\vdash A\wedge B$, we obtain the equivalence (a) of Theorem 3.3.1.

Let us prove (b): $\vdash \forall xB\vee\forall xC\rightarrow\forall x(B\vee C)$.

(1)	$B\rightarrow BvC$	Axiom L_6.
(2)	$\forall x(B\rightarrow B\vee C)$	From (1), by Gen.

(3)	$\forall xB \to \forall x(B \vee C)$	From (2), by Theorem 3.1.1(a) [L_1, L_2, L_{12}, L_{14}, MP, Gen].
(4)	$C \to B \vee C$	Axiom L_7.
(5)	$\forall x(C \to B \vee C)$	From (4), by Gen.
(6)	$\forall xC \to \forall x(B \vee C)$	From (5), by Theorem 3.1.1(a) [L_1, L_2, L_{12}, L_{14}, MP, Gen].
(7)	$\forall xB \vee \forall xC \to \forall x(B \vee C)$	From (3) and (6), by Axiom L_8.

The summary is [L_1, L_2, L_6-L_8, L_{12}, L_{14}, MP, Gen]. Q.E.D.

Theorem 3.3.2. a) [L_1-L_8, L_{12}-L_{15}, MP, Gen]: $\vdash \exists x(B \vee C) \leftrightarrow \exists xB \vee \exists xC$.

b) [L_1-L_5, L_{13}-L_{15}, MP, Gen]: $\vdash \exists x(B \wedge C) \to \exists xB \wedge \exists xC$. The converse implication $\exists xB \wedge \exists xC \to \exists x(B \wedge C)$ cannot be true. Explain, why.

For the proof of $\exists x(B \vee C) \to \exists xB \vee \exists xC$, see Exercise 3.3.2(a) below.

Let us prove $\exists xB \vee \exists xC \to \exists x(B \vee C)$.

(1)	$B \to B \vee C$	Axiom L_6.
(2)	$\forall x(B \to B \vee C)$	By Gen.
(3)	$\exists xB \to \exists x(B \vee C)$	By Theorem 3.1.1(b): [L_1, L_2, L_{12}-L_{15}, MP, Gen] $\vdash \forall x(B \to C) \to (\exists xB \to \exists xC)$
(4)	$C \to B \vee C$	Axiom L_7.
(5)	$\forall x(C \to B \vee C)$	By Gen.
(6)	$\exists xC \to \exists x(B \vee C)$	By Theorem 3.1.1(b).
(7)	$\exists xB \vee \exists xC \to \exists x(B \vee C)$	From (3) and (6), by Axiom L_8.

The summary is [L_1, L_2, L_6-L_8, L_{12}-L_{15}, MP, Gen].

Now, by Theorem 2.2.1(a) [L_5]: A, B $\vdash A \wedge B$, we obtain the equivalence (a). Q.E.D.

Exercise 3.3.2. a) Prove (a→) of Theorem 3.3.2 (Hint: start with L_{13} and finish by applying L_{15}.)

b) Prove (b) of Theorem 3.3.2. (Hint: first, assume $B \wedge C$, derive $\exists x B \wedge \exists x C$, and apply Deduction Theorem 1). The converse implication $\exists x B \wedge \exists x C \rightarrow \exists x (B \wedge C)$ cannot be true. Explain, why.

3.4. Replacement Theorems

An example: we know that $\log xy = \log x + \log y$ and $\log x^y = y \cdot \log x$. Hence,

$$\log 2^a 3^b = \log 2^a + \log 3^b = a \cdot \log 2 + b \cdot \log 3 \quad .$$

Another example: we know that (in the classical logic): $\vdash (A \rightarrow B) \leftrightarrow \neg A \vee B$. Hence, the formula (X→Y)→Z "should be" equivalent to $\neg (X \rightarrow Y) \vee Z$, and to $\neg (\neg X \vee Y) \vee Z$. We know also that $\neg (A \vee B) \leftrightarrow \neg A \wedge \neg B$, hence, we can continue: (X→Y)→Z "should be" equivalent to $(\neg \neg X \wedge \neg Y) \vee Z$, and to $(X \wedge \neg Y) \vee Z$ (since ⊢¬¬A↔A).

Until now, in our logic, we could not use this very natural kind of mathematical argument.

In this section we will prove meta-theorems that will allow replacing sub-formulas by equivalent formulas. For example, if have proved the formula ∃xB→D, and we know that ⊢B↔C, then we can replace B by C, obtaining the formula ∃xC→D. These theorems will make the above treatment of the formula (X→Y)→Z completely legal.

We will prove also that the meaning of a formula does not depend on the names of bound variables used in it. For example,

$$\vdash (\exists x B(x) \rightarrow C) \leftrightarrow (\exists y B(y) \rightarrow C).$$

Note. To prove all these replacement theorems we will need only the **minimal logic** [L_1-L_9, L_{12}-L_{15}, MP, Gen].

Sub-formulas and Occurrences

Intuitively, B is a sub-formula of C, if B is a formula, and B is a part (substring) of C. But note that a sub-formula may appear in the same formula more than once, as, for example, in the following instance of the axiom L_1: **∃xB(x)**→(∃xC(x)→**∃xB(x)**). Thus, it would be more correctly to speak about **occurrences** of sub-formulas. In the above example, there are two occurrences of the formula ∃xB(x).

The formal definition is as follows:

a) o(B) is an occurrence in B in B.

b) If o(B) is an occurrence of B in C, then o(B) is an occurrence of B in ¬C, $C \wedge D$, $D \wedge C$, $C \vee D$, $D \vee C$, C→D, and D→C.

b) If o(B) is an occurrence of B in C, then o(B) is an occurrence of B in ∃xC, and ∀xC.

We can define also the notion of **propositional** occurrences:

a) o(B) is a propositional occurrence in B in B.

b) If o(B) is a propositional occurrence of B in C, then o(B) is a propositional occurrence of B in ¬C, $C \wedge D$, $D \wedge C$, $C \vee D$, $D \vee C$, C→D, and D→C.

Intuitively, o(B) is a propositional occurrence of B in C, if, in C, no quantifiers stand over o(B).

Replacement Lemma 1. In the minimal logic, [L_1-L_9, MP]:

(a)	A↔B ⊢(A→C)↔(B→C)	[L_1-L_5, MP]
(b)	A↔B ⊢(C→A)↔(C→B)	[L_1-L_5, MP]
(c)	A↔B ⊢ $A \wedge C \leftrightarrow B \wedge C$	[L_1-L_5, MP]
(d)	A↔B ⊢ $C \wedge A \leftrightarrow C \wedge B$	[L_1-L_5, MP]
(e)	A↔B ⊢ $A \vee C \leftrightarrow B \vee C$	[L_1-L_8, MP]
(f)	A↔B ⊢ $C \vee A \leftrightarrow C \vee B$	[L_1-L_8, MP]
(g)	A↔B ⊢¬A↔¬B	[L_1-L_9, MP]

Case (a). We will first prove that [L_1, L_2, L_4, MP]: A↔B ⊢(A→C)→(B→C).

(1)	$(A \rightarrow B) \wedge (B \rightarrow A)$	A↔B – hypothesis.
(2)	A→C	Hypothesis.
(3)	B→A	From (1), by Axiom L_4.
(4)	B→C	From (3) and (2), by transitivity of implication [L_1, L_2, MP].

Thus, by [L_1, L_2, MP] Deduction Theorem 1, [L_1, L_2, L_4, MP]: A↔B ⊢ (A→C)→(B→C).

In a similar way, we can prove that

$$[L_1, L_2, L_3, MP]: A \leftrightarrow B \vdash (B \rightarrow C) \rightarrow (A \rightarrow C).$$

Now, by Theorem 2.2.1(a), we obtain (a).

Q.E.D.

Exercise 3.4.1. Prove (b, c, d) of Replacement Lemma 1.

Exercise 3.4.2. Prove (e, f , g) of Replacement Lemma 1.

This completes our proof of the Replacement Lemma 1.

Replacement Theorem 1. Let us consider three formulas: B, B', C, where B is a sub-formula of C, and o(B) is a **propositional** occurrence of B in C (i.e. no quantifiers stand over o(B)). Let us denote by C' the formula obtained from C by replacing o(B) by B' . Then, in the minimal logic,

$$[L_1\text{-}L_9, MP]: B \leftrightarrow B' \vdash C \leftrightarrow C'.$$

Proof. Induction by the "depth" of the propositional occurrence o(B).

Induction base: depth = 0. Then C is B, and C' is B'. The conclusion is obvious.

Induction step. If C is not B, then one of the following holds:

a) C is $F \rightarrow G$, and o(B) is in F.

b) C is $F \rightarrow G$, and o(B) is in G.

c) C is $F \wedge G$, and o(B) is in F.

d) C is $F \wedge G$, and o(B) is in G.

e) C is $F \vee G$, and o(B) is in F.

f) C is $F \vee G$, and o(B) is in G.

g) C is $\neg F$, and o(B) is in F.

Case (a). By induction assumption, $[L_1\text{-}L_9, MP]: B \leftrightarrow B' \vdash F \leftrightarrow F'$. By Replacement Lemma 1(a), $[L_1\text{-}L_9, MP]: F \leftrightarrow F' \vdash (F \rightarrow G) \leftrightarrow (F' \rightarrow G)$. Thus,

$$[L_1\text{-}L_9, MP]: B \leftrightarrow B' \vdash C \leftrightarrow C'.$$

Exercise 3.4.3. Repeat the above argument for the remaining cases (b, c, d, e, f, g).

Q.E.D.

Now, we can use the replacement argument mentioned at the beginning of this section – at least, for propositional occurrences of equivalent sub-formulas.

Replacement Lemma 2. In the minimal logic, [L_1-L_9, L_{12}-L_{15}, MP, Gen]:

(a) $B \leftrightarrow C \vdash \forall xB \leftrightarrow \forall xC$ [L_1-L_5, L_{12}, L_{14}, MP, Gen]

(b) $B \leftrightarrow C \vdash \exists xB \leftrightarrow \exists xC$ [L_1-L_5, L_{12}-L_{15}, MP, Gen]

Exercise 3.4.4. Prove Replacement Lemma 2.

Replacement Theorem 2. Let us consider three formulas: B, B', C, where B is a sub-formula of C, and o(B) is **any** occurrence of B in C. Let us denote by C' the formula obtained from C by replacing o(B) by B' . Then, in the minimal logic,

$$[L_1\text{-}L_9, L_{12}\text{-}L_{15}, \text{MP, Gen}]: B \leftrightarrow B' \vdash C \leftrightarrow C'.$$

Proof. Induction by the "depth" of the occurrence o(B).

Induction base: depth = 0. Then C is B, and C' is B'. The conclusion is obvious.

Induction step. If C is not B, then one of the following holds:

a)-g) – as in the proof of Replacement Theorem 1.

h) C is $\forall xF$, and o(B) is in F.

i) C is $\exists xF$, and o(B) is in F.

Case (h). By induction assumption, [L_1-L_9, L_{12}-L_{15}, MP, Gen]: $B \leftrightarrow B' \vdash F \leftrightarrow F'$. By Replacement Lemma 2(a), [L_1-L_9, L_{12}-L_{15}, MP, Gen]: $F \leftrightarrow F' \vdash \forall xF \leftrightarrow \forall xF'$. Thus, [$L_1$-$L_9$, L_{12}-L_{15}, MP, Gen]: $B \leftrightarrow B' \vdash C \leftrightarrow C'$.

Case (i). By induction assumption, [L_1-L_9, L_{12}-L_{15}, MP, Gen]: $B \leftrightarrow B' \vdash F \leftrightarrow F'$. By Replacement Lemma 2(b), [L_1-L_9, L_{12}-L_{15}, MP, Gen]: $F \leftrightarrow F' \vdash \exists xF \leftrightarrow \exists xF'$. Thus, [$L_1$-$L_9$, L_{12}-L_{15}, MP, Gen]: $B \leftrightarrow B' \vdash C \leftrightarrow C'$.

Q.E.D.

Now (only now!) , we may use in our proofs the replacement argument mentioned at the beginning of this section. And now, for any equivalent sub-formulas!

Finally, let us prove that the meaning of a formula does not depend on the names of bound variables used in it. Intuitively, it "must be so", but now we can prove this intuition as a meta-theorem.

Replacement Lemma 3. If the formula B does not contain the variable y, then (in the minimal logic):

a) $[L_5, L_{12}, L_{14}, MP, Gen]$: $\vdash \forall xB(x) \leftrightarrow \forall yB(y)$

b) $[L_5, L_{13}, L_{15}, MP, Gen]$: $\vdash \exists xB(x) \leftrightarrow \exists yB(y)$.

First, let us prove $[L_{12}, L_{14}, MP, Gen]$: $\vdash \forall xB(x) \rightarrow \forall yB(y)$.

(1)	$\vdash \forall xB(x) \rightarrow B(y)$	Axiom L_{12}: $\forall xF(x) \rightarrow F(t)$. B(x) does not contain y, hence, B(x/y) is an admissible substitution.
(2)	$\vdash \forall y(\forall xB(x) \rightarrow B(y))$	By Gen.
(3)	$\vdash \forall y(\forall xB(x) \rightarrow B(y)) \rightarrow (\forall xB(x) \rightarrow \forall yB(y))$	Axiom L_{14}: $\forall x(G \rightarrow F(x)) \rightarrow (G \rightarrow \forall xF(x))$. $\forall xB(x)$ does not contain y.
(4)	$\vdash \forall xB(x) \rightarrow \forall yB(y)$	By MP.

Now, let us prove $[L_{12}, L_{14}, MP, Gen(x)]$: $\vdash \forall yB(y) \rightarrow \forall xB(x)$.

(1)	$\vdash \forall yB(y) \rightarrow B(x)$	Axiom L_{12}: $\forall xF(x) \rightarrow F(t)$. B(x) does not contain y, hence, B(y) contains only free occurrences of y, i.e. B(y/x) is an admissible substitution.
(2)	$\vdash \forall x(\forall yB(y) \rightarrow B(x))$	By Gen.
(3)	$\vdash \forall x(\forall yB(y) \rightarrow B(x)) \rightarrow (\forall yB(y) \rightarrow \forall xB(x))$	Axiom L_{14}: $\forall x(G \rightarrow F(x)) \rightarrow (G \rightarrow \forall xF(x))$. $\forall yB(y)$ does not contain x as a free variable.
(4)	$\vdash \forall yB(y) \rightarrow \forall xB(x)$	By MP.

Now, by Theorem 2.2.1(a), we obtain (a).

To prove (b), first, let us prove $[L_{13}, L_{15}, MP, Gen(y)]$: $\vdash \exists yB(y) \rightarrow \exists xB(x)$.

(1)	$\vdash B(y) \rightarrow \exists xB(x)$	Axiom L_{13}: $F(t) \rightarrow \exists xF(t)$. B(x) does not contain y, hence, B(x/y) is an admissible substitution.

(2)	$\vdash\forall y(B(y)\rightarrow\exists xB(x))$	By Gen.
(3)	$\vdash\forall y(B(y)\rightarrow\exists xB(x))\rightarrow(\exists yB(y)\rightarrow\exists xB(x))$	Axiom L_{15}: $\forall x(F(x)\rightarrow G)\rightarrow(\exists xF(x)\rightarrow G)$. $\exists xB(x)$ does not contain y.
(4)	$\vdash\exists yB(y)\rightarrow\exists xB(x)$	By MP.

Now, let us prove $[L_{13}, L_{15}, MP, Gen(x)]$: $\vdash\exists xB(x)\rightarrow\exists yB(y)$.

(1)	$\vdash B(x)\rightarrow\exists yB(y)$	Axiom L_{13}: $F(t)\rightarrow\exists xF(x)$. B(x) does not contain y, hence, B(y) contains only free occurrences of y, i.e. B(y/x) is an admissible substitution.
(2)	$\vdash\forall x(B(x)\rightarrow\exists yB(y))$	By Gen.
(3)	$\vdash\forall x(B(x)\rightarrow\exists yB(y))\rightarrow(\exists xB(x)\rightarrow\exists yB(y))$	Axiom L_{15}: $\forall x(F(x)\rightarrow G)\rightarrow(\exists xF(x)\rightarrow G)$. $\exists yB(y)$ does not contain x as a free variable.
(4)	$\vdash\exists xB(x)\rightarrow\exists yB(y)$	By MP.

Now, by Theorem 2.2.1(a), we obtain (b).

Q.E.D.

Replacement Theorem 3. Let *y* be a variable that does not occur in a formula F, containing an occurrence of a quantifier $\forall x$ (or $\exists x$). Let us replace by *y* all occurrences of the variable *x* bound by this particular quantifier occurrence. Let us denote the resulting formula by F'. Then, in the minimal logic,

$$[L_1\text{-}L_9, L_{12}\text{-}L_{15}, MP, Gen]: \vdash F\leftrightarrow F'.$$

Proof. Thus, the formula F contains a sub-formula $\forall xB(x)$ (or $\exists xB(x)$), and we wish to replace it by $\forall y(B(y)$ (or $\exists yB(y)$), where y does not occur in F. By Replacement Lemma 3, in the minimal logic, $\vdash\forall xB(x)\leftrightarrow\forall yB(y)$, and $\vdash\exists xB(x)\leftrightarrow\exists yB(y)$. Hence, by Replacement Lemma 2, in the minimal logic, $\vdash F\leftrightarrow F'$. Q.E.D.

Now let us repeat our example. We know that (in the classical logic): $\vdash (A\rightarrow B)\leftrightarrow\neg A\vee B$. Hence, the formula $(X\rightarrow Y)\rightarrow Z$ **is** equivalent to $\neg(X\rightarrow Y)\vee Z$, and to $\neg(\neg X\vee Y)\vee Z$. We know also that $\neg(A\vee B)\leftrightarrow\neg A\wedge\neg B$, hence, we can continue: $(X\rightarrow Y)\rightarrow Z$ **is** equivalent to $(\neg\neg X\wedge\neg Y)\vee Z$, and to $(X\wedge\neg Y)\vee Z$ (since $\vdash\neg\neg A\leftrightarrow A$).

Now, in our logic, we can use this very natural kind of mathematical argument.

3.5. Constructive Embedding

Glivenko's Theorem (see Section 2.7) provides a simple "constructive embedding" for the classical propositional logic: any classically provable formula can be "proved" in the constructive logic, if you put two negations before it. This theorem does not hold for the predicate logic. For example (see Section 3.2),

II-5. In the classical logic, [L_1-L_{11}, L_{13}, L_{14}, MP, Gen]: $\vdash \neg\forall x \neg B \rightarrow \exists x B$.

The double negation of this formula, i.e. the formula $\neg\neg(\neg\forall x \neg B \rightarrow \exists x B)$ cannot be proved in the constructive predicate logic (see Section 4.5). Thus, instead of the simple operation $\neg\neg F$, we must search for a more complicated embedding operation.

However,

Exercise 3.5.0 (optional, for smart students). Verify that a formula F is provable in the classical predicate logic, if and only if $\neg\neg F$ is provable in the constructive predicate logic plus the following axiom schema: $\forall x \neg\neg B \rightarrow \neg\neg \forall x B$ (the so-called *Double Negation Shift s*chema, see Intuitionistic Logic by Joan Moschovakis in Stanford Encyclopedia of Philosophy.

The first embedding operation was introduced by Andrey Nikolaevich Kolmogorov (1903-1987) in

A.N.Kolmogorov. On the principle tertium non datur. *Matem. sbornik*, 1925, vol.32, pp.646-667 (in Russian).

A quote from A Short Biography of A.N. Kolmogorov by Paul M.B. Vitanyi follows:

"K. got interested in mathematical logic, and in 1925 published a paper in Mathematicheskii Sbornik on the law of the excluded middle, which has been a continuous source for later work in mathematical logic. This was the first Soviet publication on mathematical logic containing (very substantial) new results, and the first systematic research in the world on intuitionistic logic. K. anticipated to a large extent A. Heyting 's formalization of intuitionistic reasoning, and made a more definite correlation between classical and intuitionistic mathematics. K. defined an operation for `embedding' one logical theory in another. Using this – historically the first such operation, now called the `Kolmogorov operation' – to embed classical logic in intuitionistic logic, he proved that application of the law of the excluded middle in itself cannot

lead to a contradiction. In 1932 K. published a second paper on intuitionistic logic, in which for the first time a semantics was proposed (for this logic), free from the philosophical aims of intuitionism. This paper made it possible to treat intuitionistic logic as constructive logic."

See also Kolmogorov Centennial.

We will investigate the following version of an embedding operation: to obtain O(F), in a formula F, put two negations before: a) every atomic formula, b) every disjunction, c) every existential quantifier. More precisely, let us define the following embedding operation O (you may wish to compare it with some other versions possessing similar properties):

Operation O Detlovs [1964]	**Operation K** Kolmogorov [1925]	**Operation O'** Gödel [1933], see Kleene [1952]	**Operation O^o** Gentzen [1936], see Kleene [1952]
If F is an atomic formula, then O(F) is ¬¬F.	K(F) is ¬¬F.	O'(F) is F.	O^o(F) is F.
O(F→G) is O(F)→O(G).	¬¬(K(F)→K(G))	$\neg(O'(F)\wedge\neg O'(G))$	O^o(F)→O^o(G)
$O(F\wedge G)$ is $O(F)\wedge O(G)$.	$\neg\neg(K(F)\wedge K(G))$	$O'(F)\wedge O'(G)$	$O^o(F)\wedge O^o(G)$
$O(F\vee G)$ is $\neg\neg(O(F)\vee O(G))$	$\neg\neg(K(F)\vee K(G))$	$\neg(\neg O'(F)\wedge\neg O'(G))$	$\neg(\neg Oo(F)\wedge\neg Oo(G))$
O(¬F) is ¬O(F).	¬¬¬K(F), or ¬K(F)*	¬O'(F)	¬O^o(F)
O(∀xF) is ∀xO(F).	¬¬∀xK(F)	∀xO'(F)	∀xO^o(F)
O(∃xF) is ¬¬∃xO(F).	¬¬∃xK(F)	¬∀x¬O'(F)	¬∀x¬O^o(F)

(*) By Theorem 2.4.5, [L_1-L_9, MP]: ⊢¬¬¬K(F)↔¬K(F).

For example, let us take the above formula ¬∀x¬B→∃xB. If B is an atomic formula, then

O(¬∀x¬B→∃xB) is ¬∀x¬¬¬B→¬¬∃x¬¬B, i.e. ¬∀x¬B→¬¬∃x¬¬B

The latter formula is constructively provable (see Section 3.2, Group II).

Lemma 3.5.1. For any formula F, in the classical logic, ⊢F↔O(F).

Proof. By induction. Let us remind Theorem 2.6.1: [L_1-L_{11}, MP] ⊢¬¬A ↔ A.

1. Induction base: F is an atomic formula. Then O(F) is ¬¬F. By Theorem

2.6.1, [L_1-L_{11}, MP] $\vdash \neg\neg F \leftrightarrow F$, hence, in the classical logic, $\vdash O(F) \leftrightarrow F$.

2. Induction step.

Case 2a: F is BvC. Then O(F) is $\neg\neg(O(B) \vee O(C))$.

(1)	$O(B) \leftrightarrow B$	Induction assumption.
(2)	$O(C) \leftrightarrow C$	Induction assumption.
(3)	$B \vee C \leftrightarrow O(B) \vee C$	From (1), by Replacement Theorem 1.
(4)	$O(B) \vee C \leftrightarrow O(B) \vee O(C)$	From (2), by Replacement Theorem 1.
(5)	$O(B) \vee O(C) \leftrightarrow \neg\neg(O(B) \vee O(C))$	Theorem 2.6.1: [L_1-L_{11}, MP] $\vdash \neg\neg A \leftrightarrow A$.
(6)	$B \vee C \leftrightarrow \neg\neg(O(B) \vee O(C))$, i.e. $F \leftrightarrow O(F)$	By transitivity of implication.

Case 2b: F is $\exists x B$. Then O(F) is $\neg\neg\exists x O(B)$.

(1)	$O(B) \leftrightarrow B$	Induction assumption.
(2)	$\exists x B \leftrightarrow \exists x O(B)$	From (1), by Replacement Theorem 2.
(3)	$\exists x O(B) \leftrightarrow \neg\neg\exists x O(B)$	Theorem 2.6.1: [L_1-L_{11}, MP] $\vdash \neg\neg A \leftrightarrow A$.
(4)	$\exists x B \leftrightarrow \neg\neg\exists x O(B)$, i.e. $F \leftrightarrow O(F)$	By transitivity of implication.

Case 2c: F is $B \rightarrow C$.

Case 2d: F is B&C.

Case 2e: F is $\neg B$.

Case 2f: F is $\forall x B$.

Exercise 3.5.1.Prove (c, d, e, f).

Q.E.D.

Still, the key feature of the formulas having the form O(F) is given in

Lemma 3.5.2. For any formula F, there is a proof of

$$[L_1\text{-}L_9, L_{12}, L_{14}, MP,Gen]: \vdash \neg\neg O(F) \leftrightarrow O(F).$$

I.e., in the minimal logic, we may drop the double negation before O(F) (before an arbitrary formula, we can do this only in the classical logic).

Note. In some other textbooks, if ¬¬G↔G can be proved in the constructive logic, then G is called a **stable formula**. Thus, the embedding O(F) is a stable formula for any F.

Proof. By Theorem 2.4.4, [L_1, L_2, L_9, MP]: ⊢A→¬¬A. Thus, it remains to prove ⊢¬¬O(F)→O(F). Let us proceed by induction.

1. Induction base: F is an atomic formula. Then O(F) is ¬¬F, and ¬¬O(F)→O(F) is ¬¬¬¬F→¬¬F. Let us remind Theorem 2.4.5: [L_1-L_9, MP] ⊢ ¬¬¬A↔¬A. Hence, by taking A = ¬F:

[L_1-L_9, MP] ⊢¬¬¬¬F→¬¬F, i.e. [L_1-L_9, MP] ⊢¬¬O(F)→O(F).

2. Induction step.

Case 2a: F is $B \vee C$, or ∃xB, or ¬B. Then O(F) is $\neg\neg(O(B) \vee O(C))$, or ¬¬∃xO(B), or ¬O(B). Hence, ¬¬O(F)→O(F) is ¬¬¬G→¬G, where G is $\neg(O(B) \vee O(C))$, or ¬∃xO(B), or O(B). Let us remind Theorem 2.4.5: [L_1-L_9, MP] ⊢¬¬¬A↔¬A. Hence,

[L_1-L_9, MP] ⊢¬¬¬G→¬G, i.e. [L_1-L_9, MP] ⊢¬¬O(F)→O(F).

Case 2b: F is B→C. Then O(F) is O(B)→O(C). By induction assumption,

[L_1, L_2, L_{12}, L_{14}, MP, Gen]: ⊢¬¬O(B)→O(B), and ⊢¬¬O(C)→O(C).

(1)	¬¬O(C)→O(C)	Induction assumption.
(2)	¬¬(O(B)→O(C))	¬¬O(F) – hypothesis.
(3)	¬¬O(B)→¬¬O(C)	By Theorem 2.4.7(b): [L_1-L_9, MP] ⊢ ¬¬(A→B)→(¬¬A→¬¬B).
(4)	O(B)→¬¬O(B)	By Theorem 2.4.4, [L_1, L_2, L_9, MP]: ⊢A→¬¬A.
(5)	O(B)→O(C), i.e. O(F)	From (4), (3) and (1), by transitivity of implication [L_1, L_2, MP].

Hence, since Gen is not applied here at all, by Deduction Theorem 1 [L_1, L_2, MP] we obtain that [L_1-L_9, L_{12}, L_{14}, MP, Gen] ⊢¬¬O(F)→O(F).

Case 2c: F is $B \wedge C$. Then O(F) is $O(B) \wedge O(C)$. By induction assumption,

[L_1, L_2, L_{12}, L_{14}, MP, Gen]: $\vdash \neg\neg O(B) \rightarrow O(B)$, and $\vdash \neg\neg O(C) \rightarrow O(C)$.

(1)	$\neg\neg(O(B) \wedge O(C))$	$\neg\neg$O(F) – hypothesis.
(2)	$\neg\neg O(B) \wedge \neg\neg O(C)$	From (1), by Theorem 2.4.8(a), [L_1-L_9, MP] $\vdash \neg\neg(A \wedge B) \leftrightarrow (\neg\neg A \wedge \neg\neg B)$.
(3)	$\neg\neg$O(B)	From (2), by Axiom L_3.
(4)	$\neg\neg$O(C)	From (2), by Axiom L_4.
(5)	O(B)	From (3), by induction assumption.
(6)	O(C)	From (4), by induction assumption.
(7)	$O(B) \wedge O(C)$, i.e. O(F)	From (5) and (6), by Axiom L_5.

Hence, since Gen is not applied here at all, by Deduction Theorem 1 [L_1, L_2, MP] we obtain that [L_1-L_9, L_{12}, L_{14}, MP, Gen] $\vdash \neg\neg$O(F)$\rightarrow$O(F).

Case 2d: F is $\forall$xB. Then O(F) is $\forall$xO(B). By induction assumption, [L_1-L_9, L_{12}, L_{14}, MP, Gen]: $\vdash \neg\neg$O(B)$\rightarrow$O(B). We must prove that $\vdash \neg\neg\forall$xO(B)$\rightarrow\forall$xO(B).

(1)	$\neg\neg\forall$xO(B)$\rightarrow\forall$x$\neg\neg$O(B)	Section 3.2, I-2: [L_1-L_9, L_{12}, L_{14}, MP, Gen] $\vdash \neg\neg\forall$xB$\rightarrow\forall$x$\neg\neg$B
(2)	$\vdash \neg\neg$O(B)$\rightarrow$O(B)	Induction assumption
(3)	$\vdash \forall$x($\neg\neg$O(B)$\rightarrow$O(B))	By Gen.
(4)	$\vdash \forall$x$\neg\neg$O(B)$\rightarrow\forall$xO(B)	From (3), by Theorem 3.1.1(a), [L_1, L_2, L_{12}, L_{14}, MP, Gen] $\vdash \forall$x(B$\rightarrow$C)$\rightarrow$($\forall$xB$\rightarrow\forall$xC).
(5)	$\vdash \neg\neg\forall$xO(B)$\rightarrow\forall$xO(B)	From (1) and (4), by transitivity of implication [L_1, L_2, MP].

Q.E.D.

Lemma 3.5.3. If F is one of the (classical) axioms L_1-L_{11}, L_{12}-L_{15}, then, in the constructive logic, [L_1-L_{10}, L_{12}-L_{15}, MP, Gen]: $\vdash$O(F).

Note. The axiom L_{10} will be used in the proof of Lemma 3.5.3 only once – to

prove that $O(L_{10})$ is provable in the constructive logic. But, of course, $O(L_{10})$ cannot be proved in the minimal logic, hence, in the Lemma 3.5.3, the constructive logic cannot be replaced by the minimal one.

Proof.

Case 1. F (as an axiom schema) does not contain disjunctions and existential quantifiers, i.e. if F is L_1, L_2, L_3, L_4, L_5, L_9, L_{10}, L_{12}, or L_{14}., then O(F) is an instance of the same axiom as F, i.e. [F]: $\vdash$O(F). For example, if F is L_1, i.e. B→(C→B), then O(F) is O(B)→(O(C)→O(B)), i.e. O(F) is an instance of the same axiom L_1.

Case 2a. F is L_6: $B \to B \vee C$. Then O(F) is $O(B) \to \neg\neg(O(B) \vee O(C))$, and [[L_1, L_2, L_6, L_9, MP] $\vdash$O(F). Indeed:

(1)	$O(B) \to O(B) \vee O(C)$	Axiom L_6.
(2)	$O(B) \vee O(C) \to \neg\neg(O(B) \vee O(C))$	By Theorem 2.4.4, [L_1, L_2, L_9, MP]: $\vdash$A→¬¬A.
(3)	$O(B) \to \neg\neg(O(B) \vee O(C))$	By transitivity of implication [L_1, L_2, MP].

Case 2b. F is L_7: $C \to B \vee C$. Then O(F) is $O(C) \to \neg\neg(O(B) \vee O(C))$, and [[L_1, L_2, L_7, L_9, MP] $\vdash$O(F). Proof is similar to Case 2a.

Case 2c. F is L_8: $(B \to D) \to ((C \to D) \to (B \vee C \to D))$. Then O(F) is

$$(O(B) \to O(D)) \to ((O(C) \to O(D)) \to (\neg\neg(O(B) \vee O(C)) \to O(D)))$$.

(1)	¬¬O(D)→O(D)	By Lemma 3.5.2, [L_1-L_9, L_{12}, L_{14}, MP,Gen]: $\vdash$¬¬O(F)→O(F).
(2)	O(B)→O(D)	Hypothesis.
(3)	(O(C)→O(D)	Hypothesis.
(4)	$\neg\neg(O(B) \vee O(C))$	Hypothesis.
(5)	$(O(B) \to O(D)) \to ((O(C) \to O(D)) \to (O(B) \vee O(C) \to O(D)))$. Axiom L_8.	
(6)	$O(B) v O(C) \to O(D)$	By MP.

(7)	$\neg\neg(O(B)\vee O(C))\rightarrow\neg\neg O(D)$	From (6), by Theorem 2.4.7(a), $[L_1\text{-}L_9, MP] \vdash$ $(A\rightarrow B)\rightarrow(\neg\neg A\rightarrow\neg\neg B)$
(8)	$\neg\neg$O(D)	By MP.
(9)	O(D)	From (1), by MP.

Hence, since Gen is not applied after hypotheses appear in the proof, by Deduction Theorem 2A [L_1, L_2, L_{14}, MP, Gen] we obtain that [L_1-L_9, L_{12}, L_{14}, MP,Gen] $\vdash$O(F).

Case 2d. F is L_{11}: B$\vee\neg$B. Then O(F) is $\neg\neg(O(B)\vee\neg O(B))$. Let us remind Theorem 2.4.6(b): [L_1-L_9, MP] $\vdash$ $\neg\neg(A\vee\neg A)$. Hence, [L_1-L_9, MP] $\vdash$ O(F).

Case 2e. F is L_{13}: F(t)$\rightarrow\exists$xF(x). Then O(F) is O(F(t))$\rightarrow\neg\neg\exists$xO(F(x))), and [[$L_1$, L_2, L_9, L_{13}, MP] $\vdash$O(F). Indeed:

(1)	O(F(t))$\rightarrow\exists$xO(F(x))	Axiom L_{13}.
(2)	$\exists$xO(F(x))$\rightarrow\neg\neg\exists$xO(F(x))	By Theorem 2.4.4, [L_1, L_2, L_9, MP]: $\vdash$ A$\rightarrow\neg\neg$A.
(3)	$\vdash$O(F(t))$\rightarrow\neg\neg\exists$xO(F(x))	By transitivity of implication [L_1, L_2, MP].

Case 2f. F is L_{15}: $\forall$x(F(x)$\rightarrow$G)$\rightarrow$($\exists$xF(x)$\rightarrow$G). Then O(F) is

$\forall$x(O(F(x))$\rightarrow$O(G))$\rightarrow$($\neg\neg\exists$xO(F(x))$\rightarrow$O(G)).

(1)	$\neg\neg$O(G)$\rightarrow$O(G)	By Lemma 3.5.2, [L_1-L_9, L_{12}, L_{14}, MP,Gen]: $\vdash\neg\neg$O(F)$\rightarrow$O(F).
(2)	$\forall$x(O(F(x))$\rightarrow$O(G))	Hypothesis.
(3)	$\neg\neg\exists$xO(F(x))	Hypothesis.
(4)	$\forall$x(O(F(x))$\rightarrow$O(G))$\rightarrow$ ($\exists$xO(F(x))$\rightarrow$O(G)). Axiom L_{15}: $\forall$x(F(x)$\rightarrow$G)$\rightarrow$($\exists$xF(x)$\rightarrow$G).	
(5)	$\exists$xO(F(x))$\rightarrow$O(G)	By MP.

(6)	$\neg\neg\exists x O(F(x))\rightarrow\neg\neg O(G)$	From (4), by Theorem 2.4.7(a), $[L_1\text{-}L_9, MP] \vdash (A\rightarrow B)\rightarrow(\neg\neg A\rightarrow\neg\neg B)$
(7)	$\neg\neg O(G)$	By MP.
(8)	$O(G)$	From (1), by MP.

Hence, since Gen is not applied after hypotheses appear in the proof, by Deduction Theorem 2A $[L_1, L_2, L_{14}, MP, Gen]$ we obtain that $[L_1\text{-}L_9, L_{12}, L_{14}, L_{15}, MP, Gen] \vdash O(F)$.

Q.E.D.

Theorem 3.5.4. In the classical logic,

$$[L_1\text{-}L_{11}, L_{12}\text{-}L_{15}, MP, Gen]: B_1, B_2, ..., B_n \vdash C,$$

if and only if, in the constructive logic,

$$[L_1\text{-}L_{10}, L_{12}\text{-}L_{15}, MP, Gen]: O(B_1), O(B_2), ..., O(B_n) \vdash O(C).$$

In particular, a formula F is provable in the classical logic, if and only if the formula O(F) is provable in the constructive logic.

Proof.

1. Let $[L_1\text{-}L_{11}, L_{12}\text{-}L_{15}, MP, Gen]: B_1, B_2, ..., B_n \vdash C$. Induction by the length of the shortest proof.

Induction base. If C is an axiom, then, by Lemma 3.5.3, in the constructive logic, $\vdash O(C)$. If C is B_i, then $O(B_i) \vdash O(C)$ in any logic.

Induction step.

If C is derived by MP from B and $B\rightarrow C$, then, by induction assumption, in the constructive logic: $O(B_1), O(B_2), ..., O(B_n) \vdash O(B)$, and $O(B_1), O(B_2), ..., O(B_n) \vdash O(B\rightarrow C)$. Let us merge these two proofs. Since $O(B\rightarrow C)$ is $O(B)\rightarrow O(C)$, then, by MP, in the constructive logic: $O(B_1), O(B_2), ..., O(B_n) \vdash O(C)$.

If C is $\forall x B(x)$, and is derived by Gen from B(x), then, by induction assumption, in the constructive logic: $O(B_1), O(B_2), ..., O(B_n) \vdash O(B(x))$. Hence, by Gen, in the constructive logic: $O(B_1), O(B_2), ..., O(B_n) \vdash \forall x O(B(x))$, i.e. $O(B_1), O(B_2), ..., O(B_n) \vdash O(F)$.

Q.E.D.

2. Let in the constructive logic: $\vdash O(B_1), O(B_2), ..., O(B_n) \vdash O(C)$. By Lemma 3.5.1, in the classical logic, $\vdash B_i \rightarrow O(B_i)$ for all i, and $\vdash O(C) \rightarrow C$. Hence, in the classical logic, $B_1, B_2, ..., B_n \vdash C$.

Q.E.D.

Corollary 3.5.5. If, in the classical logic, $B_1, B_2, ..., B_n \vdash C \wedge \neg C$, then, in the constructive logic, $O(B_1), O(B_2), ..., O(B_n) \vdash O(C) \wedge \neg O(C)$. I.e., if the postulates $B_1, B_2, ..., B_n$ are inconsistent in the classical logic, then the postulates $O(B_1), O(B_2), ..., O(B_n)$ are inconsistent in the constructive logic. Or: if the postulates $O(B_1), O(B_2), ..., O(B_n)$ are consistent in the constructive logic, then the postulates $B_1, B_2, ..., B_n$ are consistent in the classical logic.

Corollary 3.5.6. If, for some predicate language, the classical logic is inconsistent, then so is the constructive logic. Or: **if, for some predicate language, the constructive logic is consistent, then so is the classical logic** (Gödel [1933], Gentzen [1936]).

Warning! Corollary 3.5.6 does not extend immediately to first order **theories**, having their own specific non-logical axioms. It must be verified separately for each theory! For example,

Exercise 3.5.2 (optional, for smart students). Verify that, if the constructive first order arithmetic is consistent, then so is the classical first order arithmetic (Gödel [1933], Gentzen [1936]). (Hint: verify that, a) atomic formulas of arithmetic are stable – this is the hard part of the proof, b) if F is an axiom of arithmetic, then so is O(F).)

Thus, the **non-constructivity does not add contradictions (at least) to arithmetic**. If it would, then we could derive "constructive" arithmetical contradictions as well.

K. Gödel. Zur intuitionistischen Arithmetik und Zahlentheorie. *Ergebnisse eines mathematischen Kolloquiums*, 1933, Vol. 4, pp. 34-38.

Gerhard Gentzen. Die Widerspruchsfreiheit der reinen Zahlentheorie. *Mathematische Annalen*, 1936, Vol. 112, pp. 493-565.

About constructive embedding operations as a general notion see

Nikolai A.Shanin. Embedding the classical logical-arithmetical calculus into the constructive logical-arithmetical calculus. *Dokladi AN SSSR*, 1954, vol. 94, N2, pp.193-196 (in Russian).

4. Completeness Theorems (Model Theory)

4.1. Interpretations and Models

In principle, to do the so-called pure mathematics, i.e. simply to prove theorems, one needs only **"syntax" –** axioms and rules of inference. And, for **computers**, this is the only way of doing mathematics!

But how about **"semantics" –** about our intended "vision" from which we started designing of our predicate language and formulating axioms? First of all, we must understand that there is no way of formulating (in the predicate language and axioms) of *all the features* of our domain of interest.

For example, what information can be derived about the person *Britney* from our "theory for people" of Section 1.3? All we can derive will be formulated in terms of the following predicates:

Male(x) − means "x is a male";
Female(x) − means "x is a female";
Mother(x, y) − means "x is mother of y";
Father(x, y) − means "x is father of y";
Married(x, y) − means "x and y are married";
x=y.

Thus, there is no way of obtaining from such a theory of any information that can't be formulated in these terms, for example, about the age, colour of eyes etc.

And thus, by communicating our predicate language and our axioms (in this language!) to a computer, we have communicated only a small part of all the features of our domain of interest.

Hence, neither our predicate language, nor our axioms can specify our initial "vision" completely. And, if so – in principle, one can imagine *many different* "visions" behind our language and axioms!

It may seem that if, instead of our "people's domain" that is very rich in details, we will consider, for example, natural numbers, then the situation will become better, and we will be able to describe our informal "vision" unambiguously?

For example, let us considered the language of first order arithmetic (language primitives: x, y, ..., 0, 1 +, *, =), and the following non-usual "Boolean vision" B behind it.

a) As the domain of B (the "target" set of objects), instead of the set of all natural numbers, let us consider the set of "Boolean values" $D_B = \{t, f\}$. Thus, now, the variables x, y, ... can take only values *t, f*.

b) The interpretation mapping int_B assigns: to the object constant 1 – the object *t*, to the object constant 0 – the object *t*, thus: $int_B(0)=f$, $int_B(1)=t$.

c) To the function constant "+" we assign the well-known *disjunction* truth table: $int_B(+) = '\vee'$, to the function constant "*" – the well-known *conjunction* truth table: $int_B(*) = '\wedge'$.

d) To the predicate constant "=" – the equality predicate for the set D_B, i.e. $int_B(=) = \{(t, t) , (f, f)\}$.

Is this vision "worse" than the usual one involving “real” natural numbers? It seems, it is worse, because the following axiom of arithmetic:

$$x+1=y+1 \rightarrow x=y$$

is false under this vision. Indeed, set x=0 and y=1: $0+1=1+1 \rightarrow 0=1$. Here, the premise is true: $0+1=f \vee t=t=1, 1+1=t \vee t=t=1$, but the conclusion is not: 0=1 means *f=t*.

On the other hand, the following theorem of Boolean algebra:

$$x+x=x$$

is true under the above "Boolean vision" ($t \vee t=t , f \vee f=f$), but it is false under the usual vision involving natural numbers.

Thus, if two theories share the same language (as do Boolean algebra and first order arithmetic), then the "validity" of a vision may depend on the formulas (axioms and theorems) that we expect to be true. If we consider only the **language**, then many different and even strange interpretations-visions will be possible. But if we consider a **theory** (i.e. a language plus some specific axioms), then only a part of the interpretations-visions will be valid – only those ones, under which the specific axioms of our theory will be true. Such interpretations-visions are called **models** of the theory.

Another example: in our "language for people" we used names of people (*JBritney, John, Paris, Peter, ...*) as object constants and the following predicate constants:

Male(x) − means "x is a male";
Female(x) − means "x is a female";
Mother(x, y) − means "x is mother of y";
Father(x, y) − means "x is father of y";
Married(x, y) − means "x and y are married";
x=y − means "x an y are the same person".

Now, let us fix the list of 4 names: *Britney, John, Paris, Peter*, and let us consider the following interpretation J of the language - "**small world**":

a) The domain – and the range of variables – is D_J = {br, jo, pa, pe} (4 character strings).

b) int_J(Britney)=br, int_J(John)=jo, int_J(Paris)=pa, int_J(Peter)=pe.

c) int_J(Male) = {jo, pe}; int_J(Female) = {br, pa}.

d) int_J(Mother) = {(pa, br), (pa, jo)}; int_J(Father) = {(pe, jo), (pe, br)}.

e) int_J(Married) = {(pa, pe), (pe, pa)}.

f) int_J(=) = {(br, br), (jo, jo), (pa, pa), (pe, pe)}.

An alternative way of specifying interpetations of predicate constants are truth tables, for example:

x	**Male(x)**	**Female(x)**
br	false	true
pa	false	true
jo	true	false
pe	true	false

x	**y**	**Father(x, y)**	**Mother(x, y)**	**Married(x, y)**	**x=y**
br	br	false	false	false	true
br	pa	false	false	false	false
...	...	...	...	...	
pa	br	false	true	false	false
...	...	...	...	...	...
pe	pe	false	false	false	true

Under this interpretation ("in this small world"), it is true that, "mothers are females", and that "all fathers are married people" (under this interpretation, not in the real world!). Thus, under this interpretation, the corresponding formulas $\forall x(Mother(x) \rightarrow Female(x))$ and $\forall x(Father(x) \rightarrow \exists y\ Married(x, y))$ qualify as true.

But, under this interpretation ("in this small world"), it **not** true that "each person possess a mother". The corresponding formula $\forall x \exists y\ Mother(y, x)$ qualifys as false.

Exercise 4.1.0. Build another interpretation (a "crazy" one!) of our "4 people language", under which the following formulas are true: "some people are both male and female", "there are sex-less people", "a person may marry herself", "a person may be mother of herself".

By introducing **specific non-logical axioms**, i.e. by introducing "4 people theory" instead of

pure axiom-less “4 people language” we can disqualify your “crazy” interpretation. For example, the following axioms are false under it:

$$\forall x(Male(x) \vee Female(x)); \forall x \neg(Male(x) \wedge Female(x)).$$

Model Theory

Could the notion of “arbitrary vision” be defined precisely? For a particular predicate language and particular axioms – is there **only one** “vision” possible? Trying to answer these questions, we arrive at the so-called model theory.

Model theory is a very specific approach to investigation of formal theories. Model theory is using (up to) the full power of set theory. **In model theory, we investigate formal theories by using set theory as a meta-theory.**

Paul Bernays, in 1958: "As Bernays remarks, syntax is a branch of number theory and semantics the one of set theory." See p. 470 of

Hao Wang. EIGHTY YEARS OF FOUNDATIONAL STUDIES. *Dialectica,* Vol. 12, Issue 3-4, pp. 466-497, December 1958 (available online at Blackwell Synergy).

In Sections 4.1-4.3 we will develop **model theory for the classical logic**, and in Sections 4.4-4.5 – model theory for the constructive logic.

In the classical model theory, we will replace our vague "visions” by relatively well-defined mathematical structures – the so-called **interpretations**. As we will see, interpretations are allowed to be non-constructive.

Technically, an interpretation will be a relatively well-defined way of assigning "precise meanings" to all formulas of a predicate language. Any particular predicate language allows multiple ways of assigning "precise meanings" to its formulas – *multiple interpretations*.

Interpretation of a language – the specific part

Let L be a predicate language containing object constants c_1, ..., c_k, ... , function constants f_1, ..., f_m, ..., and predicate constants p_1, ..., p_n, ... An interpretation J of the language L consists of the following two entities:

a) A **non-empty** set D_J – the domain of interpretation (it will serve first of all as the range of object variables). (Your favorite set theory comes in here.)

b) A mapping int_J that assigns:

- to each object constant c_i – a member $int_J(c_i)$ of the domain D_J (thus, object constants "denote" particular objects in D_J),

- to each function constant f_i – a function $int_J(f_i)$ from D_J x ... x D_J into D_J (of course, $int_J(f_i)$ has the same number of arguments as f_i),

- to each predicate constant p_i – a predicate $int_J(p_i)$ on D_J, i.e. a subset of D_J x ... x D_J (of course, $int_J(p_i)$ has the same number of arguments as p_i).

Thus, in a sense, the mapping int_J assigns "meaning" to the language primitives.

The most popular example – let us consider the so-called **standard interpretation** S of first order (Peano) arithmetic PA:

a) The domain is D_S = {0, 1, 2, ...} − the set of all natural numbers "as we know it" (more precisely – as you define it in your favorite set theory).

b) The mapping int_S assigns: to the object constant 0 – the number 0, to the object constant 1 – the number 1, to the function constant "+" − the function x+y (addition of natural numbers), to the function constant "*" − the function x*y (multiplication of natural numbers), to the predicate constant "=" − the predicate x=y (equality of natural numbers).

Yet another interpretation J1 of the same language:

a) The domain is D_{J1} = {e, a, aa, aaa, ...} − the set of all strings built of the letter "a" (e is the empty string).

b) The mapping int_{J1} assigns: to the object constant 0 – the empty string e, to the object constant 1 – the string "a", to the function constant "+" − the concatenation function of strings, to the function constant "*" − y times concatenation of x, to the predicate constant "=" − the string equality predicate.

Yet another interpretation J2 (there is no way to disqualify it as a formally correct interpretation of the language):

a) The domain is D_{J2} = {*o*} – a single object *o*.

b) The mapping int_{J2} assigns: to the object constant 0 – the object *o*, to the object constant 1 – the same object *o*, to the function constant "+" − the only possible function f(o,o)=o, to the function constant "*" − the only possible function f(o,o)=o, to the predicate constant "=" − the predicate {(o, o)}.

Some time later, we will use specific non-logical **axioms** to disqualify (at least some of) such "inadequate" interpretations.

Having an interpretation J of the language L, we can define the notion of **true formulas** (more precisely − the notion of formulas that are **true under the interpretation** J).

As the first step, **terms** of the language L are interpreted as members of D_J or functions over D_J. Indeed, terms are defined as object constants, or object

variables, or their combinations by means of function constants. The term c_i is interpreted as the member $int_J(c_i)$ of D_J. The variable x_i is interpreted as the function $X_i(x_i) = x_i$. And, if $t = f_i(t_1, ..., t_q)$, then $int_J(t)$ is defined as the function obtained by substituting of functions $int_J(t_1)$, ..., $int_J(t_q)$ into the function $int_J(f_i)$.

For example (first order arithmetic), the standard interpretation of the term (1+1)+1 is the number 3, the interpretation of (x+y+1)*(x+y+1) is the function $(x+y+1)^2$.

Important − non-constructivity! Note that, for an infinite domain D_J, the interpretations of function constants may be **non-computable** functions. But, if they are all computable, then we can compute the "value" of any term t for any combination of values of variables appearing in t.

As the next step, the notion of **true atomic formulas** is defined. Of course, if a formula contains variables (as, for example, the formula x+y=1), then its "truth-value" must be defined for each combination of values of these variables. Thus, to obtain the truth-value of the formula $p_i(t_1, ..., t_q)$ for some fixed values of the variables contained in $t_1, .., t_q$, we must first "compute" the values of these terms, and then substitute these values into the predicate $int_J(p_i)$.

For example (first order arithmetic), under the standard interpretation S, the formula x+y=1 will be true, if and only if either x takes the value 0, and y takes the value 1, or x takes the value 1, and y takes the value 0. Otherwise, the formula is false.

Important − non-constructivity! Note that, for an infinite domain D_J, the interpretations of predicate constants may be **non-computable** predicates. But, if they were all computable, then we could compute the "truth value" of any atomic formula F for any combination of values of variables appearing in F.

Interpretations of languages − the standard common part

And finally, we define the notion of **true compound formulas** of the language L under the interpretation J (of course, for a fixed combination of values of their free variables):

a) Truth-values of the formulas ¬B, $B \wedge C$, $B \vee C$ and B→C must be computed from the truth-values of B and C (by using the well-known **classical** truth tables – see Section 4.2 below).

b) The formula ∀xB is true under J, if and only if B(c) is true under J for all

members c of the domain D_J.

c) The formula ∃xB is true under J, if and only if there is a member c of the domain D_J such that B(c) is true under J.

For example (first order arithmetic), the formula

$$\exists y((x=y+y)\vee(x=y+y+1))$$

says that "*x* is even or odd". Under the standard interpretation S, this formula is true for all values of its free variable *x*. Similarly, ∀x∀y(x+y=y+x) is a closed formula that is true under S.

Important − non-constructivity! It may seem that, under an interpretation, any closed formula is "either true or flase". However, note that, for an infinite domain D_J, the notion of "true formulas under J" is extremely **non-constructive**: to establish, for example, the truth-value of the formula ∀xB, or the formula ∀x∀y(x+y=y+x), we must verify the truth of B(c) for infinitely many values of c (or a+b=b+a for infinitely many values of a and b). Of, course, this verification cannot be performed on a computer. It can only (sometimes) be proved... in some other theory. The "degree of constructivity" of the formulas like as ∀x∃yC(x,y), ∀x∃y∀zD(x,y,z) etc. is even less than that...

Empty Domains?

Do you think, we should consider also **empty domains** of interpretation? According to the axiom L_{13}: (B→B)→∃x(B→B), hence, ∃x(B→B). In an empty domain, this formula would be false. Thus, to cover the empty domain, we would be forced to re-consider the axioms and/or re-consider the traditional meaning of ∃x − see (c) above. Let us concentrate on non-empty domains only.

Let us say that a formula of the language L is **always true** under the interpretation J, if and only if this formula is true for all combinations of values of its free variables.

Three Kinds of Formulas

If one explores some formula F of the language L in various interpretations, then three situations are possible:

a) F is **true in all** interpretations of the language L. Formulas of this kind are called **logically valid formulas**.

b) F is **true in some** interpretations of L, and **false − in some** other interpretations of L.

c) F is **false in all** interpretations of L (then, of course, ¬F is true in all interpretations). Formulas of this kind are called **unsatisfiable formulas.**

Formulas that are "not unsatisfiable" (i.e. formulas of kinds (a) and (b)) are called, of course, **satisfiable formulas**.

Exercise 4.1.1. Verify that: a) **F is satisfiable, if and only if ¬F is not logically valid**. b) **F is logically valid, if and only if ¬F is unsatisfiable**.

Logically Valid Formulas

Some formulas are always true under all interpretations, for example:

$$(B \to C) \wedge (C \to D) \to (B \to D) \quad ,$$

$$F(x) \to \exists x F(x) \quad ,$$

$$\forall x F(x) \to F(x) \quad ,$$

$$\forall x (F(x) \to G(x)) \to (\forall x F(x) \to \forall x G(x)) \quad ,$$

$$\forall x (F(x) \to G(x)) \to (\exists x F(x) \to \exists x G(x)) \quad ,$$

$$\forall x (G(x) \wedge H(x)) \to (\forall x G(x) \wedge \forall x H(x)) \quad ,$$

$$\exists x (G(x) \vee H(x)) \to (\exists x G(x) \vee \exists x H(x)) \quad .$$

How about the axioms L_1-L_{15}?

Such formulas are called **logically valid**. More precisely, in a predicate language L, a formula is called **logically valid**, if and only if it is true in **all interpretations** of the language L for all values of its free variables.

Thus, a logically valid formula is true independently of its "meaning" − the interpretations of constants, functions and predicates used in it. But note that here, the (classical!) **interpretations of propositional connectives and quantifiers remain fixed.**

In a sense, logically valid formulas are “content-free”: they do not give us any specific information about features of objects they are “speaking” about.

Important − non-constructivity! The notion of logically valid formulas is **doubly non-constructive** in the sense that the universal quantifier "for all interpretations" is added to the (already) non-constructive definition of a true formula.

As we will see in, all the axioms of our classical logical axiom system [L_1-L_{15}, MP, Gen] are logically valid formulas. And that inference rules MP and Gen generate only logically valid formulas. I.e. we will prove that **all the formulas that can be proved in the classical logic [L_1-L_{15}, MP, Gen], are logically valid.**

As an example, let us verify that the axiom L_{12}: ∀xF(x)→F(t) is logically

valid. Let us assume the contrary, i.e. that, under some interpretation J, for some values of its free variables, L_{12} is false. According to the classical truth tables, this could be only, if and only if ∀xF(x) were true, and F(t) were false (under the interpretation J, for the same above-mentioned values of free variables). Let us "compute" the value of the term t for these values of free variables (since the substitution F(x/t) is admissible, t may contain only these variables), and denote it by c. Thus, F(c) is false. But ∀xF(x) is true, hence, F(a) is true for all a in the domain D_J, i.e. F(c) also is true. Contradiction. Hence, L_{12} is true under all interpretations for all combinations of its free variables (if any).

Exercise 4.1.2. Verify that the remaining 6 of the above formulas are logically valid. (Hint: follow the above example − assume that there is an interpretation J such that the formula under question is false for some values of its free variables, and derive a contradiction.)

Is our axiom system of logic powerful enough to prove ALL the logically valid formulas? The answer is positive − see Gödel's Completeness Theorem in Section 4.3: a formula is logically valid, if and only of it is provable in the **classical** logic [L_1-L_{11}, L_{12}-L_{15}, MP, Gen].

But, of course, there are formulas that are not logically valid. For example, negations of logically valid formulas are false in all interpretations, i.e. they are not logically valid. Such formulas are called **unsatisfiable formulas**. But there are formulas that are true in some interpretations, and false − in some other ones. An example of such formulas: the axiom of arithmetic x+1=y+1 → x=y considered above.

To conclude that some formula is **not** logically valid, we must build an interpretation J such that the formula under question is false for some values of its free variables.

As an example, let us verify that the formula

$$\forall x(p(x)\vee q(x))\rightarrow \forall x\, p(x)\vee \forall x\, q(x)$$

is not logically valid (p, q are predicate constants). Why it is not? Because the truth-values of p(x) and q(x) may behave in such a way that $p(x)\vee q(x)$ is always true, but neither ∀x p(x), nor ∀x q(x) is true. Indeed, let us take the domain D = {a, b}, and set:

x	p(x)	q(x)
a	true	false
b	false	true

In this interpretation, $p(a)\vee q(a)$ = true, $p(b)\vee q(b)$ = true, i.e. the premise $\forall x(p(x)\vee q(x))$ is true. But the formulas ∀x p(x), ∀x q(x) both are false. Hence, in this interpretation, the consequent $\forall x\, p(x)\vee\forall x\, q(x)$ is false, and thus, $\forall x(p(x)\vee q(x))\rightarrow\forall x\, p(x)\vee\forall x\, q(x)$ is false. We have built an interpretation, making false the formula under question. Q.E.D.

On the other hand, this formula **is satisfiable** – there is an interpretation under which it is true. Indeed, let us take D={a} as the domain of interpretation, and let us set p(a)=q(a)=true. Then all the formulas

$$\forall x(p(x)\vee q(x)),\ \forall x\, p(x),\ \forall x\, q(x)$$

become true, and so is the entire formula under consideration. Q.E.D.

Exercise 4.1.3. Verify that the following formulas are satisfiable, but **not** logically valid (p, q, r are predicate constants):

a) $p(x,y)\wedge p(y,z)\rightarrow p(x,z)$,

b) q(x)→∀x q(x),

c) (∀x q(x)→∀x r(x))→∀x(q(x)→r(x)),

c_1) ∃x(p(x)→B)→(∃x p(x)→B), where B does not contain x,

d) ∀x∃y p(x, y)→∃y∀x p(x, y),

e) $\exists x\, q(x)\wedge\exists x\, r(x)\rightarrow\exists x(q(x)\wedge r(x))$,

f) $\forall x\neg p(x,x)\wedge\forall x\forall y\forall z(p(x,y)\wedge p(y,z)\rightarrow p(x,z))\ \rightarrow$

$$\forall x\,\forall y(x=y\vee p(x,y)\vee p(y,x))\ .$$

Hint. For the domain D={a, b}, use table form to define your interpretation of a binary predicate letter r(x,y), for example,

x	**y**	**r(x, y)**
a	a	false
a	b	true
b	a	true
b	b	false

Exercise 4.1.4. Is the following formula logically valid, or not (p, q are predicate constants):

(∃x p(x)→∃x q(x))→∃x(p(x)→q(x)).

(Hint: follow the above example − use natural numbers or other objects trying to build an interpretation J such that the formula under question is false.)

Satisfiability

We already know that, in a predicate language L, a formula F is called **satisfiable**, if and only if **there is an interpretation** of the language L such that F is true for **some** values of its free variables (we will say also that F **is satisfied under this interpretation**). A set of formulas F_1, ..., F_n, ... is called satisfiable, if and only if there is an interpretation under which the formulas F_1, ..., F_n, ... are satisfied *simultaneously.*

Examples. a) Formula ∃x p(x) is, of course, not logically valid, but it is satisfiable, because it is true in the following interpretation J: D_J={b}, p(b) is true.

b) Formulas x*0=0, x+y=y+x and x+(y+z)=(x+y)+z are **not** logically valid (see Exercise 4.1.7 below), but they are satisfiable, because they are true under the standard intepretation of arithmetic.

Exercise 4.1.5. a) Verify that the formula ∀xy(p(x)→p(y)) is true in all one-element interpretations (i.e. when the interpretation domain consists of a single element), but is false in at least one two-element interpretation (p is a predicate constant).

b) Verify that the formula

$$\forall x \forall y \forall z[(p(x) \leftrightarrow p(y)) \vee (q(y) \leftrightarrow q(z)) \vee (r(z) \leftrightarrow r(x))]$$

is true in all one- and two-element interpretations, but is false in at least one three-element interpretation (p, q, r are predicate constants).

c) Prove that the formula ∃x∀y F(x,y) is logically valid, if and only if so is the formula ∃x F(x, g(x)), where g is a function constant that does not appear in F.

d) Prove that the formula ∀x∀y∃z F(x,y,z) is satisfiable, if and only if so is the formula ∀x∀y F(x, y, h(x,y)), where h is a function constant that does not appear in F.

Logical Consequences

"F implies G", or "the formula G follows from the formula F" − what should this mean in general? If F is true, then G is true? Always, or under some specific conditions? Let us specify **all** these "conditions" as formulas A_1, ..., A_n (the formula F included). Then, G follows from A_1, ..., A_n unconditionally

("logically"), i.e. if A_1, ..., A_n are all true, then G must be true without any other conditions. Since the notion of "true" we have formalized as "true in interpretation", we can formalize the notion of "logical consequence" as follows:

G is a **logical consequence** of A_1, ..., A_n, if and only if **G is true under any interpretation, under which A_1, ..., A_n are all true**.

Or, as follows:

G is a **logical consequence** of A_1, ..., A_n, if and only if **G is true in any model of A_1, ..., A_n**.

Exercise 4.1.6. Verify that:

a) The formula G is a logical consequence of formulas A_1, ..., A_n, if and only if the formula

$$A_1 \wedge ... \wedge A_n \rightarrow G$$

is logically valid.

b) If the set of formulas A_1, ..., A_n is satisfiable, then the formulas B, ¬B cannot both be logical consequences of A_1, ..., A_n.

c) The formula G is a logical consequence of formulas A_1, ..., A_n, if and only if the set A_1, ..., A_n, ¬G is unsatisfiable.

We will prove in Section 4.3 that G is a logical consequence of A_1, ..., A_n, if and only if

$$[L_1\text{-}L_{11}, L_{12}\text{-}L_{15}, \text{MP, Gen}]: \quad A_1 \wedge ... \wedge A_n \vdash G,$$

i.e. if the formula $A_1 \wedge ... \wedge A_n \rightarrow G$ is provable in the classical logic.

Theories and Their Models

If T is a first order theory, and J is an interpretation of its language, and if J makes true the specific axioms of T, then (traditionally) J is called a **model of** T.

For non-mathematical people, the term "model of a theory" may seem somewhat strange: in "normal" branches of science, theories serve as a basis for building models of natural phenomena, technical devices etc. But only the term is strange ("upside down") here, the process is the same as in "normal" branches of science: first order theories "generate" their models, and these models can be used for modeling natural phenomena, technical devices etc.

Specific axioms of a first order theory T are not logically valid formulas! They are not true in all interpretations, they are true **only in the models** of T. Models of T − it is a **proper subclass** of all the possible interpretations. For example, the "obvious" arithmetical axioms like as x+0=x (or, theorems like as x+y=y+x) are not logically valid. If we would interpret 0 as the number "two", then x+0 and x will be equal! Logically valid formulas must be true under **all** interpretations!

Exercise 4.1.7. a) Verify that, if a theory has a model, then the set of its specific axioms is satisfiable.

b)Verify that x=x, x*0=0, x+y=y+x and x+(y+z)=(x+y)+z are satisfiable, but not logically valid formulas.

As we already noted above, in a sense, **logically valid formulas "do not contain information"** (are “content-free”) − just because they are true in all interpretations, i.e. they are true independently of the "meaning" of language primitives. Indeed, let us consider the formulas x+0=x → x+0=x, and x+0=0 → x+0=0. Both are logically valid, but do we get more information about zero and addition after reading them? Another example: 2*2=5 → 2*2=5, or 2*2=4 → 2*2=4, these formulas also are logically valid, but do they help in computing the value of 2*2? The specific axioms of some theory T, on the contrary, do "contain information" − they separate a proper subclass of all interpretations − models of T.

Do the axioms of first order arithmetic “specify” the standard interpretation S, i.e. are the axioms of first order arithmetic true **in this interpretation only?** No, there are many **non-standard interpretations** making these axioms true! More: Non-standard arithmetic in Wikipedia.

Transitive Predicates and Recursion

Let us return to the problem that we considered already in Section 1.2.

How about the predicate **Ancestor(x, y)** − "x is an ancestor of y"? Could it be expressed as a formula of our "language for people"? The first idea − let us "define" this predicate recursively:

$$\forall x\, \forall y\, (Father(x, y) \vee Mother(x, y) \rightarrow Ancestor(x, y)) \;;$$
$$\forall x\, \forall y\, \forall z\, (Ancestor(x, y) \wedge Ancestor(y, z) \rightarrow Ancestor(x, z)) \;.$$

The second rule declares the transitivity property of the predicate. The above two formulas are **axioms**, allowing to derive essential properties of the predicate Ancestor(x, y). But how about a single formula F(x, y) in the "language for people", expressing that "x is an ancestor of y"? Such a formula should be a tricky combination of formulas Father(x, y), Mother(x, y) and x=y. And such a formula is **impossible**! For the proof – see **Carlos Areces**. Ph.D.

Thesis, 2000, Theorem 1.2.

Exercise 4.1.8 (optional, for smart students). Explain the precise meaning of the statement: in the "language for people", formula F(x, y) expresses that "x is an ancestor of y".

4.2. Classical Propositional Logic − Truth Tables

Emil Leon Post (1897-1954). "... Post's Ph.D. thesis, in which he proved the completeness and consistency of the propositional calculus described in the *Principia Mathematica* by introducing the truth table method. He then generalised his truth table method, which was based on the two values "true" and "false", to a method which had an arbitrary finite number of truth-values... In the 1920s Post proved results similar to those which Gödel, Church and Turing discovered later, but he did not publish them. He reason he did not publish was because he felt that a 'complete analysis' was necessary to gain acceptance." (According to MacTutor History of Mathematics archive).

First, let us consider the **classical propositional logic**. Here, each formula is built of some is built of some "atoms" B_1, B_2, ..., B_n by using propositional connectives only (i.e. $B \wedge C$, $B \vee C$, $\neg B$, $B \rightarrow C$). Our axioms for this logic we represented as axiom schemas L_1-L_{11}, in which the letters B, C, D could be replaced by any formulas.

Is our list L_1-L_{11} of classical propositional axiom schemas "complete"? Aren't some necessary axiom schemas missing there? If something necessary is missing, we must add it to the list.

This problem was solved by Emil L. Post in 1920. He proved that if **one would add to L_1-L_{11} as an axiom schema any formula that can't yet be proved from these axioms, then one would obtain a system, in which all formulas are provable, i.e. an inconsistent system.** Thus, nothing is missing in our list of classical propositional axioms.

Post proved his theorem by using the so-called **classical truth tables** (a specific interpetation – in terms of the above Section 4.1). Each propositional atom may take any of two truth-values – *true* and *false*. And, if we already know the truth-values of the formulas B, C, then we can use truth tables to compute the truth-values of the formulas $B \wedge C$, $B \vee C$, $\neg B$, $B \rightarrow C$.

If B is false, and C is false, then $B \wedge C$ is false.
If B is false, and C is true, then $B \wedge C$ is false.
If B is true, and C is false, then $B \wedge C$ is false.

If B is true, and B is true, then $B \wedge C$ is true.

B	**C**	$\boldsymbol{B \wedge C}$
0	0	0
0	1	0
1	0	0
1	1	1

If B is false, and C is false, then $B \vee C$ is false.
If B is false, and C is true, then $B \vee C$ is true.
If B is true, and C is false, then $B \vee C$ is true.
If B is true, and C is true, then $B \vee C$ is true.

B	**C**	$\boldsymbol{B \vee C}$
0	0	0
0	1	1
1	0	1
1	1	1

If B is false, then ¬B is true.
If B is true, then ¬B is false.

B	**¬B**
0	1
1	0

No problems so far.

If B is false, and C is false, then B→C is what? True? False? But, why?
If B is false, and C is true, then B→C is what? True? False? But, why?
If B is true, and C is false, then B→C is false, of course.
If B is true, and C is true, then B→C is what? Perhaps, not false? Hence, true?

How to answer the 3 what's? If B is false, then B→C possesses no real meaning. And, if we already know that B is true, and C is true, then B→C is no more interesting. But, if a definite "truth-value" for B→C is mandatory in all cases, then we can *greatly simplify the situation* by assuming that B→C is always true, except, if B is true, and C is false. Thus:

If B is false, and C is false, then B→C is true.

If B is false, and C is true, then B→C is true.
If B is true, and C is false, then B→C is false.
If B is true, and C is true, then B→C is true.

B	**C**	**B→C**
0	0	1
0	1	1
1	0	0
1	1	1

This definition is equivalent to saying that

B→C is true, if and only if $\neg(B \wedge \neg C)$ is true

or:

B→C is false, if and only if B is true, and C is false.

In this way, having any formula F and some assignment of truth-values to its atoms, we can compute the truth-value of F.

But what would happen to some propositional formula F, if we would try *all the possible* truth-values of all the propositional atoms occurring in F? There are three possibilities:

F takes only true values;

F takes only false values;

F takes both of values.

Lemma 4.2.1. Under the classical truth tables, all the classical propositional axioms L_1-L_{11} take only true values.

Proof. First, let us verify L_{11} and L_{10}:

B	**¬B**	$B \vee \neg B$
0	1	1
1	0	1

B	**C**	**¬B**	**B→C**	**¬B→(B→C)**
0	0	1	1	1
0	1	1	1	1

1	0	0	0	1
1	1	0	1	1

Exercise 4.2.1. Verify L_1-L_9.

See also:
"Truth Tables" from The Wolfram Demonstrations Project. Contributed by: Hector Zenil.

Lemma 4.2.2. Under the classical truth tables, if the formulas B and B→C take only true values, then so does C. I.e. from "always true" formulas, Modus Ponens allows deriving only of "always true" formulas.

Proof. Let us assume that, in some situation, C takes a false value. In the same situation, B and B→C take true values. If B is true, and C is false, then B→C is false. Contradiction. Hence, C takes only true values. Q.E.D.

Note. In the proof of Lemma 4.2.2, only the third row of implication truth table was significant: if B is true, and C is false, then B→C is false!

Theorem 4.2.3 (soundness of the classical propositional logic).

If [L_1-L_{11}, MP]: $\vdash$ F, then, under the classical truth tables, F takes only true values. In particular: the classical propositional logic is **consistent** – in the sense that one cannot prove [L_1-L_{11}, MP]: $\vdash G \wedge \neg G$, for any formula G.

Proof. By induction, from Lemmas 4.2.1 and 4.2.2.

Completeness of Classical Propositional Logic

How about the converse statement of Theorem 4.2.3: if, under the classical truth tables, formula F takes only true values, then [L_1-L_{11}, MP]: $\vdash$F? I.e., **are our axioms powerful enough to prove any formula that is taking only true values?** The answer is "yes":

Theorem 4.2.4 (completeness of the classical propositional logic). Assume, the formula F has been built of formulas B_1, B_2, ..., B_n by using propositional connectives only. If, under the classical truth tables, for any truth-values of B_1, B_2, ..., B_n, formula F takes only true values, then:

a) in the constructive logic,

$$[L_1\text{-}L_{10}, MP]: \quad B_1 \vee \neg B_1, B_2 \vee \neg B_2, \ldots, B_n \vee \neg B_n \quad \vdash F,$$

b) in the classical logic, [L_1-L_{11}, MP]: $\vdash$F.

Corollary 4.2.4. The classical propositional axioms [L_1-L_{11}, MP] are "complete" in the sense that if one would add any formula that can't yet be proved from these axioms, then one would obtain a system, in which all formulas are provable, i.e. an inconsistent system.

Of course, (b) follows from (a) immediately − all the premises $B_1 \vee \neg B_1, B_2 \vee \neg B_2, \ldots, B_n \vee \neg B_n$ are instances of the axiom L_{11}.

The corollary also follows immediately. Indeed, if some formula F can't be proved from [L_1-L_{11}, MP], then it takes false value for some combination of truth-values of its atoms. Replace each true atom by the formula A→A, and each false atom – by ¬(A→A). In this way we obtain a formula F' that takes only false values, i.e. ¬F' takes only true values, and hence, can be proved from [L_1-L_{11}, MP]. Thus, if we would add F to [L_1-L_{11}, MP] as an axiom schema, then, in this system, the formulas F' and ¬F' will be provable, and by L_{10} – any formula will be provable.

Note. Assume, the formula F is built of atoms B_1, B_2, ..., B_n by using propositional connectives only. If, under the classical truth tables, for any (possible and impossible) truth-values of B_1, B_2, ..., B_n, formula F takes only true values, then F is called a **tautology**. Theorem 4.2.4 says that **any tautology can be proved in the classical propositional logic**.

Completeness of the classical propositional logic was first proved by Emil L. Post in his 1920 Ph.D. thesis, and published as

E. Post. Introduction to a general theory of elementary propositions. *American Journal of Mathematics*, 1921, vol. 43, pp.163-185.

About the history, see also:

Richard Zach. Completeness before Post: Bernays, Hilbert, and the development of propositional logic. *The Bulletin of Symbolic Logic*, 1999, vol. 5, N3, pp.331-366 (online copy available).

Following an elegant later idea by Laszlo Kalmar we need two simple lemmas before trying to prove this theorem.

L. Kalmar. Ueber Axiomatisiebarkeit des Aussagenkalkuels. *Acta scientiarium mathematicarum (Szeged)*. 1934-35. vol. 7, pp. 222-243.

Lemma 4.2.5. In the constructive logic, one can "compute" the classical truth-values of $\neg B, B \to C, B \wedge C, B \vee C$ in the following sense:

Negation	**Implication**	**Conjunction**	**Disjunction**
[ ]:	[L_{10}, MP]:	[L_1, L_2, L_3, L_9, MP]:	[L_1-L_9, MP]:

$\neg B \vdash \neg B$	$\neg B, \neg C \vdash B \to C$	$\neg B, \neg C \vdash \neg(B \wedge C)$	$\neg B, \neg C \vdash \neg(B \vee C)$
$[L_1, L_2, L_9, MP]$: $B \vdash \neg\neg B$	$[L_{10}, MP]$: $\neg B, C \vdash B \to C$	$[L_1, L_2, L_3, L_9, MP]$: $\neg B, C \vdash \neg(B \wedge C)$	$[L_7, MP]$: $\neg B, C \vdash B \vee C$
	$[L_1, L_2, L_9, MP]$: $B, \neg C \vdash \neg(B \to C)$	$[L_1, L_2, L_4, L_9, MP]$: $B, \neg C \vdash \neg(B \wedge C)$	$[L_6, MP]$: $B, \neg C \vdash B \vee C$
	$[L_1, MP]$: $B, C \vdash B \to C$	$[L_5, MP]$: $B, C \vdash B \wedge C$	$[L_6, MP]$: $B, C \vdash B \vee C$

Note. Thus, to "compute" the classical truth-values, the axiom L_{11} is not necessary!

Proof.

$\neg B \vdash \neg B$

Immediately, in any logic.

$B \vdash \neg\neg B$

By Theorem 2.4.4. $[L_1, L_2, L_9, MP]$: $\vdash A \to \neg\neg A$.

$\neg B, C \vdash B \to C$

$\neg B, \neg C \vdash B \to C$

By axiom L_{10}: $\neg B \to (B \to C)$ we obtain $\neg B \vdash B \to C$. This covers both cases.

$B, \neg C \vdash \neg(B \to C)$

This is exactly Theorem 2.4.1(c) $[L_1, L_2, L_9, MP]$.

$B, C \vdash B \to C$

By axiom L_1: $C \to (B \to C)$ we obtain $C \vdash B \to C$.

$\neg B, \neg C \vdash \neg(B \wedge C)$

$\neg B, C \vdash \neg(B \wedge C)$

By axiom L_3: $B \wedge C \to B$ and the Contraposition Law (Theorem 2.4.2) $[L_1, L_2, L_9, MP]$: $\vdash (A \to B) \to (\neg B \to \neg A)$ we obtain $\vdash \neg B \to \neg(B \wedge C)$, and $\neg B \vdash \neg(B \wedge C)$. This covers both cases.

$B, \neg C \vdash \neg(B \wedge C)$

By axiom L_4: $B \wedge C \rightarrow C$ and the Contraposition Law (Theorem 2.4.2) [L_1, L_2, L_9, MP]: $\vdash (A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$ we obtain $\vdash \neg C \rightarrow \neg(B \wedge C)$, an $\neg C \vdash \neg(B \wedge C)$.

B, C $\vdash$ $B \wedge C$

By axiom L_5: $B \rightarrow (C \rightarrow B \wedge C)$ we obtain B, C $\vdash B \wedge C$.

$\neg$B, $\neg$C $\vdash$ $\neg(B \vee C)$

By Theorem 2.4.10(b).

$\neg$B, C $\vdash$ $B \vee C$

By axiom L_7: $C \rightarrow BvC$ we obtain C $\vdash B \vee C$.

B, $\neg$C $\vdash$ $B \vee C$

B, C $\vdash$ $B \vee C$

By axiom L_6: $B \rightarrow B \vee C$ we obtain B $\vdash B \vee C$. This covers both cases.

Q.E.D.

As the next step, we will generalize Lemma 4.2.5 by showing how to "compute" truth-values of arbitrary formula F, which is built of formulas B_1, B_2, ..., B_n by using more than one propositional connective. For example, let us take the formula $B \vee C \rightarrow B \wedge C$:

B	C	$B \vee C$	$B \wedge C$	$B \vee C \rightarrow B \wedge C$
0	0	0	0	1
0	1	1	0	0
1	0	1	0	0
1	1	1	1	1

We will show that, in the constructive logic [L_1-L_{10}, MP]:

$\neg$B, $\neg$C $\vdash B \vee C \rightarrow B \wedge C$,
$\neg$B, C $\vdash \neg(B \vee C \rightarrow B \wedge C)$,
B, $\neg$C $\vdash \neg(B \vee C \rightarrow B \wedge C)$,
$\neg$B, $\neg$C $\vdash B \vee C \rightarrow B \wedge C$.

Lemma 4.2.6. Assume, the formula F has been built of formulas B_1, B_2, ..., B_n by using propositional connectives only. Assume that, if the formulas B_1, B_2, ..., B_n take the truth-values v_1, v_2, ..., v_n respectively, then, for these

values, formula F takes the truth-value w. Then, in the constructive logic, we can "compute" the truth-value of F in the following sense:

$$[L_1\text{-}L_{10}, MP]: v_1B_1, v_2B_2, ..., v_nB_n \vdash wF,$$

where: wF denotes F, if w is true, and ¬F, if w is false, and v_iB_i denotes B_i, if v_i is true, and $\neg B_i$, if v_i is false.

Proof. By induction.

Induction base. F is one of the formulas B_i. Then $w=v_i$, and, of course, in any logic, $v_iB_i \vdash wF$.

Induction step.

Note that Lemma 4.2.5 represents the assertion of Lemma 4.2.6 for formulas built of $B_1, B_2, ..., B_n$ by using a single propositional connective.

1. **F is ¬G**. By the induction assumption,

$$[L_1\text{-}L_{10}, MP]: v_1B_1, v_2B_2, ..., v_nB_n \vdash w'G,$$

where w' represents the truth-value of G. By Lemma 4.2.5,

$[L_1\text{-}L_{10}, MP]: w'G \vdash wF$, hence, $[L_1\text{-}L_{10}, MP]: v_1B_1, v_2B_2, ..., v_nB_n \vdash wF$.

2. **F is G o H**, where o is implication, conjunction, or disjunction. By the induction assumption,

$$[L_1\text{-}L_{10}, MP]: v_1B_1, v_2B_2, ..., v_nB_n \vdash w'G,$$

where w' represents the truth-value of G, and

$$[L_1\text{-}L_{10}, MP]: v_1B_1, v_2B_2, ..., v_nB_n \vdash w''H,$$

where w'' represents the truth-value of H. By Lemma 4.2.5,

$$[L_1\text{-}L_{10}, MP]: w'G, w''H \vdash wF,$$

hence, $[L_1\text{-}L_{10}, MP]: v_1B_1, v_2B_2, ..., v_nB_n \vdash wF$.

Q.E.D.

Proof of Theorem 4.2.4(a). By Lemma 4.2.6:

$$[L_1\text{-}L_{10}, MP]: B_1, v_2B_2, ..., v_nB_n \vdash F,$$
$$[L_1\text{-}L_{10}, MP]: \neg B_1, v_2B_2, ..., v_nB_n \vdash F,$$

because F takes only true values. By $[L_1, L_2, MP]$ Deduction Theorem 1,

$$[L_1\text{-}L_{10}, MP]: v_2B_2, ..., v_nB_n \vdash B_1 \rightarrow F,$$

$$[L_1\text{-}L_{10}, MP]: v_2B_2, ..., v_nB_n \vdash \neg B_1 \rightarrow F,$$

Let us merge these two proofs and append an instance of the axiom L_8:

$$\vdash (B_1 \rightarrow F) \rightarrow ((\neg B_1 \rightarrow F) \rightarrow (B_1 \vee \neg B_1 \rightarrow F)) \quad .$$

Hence, by MP:

$$[L_1\text{-}L_{10}, MP]: v_2B_2, ..., v_nB_n \vdash B_1 \vee \neg B1 \rightarrow F \quad ,$$

and

$$[L_1\text{-}L_{10}, MP]: B_1 v \neg B_1, v_2B_2, ..., v_nB_n \vdash F.$$

By repeating this operation we obtain Theorem 4.2.4(a):

$$[L_1\text{-}L_{10}, MP]: \quad B_1 \vee \neg B_1, B_2 \vee \neg B_2, ..., B_n \vee \neg B_n \quad \vdash F.$$

Q.E.D.

Computational Complexity of the Problem

From now on, we could forget our ability of proving formulas in the classical propositional logic, learned in Section 2. Indeed, in order to verify, is a formula provable in $[L_1\text{-}L_{11}, MP]$, or not, we can simply check, under the classical truth tables, takes this formula only true values, or not. Is this checking really simpler than proving of formulas in $[L_1\text{-}L_{11}, MP]$?

If the formula contains *n* different atoms A, B, C, ..., then its truth table contains 2^n rows that must be checked one by one. Of course, if the formula contains 2 atoms (like as $(A \rightarrow B) \rightarrow \neg A \vee B$, or 3 atoms (like as the Axiom L_2), then its truth table consists of 4 or 8 rows − for most people this is a feasible task. But the "truth table" for a formula containing 32 atoms contains four billions of rows to check... So, let us try inventing a more efficient algorithm?

It seems, we will never succeed − the problem of determining the **classical** provability of propositional formulas belongs to the so-called complexity class “co-NP-complete”, see Boolean satisfiability problem in Wikipedia. And the problem of determining the **constructive** provability of propositional formulas is even harder – it belongs to the complexity class “PSPACE-complete”, see:

Richard Statman. Intuitionistic propositional logic is polynomial-space complete, Theoretical Computer Science 9 (1979), pp. 67–72 (online copy available).

4.3. Classical Predicate Logic − Gödel's Completeness Theorem

Kurt Gödel (1906-1978) "He is best known for his proof of Gödel's Incompleteness Theorems. In 1931 he published these results in *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme* . ...Gödel's results were a landmark in 20th-century mathematics, showing that mathematics is not a finished object, as had been believed. It also implies that a computer can never be programmed to answer all mathematical questions." (According to MacTutor History of Mathematics archive).

As David Hilbert and Wilhelm Ackermann published in

D.Hilbert, W.Ackermann. Grundzuege der theoretischen Logik. Berlin (Springer), 1928

their, in a sense, "final" version of the axioms of classical logic, they observed: "Whether the system of axioms is complete at least in the sense that all the logical formulas which are correct for each domain of individuals can actually be derived from them, is still an unsolved question."

(quoted after
S. C. Kleene. The Work of Kurt Gödel. "The Journal of Symbolic Logic", December 1976, Vol.41, N4, pp.761-778
See also:
Hilbert and Ackermann's 1928 Logic Book by Stanley N. Burris).

Indeed, as we will verify below, a) all axioms of the classical logic (L_1-L_{11}, L_{12}-L_{15}) are logically valid, b) the inference rules MP, Gen allow to prove (from logically valid formulas) only logically valid formulas. Hence, in this way only logically valid formulas can be proved. Still, is our list of logical axioms complete in the sense that **all logically valid formulas can be proved?** − the question asked by Hilbert and Ackermann in 1928. The answer is "yes" − as Kurt Gödel established in 1929, in his doctoral dissertation "Ueber die Vollständigkeit des Logikkalkuels"(visit Gödel's Archive in the Princeton University Library). The corresponding paper appeared in 1930:

K. Gödel. Die Vollständigkeit der Axiome des logischen Funktionenkalkuels. "Monatshefte fuer Mathematik und Physik", 1930, Vol.37, pp.349-360.

Gödel's Completeness Theorem. In any predicate language, a formula is logically valid, if and only if it can be proved by using the classical logic [L_1-L_{11}, L_{12}-L_{15}, MP, Gen].

In fact, a more general theorem can be proved:

Theorem 4.3.0 (Thanks to Sune Foldager for the idea.). If T is a first order theory with classical logic, then some formula F is always true in all models of

T, if and only if T proves F.

Gödel's Completeness Theorem follows from Theorem 4.3.0, if the set of specific axioms of T is empty.

First, let us prove the easy part (sometimes called the **soundness theorem**) − that all the formulas that can be proved by using the classical logic [L_1-L_{11}, L_{12}-L_{15}, MP, Gen] are logically valid.

Lemma 4.3.1. All the axioms of the classical logic (L_1-L_{11}, L_{12}-L_{15}) are logically valid.

Proof.

1) Under the classical truth tables, the propositional axioms L_1-L_{11} take only true values (Lemma 4.2.1). Hence, these axioms are true under all interpretations.

2a) $\mathbf{L_{12}}$: $\forall xF(x)\rightarrow F(t)$, where F is any formula, and t is a term such that the substitution F(x/t) is admissible.

Let us assume that, under some interpretation J, for some values of its free variables, L_{12} is false. According to the classical truth tables, this could be only, if and only if $\forall xF(x)$ were true, and F(t) were false (under the interpretation J, for the same above-mentioned values of free variables). Let us "compute" the value of the term t for these values of free variables (since the substitution F(x/t) is admissible, t may contain only these variables), and denote it by c. Thus, F(c) is false. But $\forall xF(x)$ is true, hence, F(a) is true for all $a\in D_J$, i.e. F(c) also is true. Contradiction. Hence, L_{12} is true under all interpretations for all combinations of its free variables.

2b) $\mathbf{L_{13}}$: $F(t)\rightarrow\exists xF(x)$, where F is any formula, and t is a term such that the substitution F(x/t) is admissible.

Similarly, see Exercise 4.3.1.

2c) $\mathbf{L_{14}}$: $\forall x(G\rightarrow F(x))\rightarrow(G\rightarrow\forall xF(x))$, where F is any formula, and G is a formula that does not contain x as a free variable.

Let us assume that, under some interpretation J, for some values of its free variables, L_{14} is false. According to the classical truth tables, this could be only, if and only if $\forall x(G\rightarrow F(x))$ were true, and $G\rightarrow\forall xF(x)$ were false (under the interpretation J, for the same above-mentioned values of free variables)

If $\forall x(G\rightarrow F(x))$ is true, then $G\rightarrow F(c)$ is true for all $c\in D_J$. Since G does not contain x, this means that if G is true, then F(c) is true for all $c\in D_J$.

From the orher side, if G→∀xF(x) is false, then G is true, and ∀xF(x) is false. And finally, if ∀xF(x) is false, then F(c) is false for some $c \in D_J$. But, as we established above, if G is true, then F(c) is true for all $c \in D_J$. Contradiction. Hence, under all interpretations, L_{14} is true for all combinations of its free variables.

2d) $\mathbf{L_{15}}$: ∀x(F(x)→G)→(∃xF(x)→G), where F is any formula, and G is a formula that does not contain x as a free variable.

Similarly, see Exercise 4.3.1.

Q.E.D.

Exercise 4.3.1. Verify that the axioms L_{13} and L_{15} are logically valid.

Lemma 4.3.2. From logically valid formulas, inference rules MP and Gen allow deriving only of logically valid formulas..

Proof.

1. **Modus Ponens**. Assume, B and B→C are logically valid formulas. By MP, we derive C. Assume, C is **not** logically valid, i.e., under some interpretation J, for some values of its free variables, C is false. Under this interpretation J, for these values of free variables of C, the formulas B and B→C are true. Then, according to the classical truth tables, C also must be true. Contradiction. Hence, C is logically valid.

2. **Generalization**. Assume, F(x) is logically valid, but ∀xF(x) is not, i.e., under some interpretation J, for some values of its free variables, ∀xF(x) is false. Hence, under this interpretation J, for these values of free variables of ∀xF(x), there is $c \in D_J$ such that F(c) is false. But F(x) is logically valid, i.e. F(c) is true. Contradiction. Hence, ∀xF(x) is logically valid.

Q.E.D.

Corollary 4.3.3 (soundness of the classical predicate logic). All the formulas that can be proved by using the classical logic [L_1-L_{11}, L_{12}-L_{15}, MP, Gen], are logically valid.

Proof. Immediately, by Lemmas 4.3.1 and 4.3.2.

Exercise 4.3.1X. Verify that if, under an interpretation J, all specific **axioms** of a theory T are true, then all **theorems** of T also are true under J. (Hint: each theorem C is proved by using some finite set of specific axioms, let us denote by B the conjunction of these axioms, consider the formula B→C, and use Corollary 4.3.3.)

Of course, the above soundness theorem is the easy half of Gödel's Completeness Theorem. To complete the proof, we must prove the converse: if

some formula is logically valid, then it can be proved by using the classical logic [L_1-L_{11}, L_{12}-L_{15}, MP, Gen].

Model Existence Theorem

Gödel's initial proof was simplified in 1947, when Leon Henkin presented in his Ph.D. thesis a new proof of the so-called Model Existence Theorem (see below). The result was published in 1949:

L. Henkin. The completeness of the first-order functional calculus. "J. Symbolic Logic", 1949, vol.14, pp.159-166.

See also Henkin's later account of his discovery:

L. Henkin. The discovery of my completeness proofs. "The Bulletin of Symbolic Logic", 1996, vol.2, N2, pp.127-158.

An even simpler version Henkin's proof was found independently and almost simultaneously by Gisbert Hasenjäger, however, when publishing, he acknowledged Henkin's priority:

G. Hasenjäger. Eine Bemerkung zu Henkin's Beweis fuer die Vollständigkeit des Prädikatenkalkuels der ersten Stufe. "J. Symbolic Logic", 1953, vol.18, pp.42-48.

If T is an inconsistent theory, then there are no models of T. Indeed, if T proves a contradiction, i.e. a formula of the kind $B \wedge \neg B$, then, in a model of T, the formula B must be true and false simultaneously. This is imposssible.

Hence, if there is a model of T, then T is consistent.

The converse question: could it be possible that T is a consistent theory, but there are no models of T? The answer is given in the

Model Existence Theorem. If a first order classical formal theory is consistent (in the sense that, by using the classical logic, it does not prove contradictions), then there is a finite or countable model of this theory (i.e. an interpretation with a finite or countable domain, under which all axioms and theorems of the theory are always true).

In the 1920s, some people insisted that mere consistency of a theory (in the syntactic sense of the word − as the lack of contradictions) is not sufficient to regard it as a meaningful theory − as a "theory of something". Model Existence Theorem says the contrary − (syntactic!) consistency of a theory **is** sufficient: **if a theory does not contain contradictions, then it is a "theory of something"** − it describes at least some kind of "mathematical reality". For example, you may think that Euclidean geometry is "meaningless" − because it does not describe 100% correctly the spacial properties of the Universe. But it's your problem, not Euclid's − use another theory, if necessary. Euclidean geometry describes its own kind of "mathematical reality" – and 100%

correctly!

Let us assume the Model Existence Theorem (we will prove it later in this Section).

Proof of Theorem 4.3.0.

If T proves F, then F is always true in all models of T (Exercise 4.3.1X).

Now, let us assume that some formula F is always true in all models of theory T, yet it cannot be proved in T. Let us consider the theory T' in the language of T which contains (besides the axioms of T) an additional non-logical axiom − the negation of F, i.e. the formula $\neg\forall x_1...\forall x_n F$, where x_1, .., x_n are exactly all the free variables of F (if F contains free variables x_1, .., x_n, then, to negate its assertion, we must add to F the quantifiers $\forall x_1...\forall x_n$). Since F cannot be derived from the axioms of T, **T' is a consistent theory.**

Indeed, if T' would be inconsistent, i.e. we could prove in T' some formula C and its negation ¬C, then we had proofs of [T]: $\neg\forall x_1...\forall x_n F \vdash C$, and [T]: $\neg\forall x_1...\forall x_n F \vdash \neg C$. Since $\neg\forall x_1...\forall x_n F$ is a closed formula, by Deduction Theorem 2, [T]: $\vdash \neg\forall x_1...\forall x_n F \rightarrow C$, and [T]: $\vdash \neg\forall x_1...\forall x_n F \rightarrow \neg C$. Now, by axiom L_9: $(B\rightarrow C)\rightarrow(B\rightarrow\neg C)\rightarrow\neg B$, we obtain that [T]: $\vdash \neg\neg\forall x_1...\forall x_n F$. By the (classical) Double Negation Law, this implies [T]: $\vdash \forall x_1...\forall x_n F$, and by axiom L_{12}: $\forall x B(x)\rightarrow B(x)$ − [T]: ⊢ F. But, by our assumption, F cannot be proved in T. Hence, T' is a consistent theory.

Now, by the Model Existence Theorem, there is a model of T', i.e. an interpretation J that makes all its axioms always true. Under this interpretation, all axioms of T are always true, i.e. J is a model of T. And the formula $\neg\forall x_1...\forall x_n F$ (as an axiom of T') also is true under J. On the other hand, since F is always true in all models of T, it is always true also under the interpretation J. Hence, formulas $\forall x_1...\forall x_n F$ and $\neg\forall x_1...\forall x_n F$ both are always true under J. This is impossible, hence, F must be provable in T. Q.E.D.

1. Such a simple proof seems almost impossible! We are proving that the logical axioms and rules of inference are strong enough, but where come these axioms in? They come in − in the proof of the Model Existence Theorem. This theorem says that if some formal theory T does not have models, then the logical axioms and rules of inference are strong enough to derive a contradiction from the axioms of T. But the proof of the Model Existence Theorem that we will consider below, is positive, not negative!

2. The above simple proof seems to be extremely non-constructive! "If F is

always true in all models of T, then it can be proved in T". How could we obtain this proof? Still, how do we know that F is true in all models of T? Only, if we had a constructive procedure that is verifying this, we could ask for an algorithm converting such procedures into proofs in T!

Proof of the Model Existence Theorem

Exercise 4.3.3 (optional, for smart students). Prove the Model Existence Theorem by using the following smart ideas due to L. Henkin and G. Hasenjäger. Let T be a consistent theory. We must build a model of T. What kind of "bricks" should we use for this "building"? **Idea #1:** let us use object constants of the language! So, let us add to the language of T an infinite set of new object constants d_1, d_2, d_3, ... (and adopt the corresponding additional instances of logical axioms). Prove that this extended theory T_0 is consistent. The model we are building must contain all "objects" whose existence can be proved in T_0. **Idea #2:** for each formula F of T_0 having exactly one free variable (for example, x) let us add to the theory T_0 the axiom $\exists xF(x) \rightarrow F(d_i)$, where the constant d_i is unique for each F. If T_0 proves $\exists xF(x)$, then this constant d_i will represent in our model the "object" x having the property F. Prove that this extended theory T_1 is consistent. Idea #3: prove the (non-constructive) Lindenbaum's lemma: the axiom set of any consistent theory can be extended in such a way, that the extended theory is consistent and complete (the axiom set of the extended theory may be not algorithmically solvable). After this, extend T_1 to a consistent complete theory T_2. **Idea #4:** let us take as the domain of the interpretation M the set of all those terms of T_0 that do not contain variables. And let us interpret each function constant f as the "syntactic constructor function" f', i.e. let us define the value $f'(t_1, ..., t_n)$ simply as the character string "$f(t_1, ..., t_n)$". Finally, let us interpret each predicate constant p as the relation p' such that $p'(t_1, ..., t_n)$ is true in M, if and only if T_2 proves $p'(t_1, ..., t_n)$. To complete the proof, prove that an arbitrary formula G is always true in M, if and only if T_2 proves G. Hence, all theorems of the initial theory T are always true in M.

Adolf Lindenbaum (1904-1941), his wife Janina Hosiasson-Lindenbaum (1899-1942).

Lindenbaum's Lemma. Any consistent first order theory can be extended to a consistent complete theory. More precisely, if T is a consistent first order theory, then, in the language of T, there is a set A of closed formulas such that T+A is a consistent complete theory. (In general, T+A is not a formal theory in the sense of Section 1.1, see below.)

Note. By T+A we denote the first order theory in the language of T, obtained from T by adding the formulas of the set A as non-logical axioms.

Exercise 4.3.4. Verify that, in any predicate language L, only countably many formulas can be generated. I.e. produce an algorithm for printing out a sequence F_0, F_1, F_2, ... containing all the formulas of L.

Proof of Lindenbaum's Lemma (**Attention: non-constructive reasoning!)**

Let us use the algorithm of the Exercise 4.3.4 printing out the sequence F_0, F_1, F_2, ... of all formulas in the language of T, and let us run through this sequence, processing only those formulas F_i that are closed.

At the very beginning, the set of new axioms A_0 is empty.

At the step i, we already have some set A_{i-1} of new axioms. If the formula F_i is not closed, let us ignore it, and set $A_i=A_{i-1}$. Now, let us suppose that F_i is a closed formula. If $T+A_{i-1}$ proves F_i, or $T+A_{i-1}$ proves $\neg F_i$, then we can ignore this formula, and set $A_i=A_{i-1}$. If T+A does not prove neither F_i, nor $\neg F_i$, then let us simply add F_i (or $\neg F_i$, if you like it better) to our set of new axioms, i.e. set $A_i = A_{i-1} \cup \{Fi\}$.

Etc., ad infinitum. As the result of this process we obtain a set of closed formulas $A = A_0 \cup A_1 \cup A_2 \cup ... \cup A_i \cup ...$.

Let us prove that T+A is a consistent complete theory.

Consistency. If T+A would be inconsistent, we would have a proof of $[T+A] \vdash C \wedge \neg C$ for some formula C. If, in this proof, no axioms from the set A would be used, we would have a proof of $[T] \vdash C \wedge \neg C$, i.e. T would be inconsistent.

Otherwise, the proof of $[T+A] \vdash C \wedge \neg C$ could contain a finite number of axioms B_1, ..., B_k from the set A. Let us arrange these axioms in the sequence, as we added them to the set A. Thus we have a proof of [T]: B_1, ..., $B_k \vdash C \wedge \neg C$. Let us remind Theorem 2.4.1(a):

If A_1, A_2, ..., A_n, $B \vdash C \wedge \neg C$, then A_1, A_2, ..., $A_n \vdash \neg B$.

Hence, we have a proof of [T]: B_1, ..., $B_{k-1} \vdash \neg B_k$. But this is impossible − we added B_k to the set A just because $T+A_{i-1}$ could not prove neither B_k, nor $\neg B_k$. Q.E.D.

Completeness. We must verify that, for any closed formula F in the language of T, either $T+A \vdash F$, or $T+A \vdash \neg F$. Let us assume, this is not the case for some

closed formula F. Of course, F appears in the above sequence F_0, F_1, F_2, ... as some F_i. If neither $T+A \vdash F$, nor $T+A \vdash \neg F$, then neither $T+A_{i-1} \vdash F_i$, nor $T+A_{i-1} \vdash \neg F_i$. In such a situation we would add F to the set A, hence, we would have $T+A \vdash F$. Q.E.D.

This completes the proof of Lindenbaum's Lemma.

Attention: non-constructive reasoning! T+A is a somewhat strange theory, because, in general, we do not have an algorithmical decision procedure for its axiom set. Indeed, to decide, is some closed formula F an axiom of T+A, or not, we must identify F in the sequence F_0, F_1, F_2, ... as some F_i, and after this, we must verify, whether $T+A_{i-1}$ proves F_i, or $T+A_{i-1}$ proves $\neg F_i$, or none of these. Thus, in general, T+A is not a formal theory in the sense of ion 1.1.

Proof of the Model Existence Theorem (Attention: non-constructive reasoning!)

Inspired by the beautiful exposition in Mendelson [1997].

Step 1. We must build a model of T. What kind of "bricks" should we use for this "building"? Idea #1: let us use object constants of the language! So, in order to prepare enough "bricks", let us add to the language of T a countable set of new object constants d_1, d_2, d_3, ... (and extend the definitions of terms, atomic formulas and formulas accordingly, and add new instances of logical axioms accordingly). Let us prove that, if T is consistent, then this extended theory T_0 also is consistent.

If T_0 would be inconsistent, then, for some formula C, we could obtain a proof of $[T_0]$: $\vdash C \wedge \neg C$. If, in this proof, object constants from the set $\{d_1, d_2, d_3, ...\}$ would not appear at all, then, in fact, we had a proof of [T]: $\vdash C \wedge \neg C$, i.e. we could conclude that T is inconsistent. But, if the new object constants do appear in the proof of $[T_0]$: $\vdash C \wedge \neg C$? Then, let us replace these constants by any variables of T that do not appear in this proof (this is possible, since each predicate language contains a countable set of object variables). After these substitutions, the proof becomes a valid proofs of T, because:

a) The logical axioms remain valid.

b) The non-logical axioms of T do not contain the object constants d_1, d_2, d_3, ..., i.e. they do not change.

c) Applications of inference rules MP and Gen remain valid.

Hence, [T]: $\vdash C' \wedge \neg C'$, where the formula C' has been obtained from C by

the above substitutions. I.e., if T_0 would be inconsistent, then T also would be inconsistent.

Step 2. The model we are building must contain all "objects" whose existence can be proved in T_0. Idea #2: for each formula F of T_0 having exactly one free variable (for example, x) let us add to the theory T_0 the axiom $\exists xF(x)\rightarrow F(d_i)$, where the constant d_i is unique for each F. If T_0 proves $\exists xF(x)$, then this d_i will represent in our model the "object" x having the property F. Let us prove that, if T is consistent, then this extended theory T_1 also is consistent. Note that in T_1 the same language is used as in T_0.

To implement the Idea #2 correctly, first let us use the algorithm of the Exercise 4.3.4 printing out the sequence F_0, F_1, F_2, ... of all formulas in the language of T_0, and let us run through this sequence, processing only those formulas F_i that have exactly one free variable. Let us assign to each such formula F_i a unique constant $d_{c(i)}$ in such a way that $d_{c(i)}$ does not appear neither in the non-logical axioms of T, nor in F_i, nor in the axioms $\exists yF_j(y)\rightarrow F_j(d_{c(j)})$ for all formulas F_j preceding F_i in the sequence F_0, F_1, F_2, ... And, if x is the (only) free variable of F_i, let us adopt $\exists xF_i(x)\rightarrow F_j(d_{c(i)})$ as an axiom of T_1.

Now, let us assume that the extended theory T_1 is inconsistent, i.e. that, for some formula C of T_0, we have a proof of $[T_1]$: $\vdash C\wedge\neg C$. In these proofs, only a finite number n of axioms $\exists xF(x)\rightarrow F(d_{c(F)})$ could be used. If n=0, then we have $[T_0]$: $\vdash C\wedge\neg C$, i.e. then T_0 is inconsistent.

If n>0, then let us mark the axiom $\exists xF(x)\rightarrow F(d_{c(F)})$ with F having the largest index in the sequence F_0, F_1, F_2, ... And, in the proof of $[T_1]$: $\vdash C\wedge\neg C$, let us replace the constant c(F) by some variable y that does not appear in this proof (this is possible, since each predicate language contains a countable set of variables). After this substitution, the proof remain a valid proof of T_1, because:

a) The logical axioms remain valid.

b) The non-logical axioms of T do not contain the constant c(F), i.e. they do not change.

c) The axiom $\exists xF(x)\rightarrow F(d_{c(F)})$ becomes $\exists xF(x)\rightarrow F(y)$. Since F does not contain the constant c(F), the premise $\exists xF(x)$ does not change.

d) The remaining n-1 axioms $\exists yF_j(y)\rightarrow F_j(d_{c(j)})$ of T_1 do not contain the constant c(F), i.e. they do not change.

e) Applications of inference rules MP and Gen remain valid.

Thus we have now another proof of a contradiction − [T_1]: $\vdash C' \wedge \neg C'$, where the formula C' has been obtained from C by substituting y for c(F). Let us remind Theorem 2.4.1(a):

$$\text{If } A_1, A_2, ..., A_n, B \vdash C \wedge \neg C \text{ , then } A_1, A_2, ..., A_n \vdash \neg B.$$

Let us take the formula $\exists xF(x)\rightarrow F(y)$ for B, and C'- for C. Thus, there is a proof of $\neg(\exists xF(x)\rightarrow F(y))$, where only logical axioms, non-logical axioms of T, and the remaining n-1 axioms $\exists yF_j(y)\rightarrow F_j(d_{c(j)})$ of T_1 are used. Let us remind the Exercise 2.6.3(b) [L_1-L_{11}, MP]: $\vdash \neg(A \rightarrow B) \leftrightarrow A \wedge \neg B$. Thus, $\neg(\exists xF(x)\rightarrow F(y))$ is equivalent to $\exists xF(x) \wedge \neg F(y)$, and, in fact, we have a proof of $\exists xF(x)$, and a proof of $\neg F(y)$. By applying Gen to the second formula, we obtain a proof of $\forall y\neg F(y)$, which is equivalent to $\neg\exists yF(y)$ (indeed, let us remind Section 3.2, Table 3.2, Group IV, constructively, $\vdash \forall x\neg B \leftrightarrow \neg\exists xB$). By Replacement Theorem 3, $\neg\exists yF(y)$ is equivalent to $\neg\exists xF(x)$. Thus, we have a proof of a contradiction $\exists xF(x) \wedge \neg \exists xF(x)$, where only logical axioms, non-logical axioms of T, and the remaining n-1 axioms $\exists yF_j(y)\rightarrow F_j(d_{c(j)})$ of T_1 are used.

Let us repeat the above chain of reasoning another n-1 times to eliminate **all** occurrences of the axioms $\exists xF(x)\rightarrow F(d_{c(F)})$ from our proof of a contradiction. In this way we obtain a proof of a contradiction in T_0, which is impossible (see Step 1). Hence, T_1 is a consistent theory.

Step 3. Idea #3: let us use the (non-constructive!) Lindenbaum's lemma, and extend T_1 to a consistent complete theory T_2. Note that in T_2 the same language is used as in T_0.

Step 4. Let us define an interpretation M of the language of T_0, in which all theorems of T_2 will be always true. Since all theorems of the initial theory T are theorems of T_2, this will complete our proof.

Idea #4: let us take as the domain D_M of the interpretation M the (countable! − verify!) set of all constant terms of T_0, i.e. terms that do not contain variables (this set of terms is not empty, it contains at least the countable set of object constants added in Step 1). And let us define interpretations of object constants, function constants and predicate constants as follows.

a) The interpretation of each object constant c is the constant c itself.

b) The interpretation of a function constant f is the "syntactic constructor function" f', i.e., if f is an n-ary function constant, and t_1, ..., t_n are constant terms, then the value f'(t_1, ..., t_n) is defined simply as the character string "f(t_1, ..., t_n)" (quotation marks ignored).

c) The interpretation of a predicate constant p is the relation p' such, if p is an n-ary predicate constant, and t_1, ..., t_n are constant terms, then p'(t_1, ..., t_n) is defined as true in M, if and only if T_2 proves p(t_1, ..., t_n) (note that T_2 is a consistent complete theory, i.e. it proves either p(t_1, ..., t_n), or ¬p(t_1, ..., t_n), but not both!).

Step 5. To complete the proof, we must prove that, in the language of T_0, an arbitrary formula G is always true in M, if and only if T_2 proves G (let us denote this, as usual, by $T_2 \vdash G$. This will be proved, if we will prove that, if x_1, ..., x_m is the set of at least all free variables contained in the formula G, and t_1, ..., t_m are constant terms, then G(t_1, ..., t_m) is true in M, if and only if $T_2 \vdash$ G(t_1, ..., t_m). The proof will be by induction.

Induction base: G is an atomic formula p(s_1, ..., s_n), where p is a predicate constant, and the terms s_1, ..., s_n contain some of the variables x_1, ..., x_m. In s_1, ..., s_n, let us substitute for x_1, ..., x_m the terms t_1, ..., t_m respectively. In this way we obtain constant terms s'_1, ..., s'_n. Thus G(t_1, ..., t_m) is simply p(s'_1, ..., s'_n). By definition (see Step 4), p(s'_1, ..., s'_n) is true, if and only if $T_2 \vdash$p(s'_1, ..., s'_n), i.e., if and only if $T_2 \vdash$G(t_1, ..., t_m). Q.E.D.

Induction step.

Note. Since, T_2 is a complete consistent theory, for any closed formula F, T_2 proves either F, or ¬F (but not both). Hence, if we know that F is true in M, if and only if $T_2 \vdash$F, then we can conclude that F is false in M, if and only if $T_2 \vdash$¬F. Indeed, if F is false, then F is not true, i.e. T_2 does not prove F, i.e. $T_2 \vdash$ ¬F. And, if $T_2 \vdash$¬F, then T_2 does not prove F, i.e. F is not true, i.e. F is false. And conversely: if we know that F is false in M, if and only if $T_2 \vdash$¬F, then we can conclude that F is true in M, if and only if $T_2 \vdash$F. Indeed, if F is true, then ¬F is not true, i.e. T_2 does not prove ¬F, i.e. $T_2 \vdash$F. And, if $T_2 \vdash$F, then T_2 does not prove ¬F, i.e. F is not false, i.e. F is true.

Case 1: G is ¬H. Then, according to the classical truth tables, $G(t_1, ..., t_m)$ is true in M, if and only if $H(t_1, ..., t_m)$ is false in M. By the induction assumption, $H(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash H(t_1, ..., t_m)$. Then, by the above note, since $H(t_1, ..., t_m)$ is a closed formula, $H(t_1, ..., t_m)$ is false in M, if and only if $T_2 \vdash \neg H(t_1, ..., t_m)$, i.e., if and only if $T_2 \vdash G(t_1, ..., t_m)$. Q.E.D.

Case 2: G is H→K. Then, according to the classical truth tables, $G(t_1, ..., t_m)$ is false in M, if and only if $H(t_1, ..., t_m)$ is true in M, and $K(t_1, ..., t_m)$ is false in M. By the induction assumption, $H(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash H(t_1, ..., t_m)$, and $K(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash K(t_1, ..., t_m)$. By the above note, $K(t_1, ..., t_m)$ is false in M, if and only if $T_2 \vdash \neg K(t_1, ..., t_m)$. Hence,

$G(t_1, ..., t_m)$ is false in M, if and only if

$T_2 \vdash H(t_1, ..., t_m)$, and $T_2 \vdash \neg K(t_1, ..., t_m)$,

or,

$G(t_1, ..., t_m)$ is true in M, if and only if

not ($T_2 \vdash H(t_1, ..., t_m)$, and $T_2 \vdash \neg K(t_1, ..., t_m)$).

Let us remind Theorem 2.2.1 and Exercise 2.6.3(a), [L_1-L_{11}, MP]: $\vdash (A \to B) \leftrightarrow \neg (A \wedge \neg B)$. In T_2, all the axioms of the classical logic are adopted, hence (verify!),

$G(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash H(t_1, ..., t_m) \to K(t_1, ..., t_m)$,

or,

$G(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash G(t_1, ..., t_m)$.

Q.E.D.

Case 3: G is $H \wedge K$. Then, according to the classical truth tables, $G(t_1, ..., t_m)$ is true in M, if and only if $H(t_1, ..., t_m)$ is true in M, and $K(t_1, ..., t_m)$ is true in M. By the induction assumption, $H(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash H(t_1, ..., t_m)$, and $K(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash K(t_1, ..., t_m)$. Let us remind Theorem 2.2.1. In T_2, all the axioms of the classical logic are adopted, hence (verify!),

$G(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash H(t_1, ..., t_m) \wedge K(t_1, ..., t_m)$,

or,

$G(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash G(t_1, ..., t_m)$.

Q.E.D.

Case 4: G is $H \vee K$. Then, according to the classical truth tables, $G(t_1, ..., t_m)$ is false in M, if and only if $H(t_1, ..., t_m)$ is false in M, and $K(t_1, ..., t_m)$ is false in M. By the induction assumption, and by the above note, $H(t_1, ..., t_m)$ is false in M, if and only if $T_2 \vdash \neg H(t_1, ..., t_m)$, and $K(t_1, ..., t_m)$ is false in M, if and only if $T_2 \vdash \neg K(t_1, ..., t_m)$. Let us remind Theorem 2.2.1 and Theorem 2.4.10(b): $[L_1\text{-}L_{10}, MP] \vdash \neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$ (the so-called Second de Morgan Law). In T_2, all the axioms of the classical logic are adopted, hence (verify!),

$G(t_1, ..., t_m)$ is false in M, if and only if $T_2 \vdash \neg(H(t_1, ..., t_m) \vee K(t_1, ..., t_m))$,

or, $G(t_1, ..., t_m)$ is false in M, if and only if $T_2 \vdash \neg G(t_1, ..., t_m)$. Thus, by the above note, $G(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash G(t_1, ..., t_m)$. Q.E.D.

Case 5: G is $\exists x H$. Then, by definition, $G(t_1, ..., t_m)$ is true in M, if and only if $H(x, t_1, ..., t_m)$ is "true for some x", i.e., if and only if $H(t, t_1, ..., t_m)$ is true in M for some constant term t in M. By the induction assumption, $H(t, t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash H(t, t_1, ..., t_m)$. Let us remind our above Step 2. Since $H(x, t_1, ..., t_m)$ is a formula containing exactly one free variable, in T_2 we have an axiom $\exists x H(x, t_1, ..., t_m) \rightarrow H(c_H, t_1, ..., t_m)$, where c_H is an object constant.

First, let us assume that $G(t_1, ..., t_m)$ is true in M. Then $H(t, t_1, ..., t_m)$ is true in M for some constant term t in M, hence, $T_2 \vdash H(t, t_1, ..., t_m)$ for this particular t. Let us remind the axiom L_{13}: $F(t) \rightarrow \exists x F(x)$. Since t is a constant term, this axiom is valid for t. We need the following instance of L_{13}: $H(t, t_1, ..., t_m) \rightarrow \exists x H(x, t_1, ..., t_m)$. In T_2, all the axioms of the classical logic are adopted, hence, $T_2 \vdash H(t, t_1, ..., t_m) \rightarrow \exists x H(x, t_1, ..., t_m)$, and, by MP, $T_2 \vdash \exists x H(x, t_1, ..., t_m)$, i.e. $T_2 \vdash G(t_1, ..., t_m)$. Q.E.D.

Now, let us assume that $T_2 \vdash G(t_1, ..., t_m)$, i.e. $T_2 \vdash \exists x H(x, t_1, ..., t_m)$. By the above-mentioned axiom, $T_2 \vdash \exists x H(x, t_1, ..., t_m) \rightarrow H(c_H, t_1, ..., t_m)$, where c_H is

an object constant. Thus, by MP, $T_2 \vdash H(c_H, t_1, ..., t_m)$. Since c_H is a constant term, by the induction assumption, if $T_2 \vdash H(c_H, t_1, ..., t_m)$, then $H(c_H, t_1, ..., t_m)$ is true in M. Hence, $H(c_H, t_1, ..., t_m)$ is true in M, i.e. $H(x, t_1, ..., t_m)$ is true "for some x", i.e. $\exists x H(x, t_1, ..., t_m)$ is true in M, i.e. $G(t_1, ..., t_m)$ is true in M. Q.E.D.

Case 6: G is $\forall x H$. By definition, $G(t_1, ..., t_m)$ is true in M, if and only if $H(x, t_1, ..., t_m)$ is "true for all x", i.e., if and only if $H(t, t_1, ..., t_m)$ is true in M for all constant terms t in M. By the induction assumption, $H(t, t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash H(t, t_1, ..., t_m)$.

Let us prove that

$$G(t_1, ..., t_m) \text{ is false in M, if and only if } T_2 \vdash \exists x \neg H(x, t_1, ..., t_m).$$

First, let us assume that $G(t_1, ..., t_m)$ is false in M. Then $H(t, t_1, ..., t_m)$ is false in M for some constant term t in M. By the induction assumption, and by the above note, $T_2 \vdash \neg H(t, t_1, ..., t_m)$. As in the Case 5, let us remind the axiom L_{13}: $\neg H(t, t_1, ..., t_m) \rightarrow \exists x \neg H(x, t_1, ..., t_m)$. In T_2, all the axioms of the classical logic are adopted, hence, by MP, $T_2 \vdash \exists x \neg H(x, t_1, ..., t_m)$.

Now, let us assume that $T_2 \vdash \exists x \neg H(x, t_1, ..., t_m)$. As in the Case 5, let us remind the axiom $\exists x \neg H(x, t_1, ..., t_m) \rightarrow \neg H(c_{\neg H}, t_1, ..., t_m)$, where $c_{\neg H}$ is an object constant. Hence, by MP, $T_2 \vdash \neg H(c_{\neg H}, t_1, ..., t_m)$, i.e. T_2 does not prove $H(c_{\neg H}, t_1, ..., t_m)$. Then, by the induction assumption, $H(c_{\neg H}, t_1, ..., t_m)$ is false in M, i.e. $\forall x H(x, t_1, ..., t_m)$ is false in M, i.e $G(t_1, ..., t_m)$ is false in M.

Thus, we know that $G(t_1, ..., t_m)$ is true in M, if and only if T_2 does not prove $\exists x \neg H(x, t_1, ..., t_m)$. Since T_2 is a complete theory, $G(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash \neg \exists x \neg H(x, t_1, ..., t_m)$. Now, let us remind from Section 3.2, Table 3.2, Group I, [L_1-L_{11}, L_{12}-L_{15}, MP, Gen]: $\vdash \neg \exists x \neg B \leftrightarrow \forall x B$. In T_2, all the axioms of the classical logic are adopted, hence, $T_2 \vdash \neg \exists x \neg H(x, t_1, ..., t_m)$, if and only if $T_2 \vdash \forall x H(x, t_1, ..., t_m)$, i.e. $G(t_1, ..., t_m)$ is true in M, if and only if $T_2 \vdash G(t_1, ..., t_m)$. Q.E.D.

This completes the proof of the Model Existence Theorem. Q.E.D.

Attention: non-constructive reasoning! The above construction of the model M seems to be "almost constructive". The domain D_M consists of all constant terms from the language of T_0. The axiom set of T_1 is algorithmically solvable

(verify!). The interpretations of function constants are computable functions (verify!). But the interpretations of predicate constants? We interpreted each predicate constant p as the relation p' such that p'(t_1, ..., t_n) is true, if and only if T_2 proves p(t_1, ..., t_n). This relation would be, in general, not algorithmically solvable, even if the axiom set of T_2 would be solvable! But, in general, the axiom set of theory T_2 (obtained by means of Lindenbaum's Lemma) is not algorithmically solvable! Thus, our construction of the model M is essentially non-constructive.

Exercise 4.3.5 (optional, for smart students). Verify that the "degree of non-constructiveness" of the Model Existence Theorem is Δ_2^0 in the so-called arithmetical hierarchy. This became possible due to the improvements introduced by G. Hasenjäger. Hint: verify that all the functions necessary for the proof are "computable in the limit". A function *f(x)* is called computable in the limit, if and only if there is a computable function *g(x,y)* such that, for all x, $f(x)=\lim_{y\to\infty} g(x,y)$).

Consequences of Gödel's Completeness Theorem

Notion of Logical Consequence

As noted above (Exercise 4.1.6), some formula G is a "logical consequence" of the formulas A_1, ..., A_n, if and only if the formula A_1, ..., $A_n \to G$ is logically valid, hence, by Gödel's Completeness Theorem – if and only if, **G can be derived from A_1, ..., A_n by using the axioms and rules of inference of the classical logic.** This completes the formalization of the somewhat mystical notion of "logical consequence", and allows to consider reasoning as a process that could be performed by using computers (see below).

Consistency and Satisfiability

A set of formulas F_1, ..., F_n is called **inconsistent**, if and only if a contradiction (i.e. a formula $B \wedge \neg B$) can be derived from it. For example, the set {B, B→C, C→¬B} is inconsistent (verify).

The Model Existence Theorem allows to connect the notions of consistency and satisfiability.

Exercise 4.3.6. Verify, that a set of formulas in a predicate language: a) is consistent in the classical logic, if and only if it is satisfiable, b) is inconsistent in the classical logic, if and only if it is unsatisfiable. (Hint: use the result of

Exercise 4.1.1).

Computational Complexity of the Problem

Corollary 4.3.4. In any **predicate** language the set of all logically valid formulas is algorithmically enumerable. I.e. given a language L, we can write an algorithm that (working *ad infinitum*) prints out all the logically valid formulas of L (and only these formulas).

Proof. Immediately from Exercise 1.1.4 and Gödel's Completeness Theorem.

This makes Gödel's Completeness Theorem very significant: it shows that the "doubly non-constructive" notion of logically valid formula is at least 50% constructive – **semi-constructive**! Semi-feasible for computers!

Still, unfortunately, this notion appears to be not 100% constructive. In 1936, Alonzo Church proved that at least some predicate languages do not allow an algorithm determining, is a given formula logically valid or not (i.e. an algorithm solving the famous *Entscheidungsproblem* – the decision problem):

A. Church. A note on the Entscheidungsproblem. "Journal of Symb. Logic", 1936, vol.1, pp.40-41

After this, Laszlo Kalmar in

L. Kalmar. Die Zurueckfuehrung des Entscheidungsproblems auf den Fall von Formeln mit einer einzigen, binären Funktionsvariablen. "Compositio Math.", 1936, Vol.4, pp.137-144

improved Church's result:

Church-Kalmar Theorem. If a predicate language contains **at least one predicate constant that is at least binary**, then this language does not allow an algorithm determining, is a given formula of this language logically valid or not.

Thus, none of serious predicate languages allows such an algorithm (languages of PA and ZF included). For details, Mendelson [1997].

Sometimes, this fact (the 50% constructiveness of the notion of the logical validity) is expressed a follows: the logical validity of predicate formulas is **semi-decidable**.

Corollary 4.3.5. If a predicate language contains at least one predicate constant that is at least binary, then this language does not allow an algorithm determining, does some formula G of this language follow from some other formulas A_1, ..., A_n. In other words – the **task of reasoning in such a language is not algorithmically solvable.**

Exercise 4.3.7. Verify, this.

Church-Kalmar Theorem and Knowledge Bases

If we will build our knowledge base by using some predicate language, then, in general, the situation will be as follows:

a) We will have some set of constants registered in the knowledge base: **object constants**: c_1, c_2, ..., c_k, **function constants** and **predicate constants**: p_1, p_2, ..., p_m (with argument numbers specified).

[The **Closed World Assumption**: in the world, there exist only objects denoted by our constants c_1, c_2, ..., c_k. In fact, this assumption should be represented as an axiom $\forall x\ (x=c_1 \vee x=c_2 \vee ... \vee x=c_k)$. The **Open World Assumption**: in the world, there exist more objects than denoted by our object constants.]

b) **Facts, concepts** and **rules** ("laws") are stored in the knowledge base as a set of formulas F_1, F_2, ..., F_n. Facts are represented as atomic formulas, without or with negation, that do not contain variables: $p_i(c_{j1}, c_{j2}, ..., c_{js})$, or $\neg p_i(c_{j1}, c_{j2}, ..., c_{js})$. Facts build up a kind of "database tables". Some of the rules may serve as *integrity conditions*.

c) A **query** is simply another formula ?G. Answering of such a query means that the *query processor* of the knowledge base must determine, does G (or, maybe, $\neg$G) follow from the formulas F_1, F_2, ..., F_n, stored in the knowledge base.

If G contains a free variable x, then the query ?G(x) means the following: return all the object constants c_i, for which the formula $G(c_i)$ follows from F_1, F_2, ..., F_n.

Note. Two different strategies may be used when building a knowledge base. The so-called **Closed World Assumption** is typical for the traditional databases: the predicate $p_i(c_{j1}, c_{j2}, ..., c_{js})$ is regarded as **true**, if and only if the formula $p_i(c_{j1}, c_{j2}, ..., c_{js})$ is stored in the knowledge base. If there is no such formula in the knowledge base, then $p_i(c_{j1}, c_{j2}, ..., c_{js})$ is regarded as **false**. For example, if the knowledge base does not contain the formula *Father(John, Britney)*, then it is assumed that John is **not** father of Britney.

For knowledge bases more natural is the so-called **Open World Assumption**: if neither the formula $p_i(c_{j1}, c_{j2}, ..., c_{js})$, nor the formula $\neg p_i(c_{j1}, c_{j2}, ..., c_{js})$ is stored in the knowledge base, then the truth-value of the predicate $p_i(c_{j1}, c_{j2}, ..., c_{js})$ is regarded as **unknown**. However, a definite truth value of $p_i(c_{j1}, c_{j2}, ..., c_{js})$ may follow from other formulas stored in the knowledge base. For

example, if the knowledge base does not contain neither the formula *Father(John, Britney)*, nor ¬*Father(John, Britney),* then it is assumed that John **is not known** to be (but may be) father of Britney, unless the answer follows from other formulas stored in the database.

Thus, to build the query processor of our knowledge base, we must use some algorithm allowing to determine (as fast as possible), given the formulas F_1, F_2, ..., F_n, G, does G follow from F_1, F_2, ..., F_n, or not. Let's call this task the **reasoning task**.

According to Gödel's Completeness Theorem, G follows from F_1, F_2, ..., F_n, if and only if

$$[L_1\text{-}L_{15}, MP, Gen]: F_1, F_2, ..., F_n \vdash G,$$

i.e. if G can be derived fromF_1, F_2, ..., F_n in the classical predicate logic. This makes the reasoning task at least semi-feasible for computers (in the sense of Corollary 4.3.4). However,

Corollary (of the Church-Kalmar theorem, Corollary 4.3.5). If, when building a knowledge base, we will use the full power of some predicate language (containing at least one predicate constant that is at least binary), then the reasoning task will not be algorithmically solvable, and – for such a knowledge base – **we will fail to build a universal query processor**.

Thus, to build a really usable knowledge base, we must restrict somehow our predicate language to make the reasoning task solvable. For a successful attempt to do this see the so-called description logics.

Skolem's Paradox

Initially, the Model Existence Theorem was proved in a weaker form in 1915 (by Leopold Löwenheim) and 1919 (by Thoralf Skolem): if a first order theory has a model, then it has a finite or countable model (the famous **Löwenheim-Skolem theorem**). Proof (after 1949): if T has a model, then T is consistent, i.e. T has a finite or countable model.

L. Löwenheim. Ueber Möglichkeiten im Relativkalkuel. "Mathematische Annalen", 1915, Vol.76, pp.447-470.

Th. Skolem. Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit und Beweisbarkeit mathematischen Sätze nebst einem Theoreme über dichte Mengen. *Videnskabsakademiet i Kristiania, Skrifter* I, No. 4, 1920, pp. 1-36.

Löwenheim-Skolem theorem (and the Model Existence theorem) is steadily provoking the so-called **Skolem's Paradox**, first noted by Skolem in his address before the 5th Congress of Scandinavian Mathematicians (July 4-7,

1922):

Th. Skolem. Einige Bemerkungen zur axiomatischen Begruendung der Mengenlehre. *Matematikerkongressen i Helsingfors den 4-7 Juli 1922, Den femte skandinaviska matematikerkongressen, Redogörelse*, Akademiska Bokhandeln, Helsinki, 1923, pp. 217-232.

Skolem called the effect "relativity of set-theoretic notions". In all formal set theories (for example, in ZF) we can prove the existence of uncountable sets. Still, according to the Model Existence theorem, if our formal set theory consistent, then there is a countable model where all its axioms and theorems are true. I.e. a theory proves the existence of uncountable sets, yet it has a countable model! Is this possible? Does it mean that all formal set theories are inconsistent? Platonists put it as follows: any consistent axiomatic set theory has countable models, hence, no axiom system can represent our "intended" set theory (i.e. the Platonist "world of sets") adequately.

For a formalist, Skolem's Paradox is not a paradox at all. I would rather call it Skolem's Effect − like as the photo-effect, it is simply a **striking phenomenon**. Indeed, let J be a countable model of our formal set theory. In this theory, we can prove that the set r of all real numbers is uncountable, i.e.

$$\neg\exists f\,(f \text{ is 1-1 function from r into w}), \qquad (1)$$

where w is the set of all natural numbers. What is the meaning of this theorem in the countable model J? Interpretations of r and w are subsets of the domain D_J, i.e. they both are countable sets, i.e.

$$\exists f\,(f \text{ is 1-1 function from } r_J \text{ into } w_J). \qquad (2)$$

Interpretation of (1) in J is

$$\neg\exists f((\ f \in D_J\ ) \text{ and } (f \text{ is 1-1 function from } r_J \text{ into } w_J)).$$

Hence, the mapping f of (2) does exist, yet it exists **outside the model** J! Do you think that f of (2) "must" be located in the model? Why? If you are living (as an "internal observer") within the model J, the set r_J seems uncountable to you (because you cannot find a 1-1 function from r_J into w_J in your world J). Still, for me (an "external observer") your uncountable r_J is countable − in my world I have a 1-1 function from r_J into w_J!

Hence, indeed, Skolem's Paradox represents simply a *striking phenomenon*. It is worth of knowing, yet there is no danger in it.

Added February 9, 2007.

The inter-relationship of the Completeness Theorem and Model Existence Theorem can be represented in the following very general way.

Let us replace:

- Predicate language L − by any set S of "formulas".

- First order theory T − by any "formula" of S (assume, T contains only a finite number of axioms, and take the conjuction of them).

- The notion of interpretation − by any set M and a "predicate" T(m, F) (where m is member of M, and F − a formula of S). If you wish, you may read T(m, F) as "m makes F true", i.e. m is a "model" of F.

- The notion of provability in the classical logic − by a "predicate" P(F) (where F is a formula of S). If you wish, you may read P(F) as "F is provable in the classical logic".

Assume, for a a set of "formulas" S, we have any set M and any two "predicates" T(m, F) and P(F) (where m is a member of M, and F − a formula of S) such that only the following simple principles hold:

a) For all F, $F \in S \rightarrow \neg\neg F \in S$ (i.e. S is closed under negation).

b) For all $m \in M$ and $F \in S$, T(m, F) ↔¬T(m, ¬F).

c) For all $F \in S$, ¬¬P(¬¬F) → P (F).

"Completeness Theorem". For all F, ∀m T(m, F) → P(F).

"Model Existence Theorem". For all F, ¬P(¬F) → ∃m T(m, F).

Theorem. If a, b, c) hold, then the above "theorems" are equivalent.

Proof. 1) Assume ∀m T(m, F) → P(F) for all F. Then ¬P(F) → ¬∀m T(m, F), and by a) also, ¬P(¬F) → ¬∀m T(m, ¬F) → ∃m¬T(m, ¬F) → ∃m T(m, F) by b). Q.E.D.

2) Assume ¬P(¬F) → ∃m T(m, F) for all F.Then ¬∃m T(m, F) → ¬¬P(¬F), and by a) also ¬∃m T(m, ¬F) → ¬¬P(¬¬F). By b), ∀m T(m, F) → ∀m ¬T(m, ¬F) → ¬∃m T(m, ¬F) → ¬¬P(¬¬F) → P(F) by c). Q.E.D.

4.4. Constructive Propositional Logic – Kripke Semantics

Saul Aaron Kripke (born 1940).

"American logician and philosopher Saul Kripke is one of today's leading thinkers on thought and its manifold relations to the world. His name is attached to objects in several fields of logic from Kripke-Platek axioms in higher recursion theory to the "Brouwer-Kripke scheme" in intuitionistic mathematics. Kripke models for modal logic, a discovery he made in his teen-age years, became part of the standard vocabulary of mathematical logicians after his first article appeared in 1963, when he was just 23 years old. Kripke models and the results that depend upon them are cited today not only in philosophy and logic, but also in linguistics and computer science..." (The

Gazette. The newspaper of the John Hopkins University, May 12, 1997, Vol.26, N 34)

S. Kripke (1963). Semantical Considerations on Modal Logic, *Acta Philosophica Fennica* **16**: 83-94.

S. Kripke (1963). Semantical analysis of modal logic. I. Normal modal propositional calculi. *Z. Math. Logik Grundl. Math.*, 9:67-96, 1963.

S. Kripke (1965). Semantical analysis of intuitionistic logic. In: *J. N. Crossley, M. A. E. Dummet (eds.), Formal systems and recursive functions*. Amsterdam, North Holland, 1965, pp.92-129.

As usual, let us assume, the formula F has been built of "atomic" formulas B_1, B_2, ..., B_n by using propositional connectives only. Instead of simply computing truth values of F from truth values of B_1, B_2, ..., B_n, Kripke proposed to consider the **behavior** of F when the truth values of B_1, B_2, ..., B_n are changing gradually **from false to true** according to some "scenario".

Thus, Kripke proposed to replace the classical semantics (interpretation) of the propositional connectives (defined by the classical truth tables) by a more complicated "dynamic" semantics.

Instead of simply saying that ¬F is true, iff F is false, let us say that, at some point in a scenario, ¬F is true, if and only if, at this point, F is false and remains false, when truth values of B_1, B_2, ..., B_n are changing according to the scenario.

Let o stand for implication, conjunction or disjunction. Instead of simply saying that FoG is true, if and only if, FoG is true according to the classical truth tables, let us say that, at some point in a scenario, FoG is true, if and only if, at this point, it is true and remains true, when truth values of B_1, B_2, ..., B_n are changing according to the scenario.

Example 4.4.1. Let us consider the behavior of the classical axiom L_{11}: $B \vee \neg B$ in the scenario, where, at first, B is false, and at the next step it becomes true:

0 ---------- 1

At the starting point, B is false, ¬B also is false (here, for ¬B to be true, B must remain false at the next step, but it doesn't). This means that, at the starting point, $B \vee \neg B$ is false. At the next step: B is true, hence, ¬B is false, but, of course, $B \vee \neg B$ is true. Thus, in Kripke scenarios, $B \vee \neg B$ is **not** always true. Surprisingly, some time later (Lemma 4.4.3), we will derive from this simple fact that Bv¬B cannot be proved in the constructive logic (we

already know this from Section 2.8).

Example 4.4.2. Let us consider the behavior of the (only) classically provable half of the First de Morgan Law: $\neg(A\wedge B)\rightarrow\neg A\vee\neg B$ in the scenario, where, at first A and B are both false, and at the next step, two branches appear in the scenario: in the first branch: A remains false, and B becomes true, in the second branch: A becomes true, and B remains false:

$$\begin{array}{c} \vdash\!\!-01 \\ 00\text{-}\vdash\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!- \\ \vdash\!\!-10 \end{array}$$

At the starting point: A is false, ¬A – also is false (for ¬A to be true, A must remain false at the next step, but in the second branch it doesn't). Similarly, at the starting point: B is false, ¬B – also false (for ¬B to be true, B must remain false at the next step, but in the first branch it doesn't). This means that, at the starting point, $\neg(A\wedge B)$ is true (because $A\wedge B$ is false, and it remains false in both branches), and $\neg A\vee\neg B$ is false, hence, $\neg(A\wedge B)\rightarrow\neg A\vee\neg B$ is false. Thus, in Kripke scenarios, $\neg(A\wedge B)\rightarrow\neg A\vee\neg B$ is **not** always true. Surprisingly, some time later (Lemma 4.4.3), we will derive from this simple fact that the this half of the First de Morgan Law cannot be proved in the constructive logic. We failed to do this in Section 2.8!

Exercise 4.4.1. Investigate, in appropriate scenarios, the behavior of the following (only) classically provable formulas:

$$\neg\neg(A\vee B)\rightarrow\neg\neg A\vee\neg\neg B\ ,$$
$$(A\rightarrow B)\rightarrow((\neg A\rightarrow B)\rightarrow B),$$
$$(A\rightarrow B)\vee(B\rightarrow A)\ ,$$

and verify that, in Kripke scenarios, these formulas are **not** always true. Some time later (Lemma 4.4.3), we will derive from this simple fact that these formulas cannot be proved in the constructive logic. We failed to do this in Section 2.8! (Hint: try the most simple scenarios first: 00--01, 00-10, 00-11, etc.)

More precisely, the definition of the **Kripke semantics** for the propositional logic is as follows. Assume, the formula F has been built of "atomic" formulas B_1, B_2, ..., B_n by using propositional connectives only. Instead of considering truth values of F for all the possible tuples of truth values of B_1, B_2, ..., B_n, let us consider the behavior of F in all the possible Kripke scenarios, defined as follows.

Definition of Kripke scenarios. Each scenario s is a triple (b, ≤, t) of the following objects. First, b is a finite set of objects called nodes (or, states).

The second member ≤ is a partial ordering relationship between the nodes, i.e. for all $x, y, z \in b$: x≤y → (y≤z → x≤z) (transitivity).

The third member t of the tripple is a function (t means "true"). It associates with each node x a "growing" set t(x) of "atomic" formulas, i.e. a subset of $\{B_1, B_2, ..., B_n\}$ in such a way that for all $x, y \in b$: $x \le y \rightarrow t(x) \subseteq t(y)$.

Note. In some other textbooks, Kripke scenarios are called **Kripke models**, or **Kripke structures**.

Let us say that B_i is true at the node x, if and only if B_i is in the set t(x). We will denote this fact by x |= B_i ("at x, B_i is true", or "x forces B_i"). Since t is monotonic, if x |= B_i , then y |= B_i for all y after x, i.e. for all $y \in b$ such that y≥x. I.e. if B_i is true at some node x, then B_i remains true at all nodes after x.

Let us define x |= F ("F is true at x", or "x forces F") for any formula F that has been built of "atomic" formulas B_1, B_2, ..., B_n by using propositional connectives only.

1. Negation. Suppose, the truth value of x |= F is already defined for all $x \in b$. Then x |= ¬F is defined to be true, if and only if, for all $y \ge x \in b$, y |= F is false (i.e. ¬(y |= F) is true according to the classical truth table of the negation connective).

2. Implication, conjunction or disjunction. Suppose, the truth values of x |= F and x |= G are already defined for all $x \in b$. Then x |= FoG is defined to be true, if and only if, for all $y \ge x \in b$, (y |= F)o(y |= G) is true according to the classical truth table of the implication, conjunction or disjunction connective respectively.

Lemma 4.4.1. For any formula F, any Kripke scenario (b, ≤, t), and any node $x \in b$: if x |= F, then y |= F for all $y \in b$ such that y≥x. I.e. if, in a Kripke scenario, a formula becomes true at some node, then it remains true forever after this node.

Proof. By induction.

Induction base. See above: if x |= B_i , then y |= B_i for all y after x, i.e. for all $y \in b$ such that y≥x.

Induction step.

1. Negation. Assume, x |= ¬F, i.e., according to the classical truth table, not y |= F for all $y \ge x \in b$. If y≥x, then is y |= ¬F true or false? By definition, y |= ¬F would be true, if and only if not z |= F for all $z \ge y \in b$. By transitivity of ≤, if z≥y and y≥x, then z≥x. Thus, by our assumption, if z≥y, then not z |= F. Hence, y |= ¬F. Q.E.D.

2. Implication, conjunction or disjunction. Assume, x |= FoG, i.e., according to the corresponding classical truth table, (y |= F)o(y |= G) is true for all $y \geq x \in b$. If y≥x, then is y |= FoG true or false? By definition, y |= FoG would be true, if and only if (z |=F)o(z |= G) would be true for all $z \geq y \in b$. By transitivity of ≤, if z≥y and y≥x, then z≥x. Thus, by our assumption, if z≥x, then (z |= F)o(z |= G) is true. Hence, y |= FoG. Q.E.D.

Exercise 4.4.2. Verify that if x is a maximal node in a scenario (b, ≤, t), then x |= F, if and only if F is true at x according to the classical truth tables.

Kripke established that **a formula is provable in the constructive propositional logic, if and only if it is true at all nodes in all Kripke scenarios.**

Theorem 4.4.2 (S. Kripke, completeness of the constructive propositional logic). A formula F is provable in the constructive propositional logic (i.e. [L_1-L_{10}, MP] ⊢F), if and only if F is true at all nodes in all Kripke scenarios.

As usual, the hard part of the proof is establishing that "true is provable", i.e. if F is true at all nodes in all Kripke scenarios, then [L_1-L_{10}, MP] ⊢ F (see Corollary 4.4.7 below). The easy part of the proof is, as usual, the soundness lemma:

Lemma 4.4.3. If [L_1-L_{10}, MP] ⊢ F, then F is true at all nodes in all Kripke scenarios.

This lemma will follow from

Lemma 4.4.4. If F is any of the constructive axioms L_1-L_{10}, then, for any Kripke scenario (b, ≤, t), and any node $x \in b$: x |= F. I.e. the constructive axioms are true at all nodes in all Kripke scenarios.

and

Lemma 4.4.5. If, in a Kripke scenario (b, ≤, t), at the node $x \in b$: x |= F and x |= F→G, then x |= G. Hence, if F and F→G are true at all nodes in all Kripke scenarios, then so is G.

Proof of Lemma 4.4.3. Indeed, by Lemma 4.4.4, all the constructive axioms L_1-L_{10} are true at all nodes in all scenarios, and, by Lemma 4.4.5, the Modus Ponens rule preserves the property of being "true at all nodes in all scenarios". Q.E.D.

Note. Let us return to the above Example 4.4.2 and Exercise 4.4.1. We established that formulas

$$\neg(A \wedge B) \rightarrow \neg A \vee \neg B \quad ;$$
$$\neg\neg(A \vee B) \rightarrow \neg\neg A \vee \neg\neg B \quad ;$$

$$(A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B)$$

are not true at all nodes in all scenarios. Hence, by Lemma 4.4.3, these formulas cannot be proved in the constructive logic [L_1-L_{10}, MP]. We failed to prove this in Section 2.8!

Proof of Lemma 4.4.5. We know that x |= F→G means that (y |= F)→(y |= G) is true (according to the classical truth table) for all $y \geq x \in b$. By Lemma 4.4.1, we know that y |= F for all $y \geq x \in b$. Hence, if y |= G would be false, then (y |= F)→(y |= G) also would be false. Hence, x |= G. Q.E.D.

Proof of Lemma 4.4.4.

L_1: B→(C→B)

x |= B→(C→B) is true, if and only if (y |= B)→(y |= C→B) is true for all y≥x.

x |= B→(C→B) is false, if and only if (y |= B)→(y |= C→B) is false for some y≥x.

How could (y |= B)→(y |= C→B) be false for some y≥x? According to the classical implication truth table, this could be only, if and only if y |= B is true, and y |= C→B is false.

y |= C→B is true, if and only if (z |= C)→(z |= B) is true for all z≥y.

y |= C→B is false, if and only if (z |= C)→(z |= B) is false for some z≥y.

How could (z |= C)→(z |= B) be false for some z≥y? According to the classical implication truth table, this could be, if and only if z |= C is true, and z |= B is false.

Summary:

x |= B→(C→B) is false
if and only if
∃y≥x (**y |= B is true** and y |= C→B is false)
if and only if
∃z≥y (z |= C is true and **z |= B is false**)

Hence, if x |= B→(C→B) is false, then there are y and z such that: x≤y≤z, **y |= B is true**, z |= C is true, and **z |= B is false**. By Lemma 4.4.1, if y≤z and y |= B is true, then z |= B is true. Contradiction with "z |= B is false". Thus, x |= B→(C→B) is true.

L_{10}: ¬B→(B→C)

x |= ¬B→(B→C) is false, if and only if (y |= ¬B)→(y |= B→C) is false for some y≥x, i.e. if and only if y |= ¬B is true, and y |= B→C is false.

y |= ¬B is true, if and only if z |=B is false for all z≥y.

y |= B→C is false, if and only if (z |= B)→(z |= C) is false for some z≥y, i.e. if and only if z |= B is true, and z |= C is false.

Summary:

x |= ¬B→(B→C) is false
if and only if
∃y≥x (y |= ¬B is true and y |= B→C is false)
if and only if
∀z≥y (**z |=B is false**) and ∃z≥y (**z |= B is true** and z |= C is false)

Hence, if x |= ¬B→(B→C) is false, then there is y≥x such that: a) ∀z≥y (**z |=B is false**), and b) ∃z≥y (**z |= B is true).** Contradiction. Thus, x |= ¬B→(B→C) is true.

L_3: $B \wedge C \rightarrow B$

x |= $B \wedge C \rightarrow B$ is false
if and only if
∃y≥x (y |= $B \wedge C$ is true and **y |= B is false**)
if and only if
∀z≥y **(z |=B is true** and z |= C is true)

Hence, there is y such that x≤y and y |= B is false. From ∀z≥y **(**z |=B is true) we obtain that y |= B is true. Contradiction. Thus, x |= $B \wedge C \rightarrow C$ is true.

L_4: $B \wedge C \rightarrow C$

Similarly.

L_5: $B \rightarrow (C \rightarrow B \wedge C)$

x |= $B \rightarrow (C \rightarrow B \wedge C)$ is false
if and only if
∃y≥x (**y |=B is true** and y |= $C \rightarrow B \wedge C$ is false)
if and only if
∃z≥y **(z |=C is true** and **z |=** $B \wedge C$ **is false**)

Hence, there are y, z such that x≤y≤z, y |= B is true, and z |= C is true, and **z |=** $B \wedge C$ **is false**. Then, by Lemma 4.4.1, (u |= B is true)and(u |= C) for all u≥z. I.e. **z |=** $B \wedge C$ **is true**. Contradiction. Thus, x |= $B \rightarrow (C \rightarrow B \wedge C)$ is true.

L_6: $B \rightarrow B \vee C$

x |= $B \rightarrow B \vee C$ is false
if and only if
∃y≥x (**y |=B is true** and y |= $B \vee C$ is false)
if and only if

$\exists z \geq y$ (**z |= B is false** and z |= C is false)

Hence, there are y, z such that $x \leq y \leq z$, y |= B is true, and z |= B is false. By Lemma 4.4.1, this is a contradiction. Thus, x |= $B \to B \vee C$ is true.

L_7: $C \to B \vee C$

Similarly.

L_8: $(B \to D) \to ((C \to D) \to (B \vee C \to D))$

x |= $(B \to D) \to ((C \to D) \to (B \vee C \to D))$ is false
if and only if
$\exists y \geq x$ (**y |=B→D is true** and y |= $(C \to D) \to (B \vee C \to D)$ is false)
if and only if
$\exists z \geq y$ (**z |= C→D is true** and z |= $B \vee C \to D$ is false)
if and only if
$\exists u \geq z$ (u |= $B \vee C$ is true and **u |= D is false**)

Hence, there are y, z, u such that $x \leq y \leq z \leq u$, y |= B→D is true, z |= C→D is true, and u |= D is false. By Lemma 4.4.1, u |= B→D is true, and u |= C→D is true. Thus, if u |= B would be true, then u |= D also would be true. Hence, u |= B is false. Similarly, u |= C also is false. Hence, u |= $B \vee C$ is false. But we know that it is true. Contradiction. Thus, x |= L_8 is true.

L_2: (B→(C→D))→((B→C)→(B→D))

x |= (B→(C→D))→((B→C)→(B→D)) is false
if and only if
$\exists y \geq x$ (y |= B→(C→D) is true and y |= (B→C)→(B→D) is false)
if and only if
$\forall z \geq y$ ((z |= B)→(z |= C→D))
if and only if
$\forall u \geq z$ ((u |= B)→(u |= C))

$\exists z \geq y$ (z |= B→C is true and z |= B→D is false)
if and only if
$\exists u \geq z$ (u |= B is true and u |= D is false)

Hence, there are y, z, u such that $x \leq y \leq z \leq u$, u |= B is true and **u |= D is false**. From $\forall u \geq z$ ((u |= B)→(u |= C)) we obtain that u |=C also is true, and from $\forall z \geq y$ ((z |= B)→(z |= C→D)) – that z |= C→D is true. Then, by Lemma 4.4.1, u |= C→D also is true, i.e. $\forall v \geq u$ ((v |= C)→(v |= D)), in particular, (u |= C)→(u |= D). Hence, **u|= D is true**. Contradiction. Thus, x |= L_2 is true.

L_9: (B→C)→((B→¬C)→¬B)

x |= (B→C)→((B→¬C)→¬B) is false
if and only if
$\exists y \geq x$ (y |= B→C is true and y |= (B→¬C)→¬B is false)

if and only if
∀z≥y ((z |= B)→(z |= C))
if and only if
∃z≥y (z |= B→¬C is true and z |= ¬B is false)
if and only if
∀u≥z ((u |= B)→(u |= ¬C))
if and only if
∃u≥z (**u |= B is true**)

Hence, there are y, z, u such that x≤y≤z≤u , and **u |= B is true**. From **∀z≥y ((z |= B)→(z |= C))** we obtain that u |= C is true. From **∀u≥z ((u |= B)→(u |= ¬C))** we obtain that u |= ¬C is true, i.e. v |= C is false for some v≥u. By Lemma 4.4.1, if u |= C is true, then v |= C is true. Contradiction with "v |= C is false". Hence, x |= L_9 is true.

Exercise 4.4.3. Verify that, in the above recursive definition of x |= F, the item

2. Implication, conjunction or disjunction: x |= FoG is defined to be true, if and only if, according to the classical truth tables, (y |= F)o(y |= G) is true for all $y \geq x \in b$.

could be replaced by

2a. Implication ("non-monotonic" connective): x |= F→G is defined to be true, if and only if, according to the classical truth tables, (y |= F)→(y |= G) is true for all $y \geq x \in b$.

2b. Conjunction or disjunction ("monotonic" connectives): x |= FoG is defined to be true, if and only if, according to the classical truth tables, (x |= F)o(x |= G) is true.

End of Exercise 4.4.3.

The Hard Part of the Proof

Now, let us try proving that, if F is true at all nodes in all Kripke scenarios, then F is provable in the constructive propositional logic). We will follow the paper by

Judith Underwood. A constructive Completeness Proof for Intuitionistic Propositional Calculus. TR-90-1179, December 1990, *Department of Computer Science, Cornell University.*

based on the contructions from

Melvin Fitting. Intuitionistic Logic, Model Theory and Forcing. North-Holland, Amsterdam, 1969

The smart idea is to **generalize the problem** in the following way. Instead of considering constructive provability of single formulas, let us consider the constructive provability of $D_1, D_2, ..., D_m \vdash C_1 \vee C_2 \vee ... \vee C_n$ for arbitrary formulas $D_1, D_2, ..., D_m, C_1, C_2, ..., C_n$, i.e. let us consider ordered pairs of

sets ($\{D_1, D_2, ..., D_m\}$, $\{C_1, C_2, ..., C_n\}$). Let us call such pairs **sequents**. If S_1, S_2 are sets of formulas (S_1 may be empty), let us call the sequent (S_1, S_2) constructively provable, if and only if [L_1-L_{10}, MP]: $S_1 \vdash VS_2$, where VS_2 denotes the disjunction of formulas contained in S_2. Moreover, let us consider **sets of sequents**. This will allow to carry out a specific induction argument (considering single formulas or single sequents does not allow such an argument!).

Let us say that a Kripke scenario (b, ≤, t) contains a **counterexample** for the sequent (S_1, S_2), if and only if the sequent is false at some node in the scenario (or, more precisely, if and only if there is $x \in b$ such that x |= F for all formulas $F \in S_1$ and not x |= G for all formulas $G \in S_2$).

Additionally, let us use Corollary 6.1.2(b) of Theorem 6.1.1 to replace all negations ¬F by F→f, where f is an atomic formula, which is "always false", i.e. which, in a sequent (S_1, S_2), never belongs to S_1. Thus, formulas mentioned in the proof of the following Theorem 4.4.6 do not contain negations (but they may contain the specific atomic formula f).

Theorem 4.4.6. For any set S of sequents, either some sequent of S is constructively provable, or there is a Kripke scenario (b, ≤, t), which contains counterexamples for each sequent in S.

Proof. Let us start with a **proof overview**. We will consider the following cases:

Case 1. S contains (S_1, S_2) such that $A \wedge B \in S_1 \wedge \neg(A \in S_1 \wedge B \in S_1)$. Let us consider the set S' obtained from S by adding the "missing" formulas A, B to S_1, i.e. by replacing (S_1, S_2) by ($S_1 \cup \{A, B\}$, S_2). Let us verify that if Theorem is true for S', then it is true for S...

Case 2. S contains (S_1, S_2) such that $A \wedge B \in S_2 \wedge \neg(A \in S_2 \vee B \in S_2)$. Let us consider the following two sets: a) S' – obtained from S by adding the formula A to S_2, i.e. by replacing (S_1, S_2) by (S_1, $S_2 \cup \{A\}$). b) S" – obtained from S by adding the formula B to S_2, i.e. by replacing (S_1, S_2) by (S_1, $S_2 \cup \{B\}$). Let us verify that if Theorem is true for S' and S", then it is true for S...

Case 3. S contains (S_1, S_2) such that $A \vee B \in S_1 \wedge \neg(A \in S_1 \vee B \in S_1)$. Let us consider the following two sets: a) S' – obtained from S by adding the formula A to S_1, i.e. by replacing (S_1, S_2) by ($S_1 \cup \{A\}$, S_2). b) S" – obtained from S by adding the formula B to S_1, i.e. by replacing (S_1, S_2) by ($S_1 \cup \{B\}$, S_2). Let us verify that if Theorem is true for S' and S", then it is true for S...

Case 4. S contains (S_1, S_2) such that $A \vee B \in S_2 \wedge \neg(A \in S_2 \wedge B \in S_2)$. Let us consider the set S' obtained from S by adding the "missing" formulas A, B to S_2, i.e. by replacing (S_1, S_2) by (S_1, $S_2 \cup \{A, B\}$). Let us verify that if Theorem is true for S', then it is true for S...

Case 5. S contains (S_1, S_2) such that $A \rightarrow B \in S_1 \wedge \neg(A \in S_2 \vee B \in S_1)$. Let us consider the following two sets: a) S' – obtained from S by adding the formula A to S_2, i.e. by replacing (S_1, S_2) by (S_1, $S_2 \cup \{A\}$). b) S'' – obtained from S by adding the formula B to S_1, i.e. by replacing (S_1, S_2) by ($S_1 \cup \{B\}$, S_2). Let us verify that if Theorem is true for S' and S'', then it is true for S...

Case 6. S contains (S_1, S_2) such that $A \rightarrow B \in S_2$ and for every sequent $(T_1, T_2) \in S$, $\neg(S_1 \subseteq T_1 \wedge A \in T_1 \wedge B \in T_2)$. Let us consider the set S' obtained from S by adding the sequent ($S_1 \cup \{A\}$, B) to it. Let us verify that if Theorem is true for S', then it is true for S...

Case 7. None of the above cases hold for S. Then, Theorem is true for S – easy to verify...

The first six cases represent the induction argument: proving of Theorem for a sequent set S is reduced to proving it for some other sets – S' and S''. By iterating this reduction, we always arrive happily to the Case 7, where Theorem is easy to verify.

Indeed, let us denote by $universe(S_1, S_2)$ the set of all formulas and sub-formulas (of the formulas) contained in $S_1 \cup S_2$. Let us denote by $universe(S)$ the union of the universes of sequents from S.

Exercise 4.4.4. Verify that:

a) When, in the Cases 1-5, the sequent (S_1, S_2) is replaced by some other sequent (T_1, T_2), then

$$universe(T_1, T_2) \subseteq universe(S_1, S_2) \quad .$$

b) When, in the Case 6, because of the sequent (S_1, S_2), the sequent $(S_1 \cup \{A\}, B)$ is added to S, then

$$universe(S_1 \cup \{A\}, B) \subseteq universe(S_1, S_2) \quad .$$

c) For a given $universe(S)$, there exist no more than $N = 2^{|universe(S)|+1}$ different sequents (S_1, S_2) such that $universe(S1, S2) \leq universe(S)$. And, no more than 2^N different sets of sequents.

Thus, any chain of iterated Cases 1-6 cannot be longer than 2^N+1 – either we will arrive at a set of sequents already built at a previous step, or we will arrive

at the Case 7.

Now – the proof as it should be.

Case 1. S contains (S_1, S_2) such that $A \wedge B \in S_1 \wedge \neg(A \in S_1 \wedge B \in S_1)$. Let us consider the set S' obtained from S by adding the "missing" formulas A, B to S_1, i.e. by replacing (S_1, S_2) by $(S_1 \cup \{A, B\}, S_2)$.

Let us verify that if Theorem is true for S', then it is true for S.

Assume, some sequent of S' is constructively provable, then it is $(S_1 \cup \{A, B\}, S_2)$ or some other sequent. If it is some other sequent, then it belongs to S, i.e. some sequent of S is constructively provable. If $(S_1 \cup \{A, B\}, S_2)$ is constructively provable, then so is (S_1, S_2). Indeed, if $S_1 \cup \{A, B\} \vdash VS_2$ is constructively provable, how to prove $S_1 \vdash VS_2$? Since S_1 contains $A \wedge B$, by axioms L_3 and L_3 we can derive A and B. After this, we can apply the proof of $S_1 \cup \{A, B\} \vdash VS2$. Hence, $S_1 \vdash VS_2$ is constructively provable.

On the other side, if there is a Kripke scenario (b, ≤, t), which contains a counterexample for each sequent in S', then it contains also a counterexample for each sequent in S. Indeed, a sequent in S is either (S_1, S_2), or some other sequent. If it is some other sequent, then it belongs to S', i.e. (b, ≤, t) contains a counterexample for it. Does (b, ≤, t) contain a counterexample also for (S_1, S_2)? We know that it contains a counterexample for $(S_1 \cup \{A, B\}, S_2)$, i.e. for some $x \in b$, x |= F for all formulas $F \in S_1 \cup \{A, B\}$ and not x |= G for all formulas $G \in S_2$. Hence, (b, ≤, t) contains a counterexample also for (S_1, S_2). Q.E.D.

Case 2. S contains (S_1, S_2) such that $A \wedge B \in S_2 \wedge \neg(A \in S_2 \vee B \in S_2)$. Let us consider the following two sets:

a) S' – obtained from S by adding the formula A to S_2, i.e. by replacing (S_1, S_2) by $(S_1, S_2 \cup \{A\})$.

b) S'' – obtained from S by adding the formula B to S_2, i.e. by replacing (S_1, S_2) by $(S_1, S_2 \cup \{B\})$.

Let us verify that if Theorem is true for S' and S'', then it is true for S.

Assume, some sequent of S' and some sequent of S'' is constructively provable. The sequent of S' is $(S_1, S_2 \cup \{A\})$ or some other sequent. If it is some other sequent, then it belongs to S, i.e. some sequent of S is constructively provable.

The sequent of S'' is $(S_1, S_2 \cup \{B\})$ or some other sequent. If it is some other sequent, then it belongs to S, i.e. some sequent of S is constructively provable. So, let us consider the situation, when $(S_1, S_2 \cup \{A\})$ and $(S_1, S_2 \cup \{B\})$ both are constructively provable.

If $S_1 \vdash A \vee S_2$ and $S_1 \vdash B \vee S_2$ both are constructively provable, how to prove $S_1 \vdash VS_2$ (we know that S_2 contains $A \wedge B$)?

By Theorem 2.3.1, conjunction is distributive to disjunction:

$$[L_1\text{-}L_8, MP]: \vdash (A \wedge B) \vee C \leftrightarrow (A \vee C) \wedge (B \vee C) .$$

Hence, $[L_1\text{-}L_8, MP]: (A \vee S_2) \wedge (B \vee S_2) \rightarrow (A \wedge B) \vee S_2$. So, let us merge the proofs of $S_1 \vdash A \vee S_2$ and $S_1 \vdash B \vee S_2$, and let us append the proof of Theorem 2.3.1. Thus, we have obtained a proof of $S_1 \vdash (A \wedge B) \vee S_2$.

From Section 2.3 we know that in $[L_1\text{-}L_8, MP]$ disjunction is associative, commutative and idempotent. And, by Replacement Lemma 1(e):

$[L_1\text{-}L_8, MP]$ $A \leftrightarrow B \vdash A \vee C \leftrightarrow B \vee C$. Since S_2 contains $A \wedge B$, these facts allow, from a proof of $S_1 \vdash (A \wedge B) \vee S_2$, to derive a proof of $S_1 \vdash VS_2$.

On the other side, if there is a Kripke scenario (b, ≤, t), which contains a counterexample for each sequent in S', then it contains also a counterexample for each sequent in S. Indeed, a sequent in S is either (S_1, S_2), or some other sequent. If it is some other sequent, then it belongs to S', i.e. (b, ≤, t) contains a counterexample for it. Does (b, ≤, t) contain a counterexample also for (S_1, S_2)? We know that it contains a counterexample for $(S_1, S_2 \cup \{A\})$, i.e. for some $x \in b$, x |= F for all formulas $F \in S_1$ and not x |= G for all formulas $G \in S_2 \cup \{A\}$. Hence, (b, ≤, t) contains a counterexample also for (S_1, S_2). Q.E.D.

If there is a Kripke scenario (b, ≤, t), which contains a counterexample for each sequent in S'', then it contains also a counterexample for each sequent in S. The argument is similar to the above.

Case 3. S contains (S_1, S_2) such that $A \vee B \in S_1 \wedge \neg(A \in S_1 \vee B \in S_1)$. Let us consider the following two sets:

a) S' – obtained from S by adding the formula A to S_1, i.e. by replacing (S_1, S_2) by $(S_1 \cup \{A\}, S_2)$.

b) S'' – obtained from S by adding the formula B to S_1, i.e. by replacing (S_1, S_2) by $(S_1 \cup \{B\}, S_2)$.

Let us verify that if Theorem is true for S' and S'', then it is true for S.

Assume, some sequent of S' and some sequent of S'' is constructively provable. The sequent of S' is $(S_1 \cup \{A\}, S_2)$ or some other sequent. If it is some other sequent, then it belongs to S, i.e. some sequent of S is constructively provable. The sequent of S'' is $(S_1 \cup \{B\}, S_2)$ or some other sequent. If it is some other sequent, then it belongs to S, i.e. some sequent of S is constructively provable. So, let us consider the situation, when $(S_1 \cup \{A\}, S_2)$ and $(S_1 \cup \{B\}, S_2)$ both are constructively provable.

Let us remind Exercise 2.3.2 [L_1, L_2, L_8, MP]: if A_1, A_2, ..., A_n, B $\vdash$ D, and A_1, A_2, ..., A_n, C $\vdash$ D, then A_1, A_2, ..., A_n , $B \vee C$ $\vdash$ D. Thus, if $S_1 \cup \{A\}$ $\vdash$ VS_2 and $S_1 \cup \{B\}$ $\vdash$ VS_2 both are constructively provable, then (since S_1 contains $A \vee B$) so is $S_1 U \{B\} \vdash VS_2$.

On the other side, if there is a Kripke scenario (b, $\leq$, t), which contains a counterexample for each sequent in S', then it contains also a counterexample for each sequent in S. Indeed, a sequent in S is either (S_1, S_2), or some other sequent. If it is some other sequent, then it belongs to S', i.e. (b, $\leq$, t) contains a counterexample for it. Does (b, $\leq$, t) contain a counterexample also for (S_1, S_2)? We know that it is contains counterexample for $(S_1 \cup \{A\}, S_2)$, i.e. for some $x \in b$, x |= F for all formulas $F \in S_1 \cup \{A\}$ and not x |= G for all formulas $G \in S_2$. Hence, (b, $\leq$, t) contains a counterexample also for (S_1, S_2). Q.E.D.

If there is a Kripke scenario (b, $\leq$, t), which contains a counterexample for each sequent in S'', then it is also contains counterexample for each sequents in S. The argument is similar to the above.

Case 4. S contains (S_1, S_2) such that $A \vee B \in S_2 \wedge \neg(A \in S_2 \wedge B \in S_2)$. Let us consider the set S' obtained from S by adding the "missing" formulas A, B to S_2, i.e. by replacing (S_1, S_2) by $(S_1, S_2 \cup \{A, B\})$.

Let us verify that if Theorem is true for S', then it is true for S.

Assume, some sequent of S' is constructively provable, then it is $(S_1, S_2 \cup \{A, B\})$ or some other sequent. If it is some other sequent, then it belongs to S, i.e. some sequent of S is constructively provable. If $(S_1, S_2 \cup \{A, B\})$ is constructively provable, then so is (S_1, S_2). Indeed, if $S_1 \vdash (A \vee B) \vee S_2$ is constructively provable, how to prove $S_1 \vdash VS_2$ (where S_2 contains $A \vee B$)?

From Section 2.3 we know that in [L_1-L_8, MP] disjunction is associative,

commutative and idempotent. And, by Replacement Lemma 1(e):

[L_1-L_8, MP] A↔B ⊢ $A \vee C \leftrightarrow B \vee C$. Since that S_2 contains AvB, these facts allow, from a proof of $S_1 \vdash (A \vee B) \vee S_2$, to derive a proof of $S_1 \vdash VS_2$.

On the other side, if there is a Kripke scenario (b, ≤, t), which contains a counterexample for each sequent in S', then it contains also a counterexample for each sequent in S. Indeed, a sequent in S is either (S_1, S_2), or some other sequent. If it is some other sequent, then it belongs to S', i.e. (b, ≤, t) contains a counterexample for it. Does (b, ≤, t) contain a counterexample also for (S_1, S_2)? We know that it contains a counterexample for $(S_1, S_2 \cup \{A, B\})$, i.e. for some $x \in b$, x |= F for all formulas $F \in S_1$ and not x |= G for all formulas $G \in S_2 \cup \{A, B\}$. Hence, (b, ≤, t) contains a counterexample also for (S_1, S_2). Q.E.D.

If there is a Kripke scenario (b, ≤, t), which contains a counterexample for each sequent in S'', then it contains also a counterexample for each sequent in S. The argument is similar to the above.

Case 5. S contains (S_1, S_2) such that $A \to B \in S_1 \wedge \neg(A \in S_2 \vee B \in S_1)$. Let us consider the following two sets:

a) S' – obtained from S by adding the formula A to S_2, i.e. by replacing (S_1, S_2) by $(S_1, S_2 \cup \{A\})$.

b) S'' – obtained from S by adding the formula B to S_1, i.e. by replacing (S_1, S_2) by $(S_1 \cup \{B\}, S_2)$.

Let us verify that if Theorem is true for S' and S'', then it is true for S.

Assume, some sequent of S' and some sequent of S'' is constructively provable. The sequent of S' is $(S_1, S_2 \cup \{A\})$ or some other sequent. If it is some other sequent, then it belongs to S, i.e. some sequent of S is constructively provable. The sequent of S'' is $(S_1 \cup \{B\}, S_2)$ or some other sequent. If it is some other sequent, then it belongs to S, i.e. some sequent of S is constructively provable. So, let us consider the situation, when $(S_1, S_2 \cup \{A\})$ and $(S_1 \cup \{B\}, S_2)$ both are constructively provable.

We have two proofs: $S_1 \vdash A \vee S_2$ and S_1, B ⊢ VS_2, and we know that S_1 contains A→B. How to derive a proof of $S_1 \vdash VS_2$?

Since S_1 contains A→B, we have a proof of S_1, A ⊢ B. Together with S_1, B ⊢ VS_2 this yields a proof of S_1, A ⊢ VS_2. Of course, $VS_2 \vdash VS_2$. Now, let us

remind Exercise 2.3.2 [L_1, L_2, L_8, MP]:

If A_1, A_2, ..., A_n, B $\vdash$ D, and A_1, A_2, ..., A_n, C $\vdash$ D, then A_1, A_2, ..., A_n , $B \vee C$ $\vdash$ D. Thus, S_1, $A \vee S_2$ $\vdash$ VS_2. Since we have a proof of S_1 $\vdash$ $A \vee S_2$, we have also a proof of S_1 $\vdash$ $A \vee S_2$.

On the other side, if there is a Kripke scenario (b, ≤, t), which contains a counterexample for each sequent in S', then it contains also a counterexample for each sequent in S. Indeed, a sequent in S is either (S_1, S_2), or some other sequent. If it is some other sequent, then it belongs to S', i.e. (b, ≤, t) contains a counterexample for it. Does (b, ≤, t) contain a counterexample also for (S_1, S_2)? We know that it contains a counterexample for $(S_1, S_2 \cup \{A\})$, i.e. for some $x \in b$, x |= F for all formulas $F \in S_1$ and not x |= G for all formulas $G \in S_2 \cup \{A\}$. Hence, (b, ≤, t) contains a counterexample also for (S_1, S_2). Q.E.D.

If there is a Kripke scenario (b, ≤, t), which contains a counterexample for each sequent in S'', then it contains also a counterexample for each sequent in S. The argument is similar to the above.

Case 6. S contains (S_1, S_2) such that $A \to B \in S_2$ and for every sequent $(T_1, T_2) \in S$, $\neg(S_1 \subseteq T_1 \wedge A \in T_1 \wedge B \in T_2)$. Let us consider the set S' obtained from S by adding the sequent $(S1 U A, B)$ to it.

Let us verify that if Theorem is true for S', then it is true for S.

Assume, some sequent of S' is constructively provable, then it is $(S_1 \cup \{A\}, B)$ or some other sequent. If it is some other sequent, then it belongs to S, i.e. some sequent of S is constructively provable. If $(S_1 \cup \{A\}, B)$ is constructively provable, then so is (S_1, S_2). Indeed, if S_1, A $\vdash$B is constructively provable, then, by Deduction Theorem 1, S_1 $\vdash$A→B, and S_1 $\vdash$$VS_2$ (since S_2 contains A→B).

On the other side, if there is a Kripke scenario (b, ≤, t), which contains a counterexample for each sequent in S', then, since S is a subset of S', this scenario contains also a counterexample for each sequent in S.

Case 7. None of the above cases hold for S. Hence, for every sequent $(S_1, S_2) \in S$ the following holds:

1) If $A \wedge B \in S_1$, then $A \in S_1 \wedge B \in S_1$,

2) If $A \wedge B \in S_2$, then $A \in S_2 \vee B \in S_2$,

3) If $A \vee B \in S_1$, then $A \in S_1 \vee B \in S_1$,

4) If $A \vee B \in S_2$, then $A \in S_2 \wedge B \in S_2$,

5) If $A \rightarrow B \in S_1$, then $A \in S_2 \vee B \in S_1$,

6) If $A \rightarrow B \in S_2$, then there is

$(T1, T2) \in S$ such that $S_1 \subseteq T_1 \wedge A \in T_2 \wedge B \in T_2$.

For this kind of sequent sets we have a very simple situation:

a) If, in some sequent $(S_1, S_2) \in S$ the sets S_1, S_2 contain the same formula A, then from L_6: $A \rightarrow A \vee B$ we can derive easily that [L_1-L_8, MP]: $S_1 \vdash VS_2$.

b) If the sets S_1, S_2 are disjoint for all sequents $(S_1, S_2) \in S$, then we must (and will) build a scenario, containing a counterexample for each sequent in S.

So, let us suppose that the sets S_1, S_2 are disjoint for all sequents $(S_1, S_2) \in S$, and let us define the following Kripke scenario (b, ≤, t):

b = S,

x≤y must be defined for every two members x, y of b, i.e. for every two sequents (S_1, S_2) and (T_1, T_2) in S. Let us define (S_1, S_2) ≤ (T_1, T_2), if and only if $S_1 \subseteq T_1$. Of course, $'\subseteq'$ is a partial ordering of b.

t must be a monotonic mapping from members of b to sets of atomic formulas. Let us define t(S_1, S_2) as the set of all atomic formulas in S_1. Of course, t is monotonic for $'\subseteq'$. (And, of course, f – our atomic "false", never belongs to t(S_1, S_2)).

Thus, (b, ≤, t) is a Kripke scenario. Let us prove that it contains a counterexample for each sequent in S. In fact, we will prove that for each sequent $(S_1, S_2) \in S$, and each formula F:

If $F \in S_1$, then (S_1, S_2) |= F.

If $F \in S_2$, then not (S_1, S_2) |= F.

This will mean that, (S_1, S_2) represents a counterexample for (S_1, S_2).

Of course, our proof will be by induction along the structure of the formula F.

a) F is an atomic formula.

If $F \in S_1$, then $F \in t(T_1, T_2)$ for every $(T_1, T_2) \in S$ such that (S_1, S_2)≤(T_1, T_2). Hence, (S_1, S_2) |= F.

If $F \in S_2$, then, since S_1 and S_2 are disjoint sets, $F \notin S_1$, and $F \notin t(S_1, S_2)$, i.e. not $(S_1, S_2) \models F$.

b) F is $A \wedge B$.

If $F \in S_1$, then, by (1), $A \in S_1 \wedge B \in S_1$. Hence, by induction assumption, $(S_1, S_2) \models A$ and $(S_1, S_2) \models B$, i.e., by Exercise 4.4.3, $(S_1, S_2) \models A \wedge B$.

If $F \in S_2$, then, by (2), $A \in S_2 \vee B \in S_2$. If $A \in S_2$, then, by induction assumption, not $(S_1, S_2) \models A$, i.e., by Exercise 4.4.3, not $(S_1, S_2) \models A \wedge B$. If $B \in S_2$ – the argument is similar.

c) F is $A \vee B$.

If $F \in S_1$, then, by (3), $A \in S_1 \vee B \in S_1$. If $A \in S_1$, then, by induction assumption, $(S_1, S_2) \models A$, i.e., by Exercise 4.4.3, $(S_1, S_2) \models A \vee B$. If $B \in S_1$ – the argument is similar.

If $F \in S_2$, then, by (4), $A \in S_2 \wedge B \in S_2$. By induction assumption, not $(S_1, S_2) \models A$ and not $(S_1, S_2) \models B$, i.e., by Exercise 4.4.3, not $(S_1, S_2) \models A \vee B$.

d) F is $A \rightarrow B$.

d1) $F \in S_1$. We must prove that $(S_1, S_2) \models A \rightarrow B$, i.e. that $(T_1, T_2) \models A \rightarrow B$ for each $(T_1, T_2) \in S$ such that $(S_1, S_2) \leq (T_1, T_2)$. So, let us assume that not $(T_1, T_2) \models A \rightarrow B$, i.e. that $(U_1, U_2) \models A$ and not $(U_1, U_2) \models B$ for some $(U_1, U_2) \in S$ such that $(T_1, T_2) \leq (U_1, U_2)$.

Since $A \rightarrow B \in S_1$, then also $A \rightarrow B \in U_1$, and, by (5), $A \in U_2 \vee B \in U_1$. By induction assumption, this means that not $(U_1, U_2) \models A$ or $(U_1, U_2) \models B$. Contradiction, hence, $(S_1, S_2) \models A \rightarrow B$.

d2) $F \in S_2$. We must prove that not $(S_1, S_2) \models A \rightarrow B$, i.e. that there is $(T_1, T_2) \in S$ such that $(S_1, S_2) \leq (T_1, T_2)$ and $(T_1, T_2) \models A$ and not $(T_1, T_2) \models B$.

Since $A \rightarrow B \in S_2$, by (6), there is $(T_1, T_2) \in S$ such that $(S_1, S_2) \leq (T_1, T_2)$ and $A \in T_1$ and $B \in T_2$. By induction assumption, this means that $(T_1, T_2) \models A$ and not $(T_1, T_2) \models B$. Q.E.D.

This completes the proof of Theorem 4.4.6.

Note. The above proof contains an **algorithm** allowing to find, for each set S of sequents, either a constructive proof of some sequent of S, or a Kripke

scenario containing counterexamples for each sequent of S.

Corollary 4.4.7. If a formula F is true at all nodes in all scenarios, then

$[L_1\text{-}L_{10}, MP] \vdash F$ (i.e. F is provable in the constructive propositional logic).

Indeed, let us consider the set of sequents {(0, {F})} consisting of a single sequent (0, {F}), where 0 is empty set. By Theorem 4.4.6, either the sequent (0, {F}) is constructively provable, or there is a Kripke scenario (b, ≤, t), which contains a counterexample for (0, {F}). Since F is true at all nodes in all Kripke scenarios, it cannot have counterexamples; hence, the sequent (0, {F}) (i.e. the formula F) is constructively provable.

Together with Lemma 4.4.3 this Corollary implies the above Theorem 4.4.2 – **Kripke's theorem on the completeness of the constructive propositional logic**: a formula F is true at all nodes in all Kripke scenarios, if and only if F is provable in the constructive propositional logic.

Corollary 4.4.8 (decidability of the constructive propositional logic). There is an **algorithm** allowing to determine for any formula F, is this formula provable in the constructive propositional logic $[L_1\text{-}L_{10}, MP]$, or not.

Gerhard Gentzen established this fact in 1934:

G. Gentzen. Untersuchungen über das logische Schliessen II. *Mathematische Zeitschrift*, 1934, Vol. 39, pp. 405-431.

Corollary 4.4.9. If $F \vee G$ is true at all nodes in all scenarios, then F is true at all nodes in all scenarios, or G is true at all nodes in all scenarios.

Proof. Assume, there is a scenario $(b_1, \leq_1, t_1)$ such that $x_1 \models F$ is false for some $x_1 \in b_1$, and a scenario $(b_2, \leq_2, t_2)$ such that $x_2 \models G$ is false for some $x_2 \in b_2$. We may assume that the (node) sets b_1 and b_2 do not intersect. Let us merge these scenarios by adding a new common starting node x_0, where all B_i are false. Then, $x_0 \models F$ is false (Lemma 4.4.1), and $x_0 \models G$ is false (similarly). Hence, according to the classical disjunction truth table, $x_0 \models F \vee G$ is false. But, $x \models F \vee G$ is always true. Hence, $x \models F$ is always true, or $x \models G$ is always true. Q.E.D.

Theorem 4.4.10. (Gödel [1932]). If $[L_1\text{-}L_{10}, MP] \vdash B \vee C$, then

$[L_1\text{-}L_{10}, MP] \vdash B$ or $[L_1\text{-}L_{10}, MP] \vdash C$. (I.e. if the disjunction $B \vee C$ is constructively provable, then one of the formulas B, C also is constructively provable.)

Proof. If $[L_1\text{-}L_{10}, MP] \vdash B \vee C$, then, by Kripke's Completeness Theorem

4.4.2, $B \vee C$ is true at all nodes in all scenarios. Then, by Corollary 4.4.9, so is B or so is C. By Kripke's Completeness Theorem 4.4.2, this means that one of the formulas B, C is constructively provable. Q.E.D.

Let us remind the constructive interpretation of disjunction from Section 1.3:

- To prove $B \vee C$ constructively, you must prove B, or prove C. To prove $B \vee C$ classically, you may assume $\neg(B \vee C)$ as a hypothesis, and derive a contradiction. Having only such a "negative" proof, you may be unable to determine, which part of the disjunction $B \vee C$ is true – B, or C, or both.

According to Theorem 4.4.10, the constructive propositional logic [L_1-L_{10}, MP] supports the constructive interpretation of disjunction.

K.Gödel established this fact in 1932:

K. Gödel. Zum intuitionistischen Aussagenkalkül. *Akademie der Wissenschaften in Wien, Mathematisch- naturwissenschaftliche Klasse, Anzeiger*, 1932, Vol.69, pp.65-66.

Exercise 4.4.5 (optional, for smart students). By adding the schema $(B \to C) \vee (C \to B)$ to the axioms of the constructive logic, we obtain the so-called *Gödel-Dummett logic*. Verify, that a propositional formula F is provable in Gödel-Dummett logic, if and only if F is true at all nodes in all **linear** Kripke scenarios (i.e. in the scenarious that do not allow branching). See also Intuitionistic Logic by Joan Moschovakis in Stanford Encyclopedia of Philosophy, and Michael Dummett in Internet Encyclopedia of Philosophy.

5. Normal Forms. Resolution Method

In this section, we will try to produce a practical method allowing to derive consequences and prove theorems by using computers. In general, this task is not feasible because of its enormous computational complexity (see Section 4.3). Still, for problems of a "practical size" (arising, for example, in deductive databases and other artificial intelligence systems, or, trying to formalize real mathematical proofs), such methods are possible and some of them are already implemented successfully.

This field of research is called automated reasoning, or automated theorem-proving.

Warning! The principal results of this Section are valid only for the **classical logic**!

Main Ideas

If F_1, ..., F_n is the set of our assumptions (facts, rules, axioms, hypotheses etc.), does the assertion G follow from this set? One of the well known approaches to proving theorems in mathematics – and especially convenient for computers – are the so-called **refutation proofs** (*reductio ad absurdum*) – proofs by deriving a contradiction: assume ¬G, and try deriving a contradiction. I.e. try proving that F_1, ..., F_n, ¬G is an inconsistent set of assumptions.

Idea #1: let us derive consequences and prove theorems **only in this way**. Let us try developing the best possible method of deriving contradictions from inconsistent sets of assumptions. This (at first glance – trivial) decision is one of the most important steps in the whole story – it will allow (see Section 5.2 below) conversion of the formulas F_1, ..., F_n, ¬G into a form that does not contain existential quantifiers. And after this, having universal quantifiers only, we may simply drop them at all, and continue working with quantifier-free formulas (see Section 5.4).

Idea #2: let us "normalize" our assumption formulas as far as possible.

The first step (**idea #2a**) is reducing to the so-called **prenex normal form** – moving all the quantifiers to left. For example, the formula

$$[(\exists x B(x) \to \exists x C(x)) \to \exists x D(x)] \to \exists x F(x)$$

is equivalent (in the classical logic!) to the following formula in prenex normal form:

$$\forall x_1 \exists x_2 \forall x_3 \exists x_4 [[(B(x_1) \rightarrow C(x_2)) \rightarrow D(x_3)] \rightarrow F(x_4)].$$

(When moving quantifiers to left, some of them must be changed from ∃ to ∀, or from ∀ to ∃, see Section 5.1 below.)

The second step (**idea #2b**, due to Thoralf Skolem) allows elimination of existential quantifiers. Indeed, $\forall x_1 \exists x_2$ means that $x_2=f(x_1)$, and $\forall x_1 \forall x_3 \exists x_4$ means that $x_4=g(x_1, x_3)$, where f and g are some functions (see Section 5.2). In this way we obtain the so-called **Skolem normal form**, containing universal quantifiers only:

$$\forall x_1 \forall x_3 [[(B(x_1) \rightarrow C(f(x_1))) \rightarrow D(x_3)] \rightarrow F(g(x_1, x_3))].$$

Note that a formula and its Skolem normal form **are not equivalent** (even in the classical logic!), they are only a kind of "semi-equivalent": a set of formulas is inconsistent, if and only if so is the set of their Skolem normal forms.

Now, since, our formulas contain universal quantifiers only, we may **drop these quantifiers** (simply by assuming that all free variables are universally quantified):

$$[(B(x_1) \rightarrow C(f(x_1))) \rightarrow D(x_3)] \rightarrow F(g(x_1, x_3)).$$

The third step (**idea #2c**) – reduction of quantifier-free formulas to the so-called **conjunctive normal form** (a conjunction of disjunctions of atomic formulas – with or without negations, see Section 5.3). For example, the above formula can be reduced to the following form:

$$(\neg B(x_1) \vee C(f(x_1)) \vee F(g(x_1, x_3))) \wedge (\neg D(x_3) \vee F(g(x_1, x_3))) \quad .$$

By assuming that a set of formulas means their conjunction, we can drop the conjunction(s) obtaining a set of the so-called **clauses**:

$$\neg B(x_1) \vee C(f(x_1)) \vee F(g(x_1, x_3)) \quad ;$$
$$\neg D(x_3) \vee F(g(x_1, x_3)) \quad .$$

Each clause is a disjunctions of atomic formulas – with or without negations. To separate clearly the meaning of each clause, we must rename some of the variables – no two clauses are allowed to contain common variables:

$$\neg B(x_1) \vee C(f(x_1)) \vee F(g(x_1, x_3)) \quad ;$$
$$\neg D(x_5) \vee F(g(x_4, x_5)) \quad .$$

In this way, instead of our initial set of assumptions $F_1, ..., F_n, \neg G$, we obtain a set of separate clauses ("large cloud of simple disjunctions"), which is inconsistent, if and only if so is the initial set $F_1, ..., F_n, \neg G$.

The last step – how to work with a set of clauses (“large cloud of simple disjunctions”)?

Idea #3 (due to John Alan Robinson, see Section 5.5 and 5.7) – a set of clauses is inconsistent, if and only if a contradiction can be derived from it by using term substitution and the so-called **Robinson's Resolution rule**:

$$\frac{F \vee \boldsymbol{C}, \neg \boldsymbol{C} \vee G}{F \vee G} \quad .$$

Continue reading...

Alternative method: the so-called Method of Analytic Tableaux.

5.1. Prenex Normal Form

Warning! The principal results of this Section are valid only for the **classical logic**!

Let us consider an interpretation J of some predicate language L, such that the domain D_J contains an infinite set of "objects". Under such interpretation, the "meaning" of formulas containing quantifiers may be more or less non-constructive, or, at least, "constructively difficult".

For example, the formula ∀xB(x) will be true, if B(x) will be true for all "objects" x in the (infinite!) set D_J. Thus, it is impossible to verify directly (i.e. "empirically"), is ∀xB(x) true or not. Saying that ∀x∀y(x+y=y+x) is true under the standard interpretation of first order arithmetic, does not mean that we have verified this fact empirically – by checking x+y=y+x for all pairs of natural numbers x, y. Then, how do we know that ∀x∀y(x+y=y+x) is true? Of course, we either postulated this feature of natural numbers directly (i.e. derived it from "empirical evidence"), or proved it by using some set of axioms (i.e. derived it from other postulates). But, in general, formulas having the form ∀xB(x), are "constructively difficult".

The formula ∀x∃yC(x, y) may be even more difficult: it will be true, if for each x in D_J we will be able to find y in D_J such that C(x, y) is true. Thus, thinking constructively, we could say that ∀x∃yC(x, y) is true, only, if there is an algorithm, which, for each x in D_J can find y in D_J such that C(x, y) is true. For example, under the standard interpretation of first order arithmetic, the formula

$$\forall x \, \exists y \left(x < y \wedge prime(y) \right)$$

is true (i.e. "there are infinitely many prime numbers"). How do we know this? This fact was proved in VI century BC. But the (similarly quantified) formula

$$\forall x \exists y (x < y \wedge prime(y) \wedge prime(y+2)) \quad ,$$

i.e. the famous twin prime conjecture, is it true or not? Until now, nobody knows the answer.

Exercise 5.1.1. Verify that the "meaning" of ∀x∃y∀zD(x, y, z) and ∀x∃y∀z∃uF(x, y, z, u) may be even more non-constructive.

But how about the formula ∃xG(x)→∃yH(y)? Is it constructively more difficult than ∀x∃yC(x, y), or less? In general, we could prove that ∃xG(x)→∃yH(y) is true, if we had an algorithm, which, for each $x \in D_J$ such that G(x) is true, could find $y \in D_I$ such that G(y) is true, i.e. if ∀x∃y(G(x)→H(y)) would be true. We will establish below, that, in the classical logic, if G does not contain y, and H does not contain x, then the formula ∃xG(x)→∃yH(y) is equivalent to ∀x∃y(G(x)→H(y)). Thus, in general, the formula ∃xG(x)→∃yH(y) is constructively as difficult as is the formula ∀x∃yC(x, y)!

To generalize this approach to comparing "constructive difficulty" of formulas, the so-called **prenex normal forms** have been introduced:

a) If a formula does not contain quantifiers, then it is in the prenex normal form.

b) If x is any variable, and the formula F is in the prenex normal form, then ∀xF and ∃xF also are in the prenex normal form.

c) (If you wish so,) there are no other formulas in the prenex normal form.

I.e. a formula is in the prenex normal form, if and only if it has all its quantifiers gathered in front of a formula that does not contain quantifiers. It appears, that in the classical logic, each formula cane be "reduced" to an appropriate equivalent formula in the prenex normal form. To obtain this normal form, the following Lemmas 5.1.1-5.1.3 can be used.

Lemma 5.1.1. If the formula G does not contain x as a free variable, then:

a) [L_1, L_2, L_5, L_{12}, L_{14}, MP, Gen]: (G→∀xF(x)) ↔ ∀x(G→F(x)).

b) [L_1, L_2, L_5, L_{12}-L_{15}, MP, Gen]: (∃xF(x)→G) ↔ ∀x(F(x)→G). What does it mean precisely?

c) [L_1-L_{11}, L_{12}-L_{15}, MP, Gen]: (G→∃xF(x)) ↔ ∃x(G→F(x)). More precisely:

[L_1-L_{11}, L_{12}-L_{15}, MP, Gen]: (G→∃xF(x)) → ∃x(G→F(x)). This formula cannot be proved constructively! Why? See Section 4.5. But the converse formula can be proved constructively:

$[L_1, L_2, L_{13}\text{-}L_{15}, MP, Gen]$: $\exists x(G\rightarrow F(x)) \rightarrow (G\rightarrow \exists x F(x))$.

d) $[L_1\text{-}L_{11}, L_{12}\text{-}L_{15}, MP, Gen]$: $(\forall x F(x)\rightarrow G) \leftrightarrow \exists x(F(x)\rightarrow G)$. What does it mean precisely? More precisely:

$[L_1\text{-}L_{11}, L_{12}\text{-}L_{15}, MP, Gen]$: $(\forall x F(x)\rightarrow G) \rightarrow \exists x(F(x)\rightarrow G)$. This formula cannot be proved constructively! Why? See Section 4.5. But the converse formula can be proved constructively:

$[L_1, L_2, L_{13}\text{-}L_{15}, MP, Gen]$: $\exists x(F(x)\rightarrow G) \rightarrow (\forall x F(x)\rightarrow G)$.

Proof.

First, let us note that (a)$\leftarrow$ is an instance of the axiom L_{14}: $\forall x(G\rightarrow F(x))\rightarrow(G\rightarrow \forall x F(x))$, and that (b)$\leftarrow$ is an instance of the axiom L_{15}.

Prove (a)$\rightarrow$ and (b)$\rightarrow$ as the Exercise 5.1.2 below.

Let us prove (c)$\leftarrow$: $\exists x(G\rightarrow F(x))\rightarrow(G\rightarrow \exists x F(x))$.

(1)	$G\rightarrow F(x)$	Hypothesis.
(2)	G	Hypothesis.
(3)	$F(x)$	By MP.
(4)	$\exists x F(x)$	By Axiom L_{13}: $F(x)\rightarrow \exists x F(x)$.
(5)	$(G\rightarrow F(x))\rightarrow(G\rightarrow \exists x F(x))$	By Deduction Theorem 1.
(6)	$\forall x((G\rightarrow F(x))\rightarrow(G\rightarrow \exists x F(x)))$	By Gen.
(7)	$\exists x(G\rightarrow F(x))\rightarrow(G\rightarrow \exists x F(x))$	By Axiom L_{15}: $\forall x(F(x)\rightarrow G)\rightarrow(\exists x F(x)\rightarrow G)$, since $G\rightarrow \exists x F(x)$ does not contain x as a free variable.

Let us prove (d)$\leftarrow$: $\exists x(F(x)\rightarrow G) \rightarrow (\forall x F(x)\rightarrow G)$.

(1)	$F(x)\rightarrow G$	Hypothesis.
(2)	$\forall xF(x)$	Hypothesis.
(3)	$F(x)$	By Axiom L_{12}: $\forall xF(x)\rightarrow F(x)$.
(4)	G	By MP.
(5)	$(F(x)\rightarrow G)\rightarrow(\forall xF(x)\rightarrow G)$	By Deduction Theorem 1.
(6)	$\forall x((F(x)\rightarrow G)\rightarrow(\forall xF(x)\rightarrow G))$	By Gen.
(7)	$\exists x(F(x)\rightarrow G)\rightarrow(\forall xF(x)\rightarrow G)$	By Axiom L_{15}: $\forall x(F(x)\rightarrow G)\rightarrow(\exists xF(x)\rightarrow G)$, since $\forall xF(x)\rightarrow G$ does not contain x as a free variable.

Now, let us prove (c)→: $(G\rightarrow\exists xF(x))\rightarrow\exists x(G\rightarrow F(x))$ in the classical logic (a constructive proof is impossible, see Section 4.5).

	First, let us prove:	$\neg\mathbf{G}\rightarrow((G\rightarrow\exists xF(x))\rightarrow\exists x(G\rightarrow F(x)))$
(1)	$\neg G\rightarrow(G\rightarrow F(x))$	Axiom L_{10}.
(2)	$(G\rightarrow F(x))\rightarrow\exists x(G\rightarrow F(x))$	Axiom L_{13}: $F(x)\rightarrow\exists xF(x)$.
(3)	$\neg G\rightarrow\exists x(G\rightarrow F(x))$	From (1) and (2).
(4)	$\neg G\rightarrow((G\rightarrow\exists xF(x))\rightarrow\exists x(G\rightarrow F(x)))$	By Axiom L_1: $B\rightarrow(C\rightarrow B)$.
	Now, let us prove:	$\mathbf{G}\rightarrow((G\rightarrow\exists xF(x))\rightarrow\exists x(G\rightarrow F(x)))$
(5)	G	Hypothesis.
(6)	$G\rightarrow\exists xF(x)$	Hypothesis.
(7)	$\exists xF(x)$	From (5) and (6).
(8)	$F(x)\rightarrow(G\rightarrow F(x))$	Axiom L_1: $B\rightarrow(C\rightarrow B)$.
(9)	$\forall x(F(x)\rightarrow(G\rightarrow F(x)))$	By Gen.
(10)	$\exists xF(x)\rightarrow\exists x(G\rightarrow F(x))$	By Theorem 3.1.1(b), [L_1, L_2, L_{12}-

		L_{15}, MP, Gen] ⊢ $\forall x(B\rightarrow C)\rightarrow(\exists xB\rightarrow\exists xC)$.
(11)	$\exists x(G\rightarrow F(x))$	From (7) and (10).
(12)	$G \rightarrow ((G\rightarrow\exists xF(x))\rightarrow \exists x(G\rightarrow F(x)))$	By Deduction Theorem 2 (x is not a free variable in G and $G\rightarrow\exists xF(x)$.
(13)	$\mathbf{G \vee \neg G} \rightarrow ((G\rightarrow\exists xF(x))\rightarrow \exists x(G\rightarrow F(x)))$	From (4) and (12), by Axiom L_8. The total is [L_1, L_2, L_8, L_{10}, L_{12}-L_{15}, MP, Gen]
(14)	$(G\rightarrow\exists xF(x)\rightarrow \exists x(G\rightarrow F(x))$	By **Axiom L_{11}**: $G \vee \neg G$.

Finally, let us prove (d)→: $(\forall xF(x)\rightarrow G) \rightarrow \exists x(F(x)\rightarrow G)$ in the classical logic(a constructive proof is impossible, see Section 4.5). Let us denote this formula by H.

	First, let us prove:	$\mathbf{\forall xF(x)}\rightarrow H$
(1)	$\forall xF(x)$	Hypothesis.
(2)	$\forall xF(x)\rightarrow G$	Hypothesis.
(3)	G	From (1) and (2).
(4)	$F(x)\rightarrow G$	By Axiom L_1: $B\rightarrow(C\rightarrow B)$.
(5)	$\exists x(F(x)\rightarrow G)$	By Axiom L_{13}: $F(x)\rightarrow\exists xF(x)$.
(6)	$\forall xF(x) \rightarrow H$	By Deduction Theorem 2.
	Now, let us prove:	$\mathbf{\exists x\neg F(x)}\rightarrow H$
(5)	$\neg F(x)$	Hypothesis.
(6)	$\neg F(x)\rightarrow(F(x)\rightarrow G)$	Axiom L_{10}.
(7)	$F(x)\rightarrow G$	From (5) and (6).
(8)	$\exists x(F(x)\rightarrow G)$	By Axiom L_{13}: $F(x)\rightarrow\exists xF(x)$.
(9)	$(\forall xF(x)\rightarrow G) \rightarrow \exists x(F(x)\rightarrow G)$	By Axiom L_1: $B\rightarrow(C\rightarrow B)$.
(10)	$\neg F(x)\rightarrow H$	By Deduction Theorem 2.

(11) $\exists x\neg F(x)\rightarrow H$	By Gen and Axiom L_{15}: $\forall x(\neg F(x)\rightarrow H)\rightarrow (\exists x\neg F(x)\rightarrow H)$.
(12) $\neg\forall xF(x)\rightarrow H$	By Section 3.2, III-4. [L_1-L_{11}, L_{13}, L_{14}, MP, Gen]: $\vdash \neg\forall xF(x)\rightarrow\exists x\neg F(x)$. **Axiom L_{11} is used here!**
(13) $\forall xF(x)\ v\ \neg\forall xF(x)\rightarrow H$	From (4) and (12), by Axiom L_8.
(13) H	By Axiom L_{11}: $\forall xF(x)\ v\ \neg\forall xF(x)$

Q.E.D.

Exercise 5.1.2. a) Prove (a)→ of Lemma 5.1.1,

$$[L_1, L_2, L_{12}, MP, Gen]: (G\rightarrow\forall xF(x))\rightarrow \forall x(G\rightarrow F(x)).$$

b) Prove (b)→ of Lemma 5.1.1,

$$[L_1, L_2, L_{13}, MP, Gen]: (\exists xF(x)\rightarrow G)\rightarrow \forall x(F(x)\rightarrow G).$$

Lemma 5.1.2. If the formula G does not contain x as a free variable, then

a) [L_1-L_5, L_{12}-L_{15}, MP, Gen]: $\exists xF(x)\wedge G\leftrightarrow\exists x(F(x)\wedge G)$.

b) [L_1-L_5, L_{12}, L_{14}, MP, Gen]: $\forall xF(x)\wedge G\leftrightarrow\forall x(F(x)\wedge G)$.

c) [L_1, L_2, L_5, L_6-L_8, L_{12}-L_{15}, MP, Gen]: $\exists xF(x)\vee G\leftrightarrow\exists x(F(x)\vee G)$.

d) [L_1-L_{11}, L_{12}, L_{14}, MP, Gen]: $\forall xF(x)\vee G\leftrightarrow\forall x(F(x)\vee G)$. More precisely:

[L_1, L_2, L_5, L_6-L_8, L_{12}, L_{14}, MP, Gen]: $\forall xF(x)\vee G\rightarrow\forall x(F(x)\vee G)$, i.e. this part of the equivalence can be proved constructively. But,

[L_1-L_{11}, L_{12}, L_{14}, MP, Gen]: $\forall x(F(x)\vee G)\rightarrow\forall xF(x)\vee G$. This formula cannot be proved constructively! Why? See Section 4.5.

Proof.

Prove (a, b, c) as the Exercise 5.1.3 below.

Let us prove (d)→: $\forall xF(x)\vee G\rightarrow\forall x(F(x)\vee G)$.

(1)	$F(x)\to F(x)\vee G$	Axiom L_6.
(2)	$\forall x(F(x)\to F(x)\vee G)$	By Gen.
(3)	$\forall$x(B→C)→($\forall$xB→$\forall$xC)	Theorem 3.1.1(a) [L_1, L_2, L_{12}, L_{14}, MP, Gen].
(4)	$\forall xF(x)\to\forall x(F(x)\vee G)$	From (2) and (3).
(5)	$G\to F(x)\vee G$	Axiom L_7.
(6)	$\forall x(G\to F(x)\vee G)$	By Gen.
(7)	$G\to\forall x(F(x)\vee G)$	By Axiom L_{14}.
(8)	$\forall xF(x)\vee G\to\forall x(F(x)\vee G)$	From (4) and (7), by Axiom L_8.

Finally, let us prove (d)←: $\forall x(F(x)\vee G)\to\forall xF(x)\vee G$ in the classical logic (a constructive proof is impossible, see Section 4.5).

(1)	$\forall x(F(x)\vee G)$	Hypothesis.
(2)	$F(x)\vee G$	By Axiom L_{12}.
(3)	$G\vee F(x)$	From (2).
(4)	¬G	Hypothesis.
(4)	F(x)	By Theorem 2.5.1(b) [L_1, L_2, L_8, L_{10}, MP]: $\vdash A\vee B\to(\neg A\to B)$
(5)	$\forall$xF(x)	By Gen.
(6)	$\forall xF(x)\vee G$	By Axiom L_6.
(7)	$\neg G\to(\forall x(F(x)\vee G)\to\forall xF(x)\vee G)$	By Deduction Theorem 2 (x is not free variable in $\forall x(F(x)\vee G)$.
(8)	$G\to\forall xF(x)\vee G$	Axiom L_7.

(9) $G\to(\forall x(F(x)\vee G)\to\forall xF(x)\vee G)$ — By Axiom L_1: B→(C→B).

(10) $G\vee\neg G\to(\forall x(F(x)\vee G)\to\forall xF(x)\vee G)$ — From (7) and (9), by Axiom L_8.

(11) $\forall x(F(x)\vee G)\to\forall xF(x)\vee G$ — By **Axiom L_{11}**: $G\vee\neg G$.

Q.E.D

Exercise 5.1.3. Prove (a, b, c) of Lemma 5.1.2.

Lemma 5.1.3. a) [L_1-L_{10}, L_{12}-L_{15}, MP, Gen]: ¬∃xF(x) ↔ ∀x¬F(x).

b) [L_1-L_{11}, L_{12}-L_{15}, MP, Gen]: ¬∀xF(x) ↔ ∃x¬F(x). More precisely:

[L_1-L_{11}, L_{13}, L_{14}, MP, Gen]: ¬∀xF(x) → ∃x¬F(x). This formula cannot be proved constructively! Why? See Section 4.5. But,

[L_1-L_{10}, L_{13}, L_{14}, MP, Gen]: ∃x¬F(x) → ¬∀xF(x).

Proof.

a) See Section 3.2, Group IV.

b)→. This is exactly Section 3.2, III-4.

b)←. See Section 3.2, Group III.

Q.E.D.

Let us remind that a formula is in the prenex normal form, if and only if it has all its quantifiers gathered in front of a formula that does not contain quantifiers.

Theorem 5.1.4. In the classical logic, each formula is equivalent to an appropriate formula in the prenex normal form. More precisely, if F is a formula, then, following a simple algorithm, a formula F' can be constructed such that: a) F' is in a prenex normal form, b) F' has the same free variables as F, c) [L_1-L_{11}, L_{12}-L_{15}, MP, Gen]: F↔F'.

Proof. Let us start by an example:

∃xG(x)→∃yH(y).

If H did not contain x as a free variable, then, by Lemma 5.1.1(b): ∃xF(x)→G ↔ ∀x(F(x)→G), i.e. this formula would be equivalent to ∀x(G(x)→∃yH(y)). Now, let us consider the sub-formula G(x)→∃yH(y). If G did not contain y as a free variable, then, by Lemma 5.1.1(c): G→∃xF(x) ↔ ∃x(G→F(x)), the sub-

formula would be equivalent to $\exists y(G(x)\rightarrow H(y))$. Hence, by Replacement Theorem 2, $\forall x(G(x)\rightarrow\exists yH(y))$ would be equivalent to $\forall x\exists y(G(x)\rightarrow H(y))$.

But, if H would contain x as a free variable, and/or G would contain y as a free variable? Then our "shifting quantifiers up" would be wrong – the formula $\forall x\exists y(G(x)\rightarrow H(y))$ would **not** be equivalent to $\exists xG(x)\rightarrow\exists yH(y)$.

To avoid this problem, let us use Replacement Theorem 3, which says that the meaning of a formula does not depend on the names of bound variables used in it. Thus, as the first step, in $\exists xG(x)$, let us replace x by another variable x_1 that does not appear neither in G, nor in H. Then, by Replacement Theorem 3, $\exists xG(x)$ is equivalent to $\exists x_1G(x_1)$, and by Replacement Theorem 2, $\exists xG(x)\rightarrow\exists yH(y)$ is equivalent to $\exists x_1G(x_1)\rightarrow\exists yH(y)$. Now, $\forall x_1(G(x_1)\rightarrow\exists yH(y))$ is really equivalent to $\exists x_1G(x_1)\rightarrow\exists yH(y)$. As the next step, in $\exists yH(y)$, let us replace y by another variable y_1 that does not appear neither in G, nor in H. Then, by Replacement Theorem 3, $\exists yH(y)$ is equivalent to $\exists y_1H(y_1)$, and by Replacement Theorem 2, $G(x_1)\rightarrow\exists y_1H(y_1)$ is equivalent to $\exists y_1(G(x_1)\rightarrow H(y_1))$. And, finally, $\exists xG(x)\rightarrow\exists yH(y)$ is equivalent to $\forall x_1\exists y_1(G(x_1)\rightarrow H(y_1))$.

Now, we can start the general proof. In a formula F, let us find the **leftmost** quantifier having a propositional connective over it. If such a quantifier does not exist, the formula is in the prenex normal form. If such a quantifier exists, then F is in one of the following forms:

$$Q_qQ_q...Q_q(...(\neg QxG)...),\ \text{or}\ Q_qQ_q...Q_q(...(QxGooH)...),\ \text{or}\ Q_qQ_q...Q_q(...(GooQxH)...),$$

where $Q_qQ_q...Q_q$ are the quantifiers "already in prefix", Q is the quantifier in question, and oo is the propositional connective standing directly over Q.

In the first case, by Lemma 5.1.3, $\neg QxG$ is equivalent to $Q'x\neg G$, where Q' is the quantifier opposite to Q. By Replacement Theorem 2, $Q_qQ_q...Q_q(...(\neg QxG)...)$ is then equivalent to $Q_qQ_q...Q_q(...(Q'x\neg G)...)$, i.e. Q' has now one propositional connective less over it ((than had Q).

In the second case, as the first step, in QxG, let us replace x by another variable x_1 that does not appear in the entire formula F at all. Then, by Replacement Theorem 3, QxG is equivalent to Qx_1G_1, and by Replacement Theorem 2, $Q_qQ_q...Q_q(...(QxGooH)...)$ is equivalent to $Q_qQ_q...Q_q(...(Qx_1G_1ooH)...)$. Now, we can apply the appropriate case of Lemma 5.1.1 or Lemma 5.1.2, obtaining that Qx_1G_1ooH is equivalent to $Q'x_1(G_1ooH)$, where

Q' is the quantifier determined by the lemma applied. Then , by Replacement Theorem 2, $Q_qQ_q...Q_q(...(Qx_1G_1ooH)...)$ is equivalent to $Q_qQ_q...Q_q(...(Q'x_1(G_1ooH))...)$, i.e. Q' has now one propositional connective less over it (than had Q).

In the third case, the argument is similar.

By iterating this operation a finite number of times, we arrive at a formula F' which is in the prenex normal form, and which is (in the classical logic) equivalent to F. Q.E.D.

Note. Most formulas admit many different prenex normal forms. For example, the above formula $\exists xG(x)\rightarrow\exists yH(y)$ is equivalent not only to $\forall x_1\exists y_1(G(x_1)\rightarrow H(y_1))$, but also to $\exists y_1\forall x_1(G(x_1)\rightarrow H(y_1))$ (verify).

As an example, let us obtain a prenex normal form of the following formula:

$$\exists xB(x)\vee\forall xC(x)\rightarrow\forall xD(x)\wedge(\neg\forall xF(x)) \quad .$$

First, assign unique names to bound variables:

$$\exists x_1B(x_1)\vee\forall x_2C(x_2)\rightarrow\forall x_3D(x_3)\wedge(\neg\forall x_4F(x_4)) \quad .$$

Process disjunction:

$$\exists x_1\forall x_2(B(x_1)\vee C(x_2))\rightarrow\forall x_3D(x_3)\wedge(\neg\forall x_4F(x_4)) \quad .$$

Process negation (∀-∃):

$$\exists x_1\forall x_2(B(x_1)\vee C(x_2))\rightarrow\forall x_3D(x_3)\wedge\exists x_4\neg F(x_4) \quad .$$

Process conjunction:

$$\exists x_1\forall x_2(B(x_1)\vee C(x_2))\rightarrow\forall x_3\exists x_4(D(x_3)\wedge\neg F(x_4)) \quad .$$

Process implication premise (∃-∀, ∀-∃):

$$\forall x_1\exists x_2(B(x_1)\vee C(x_2)\rightarrow\forall x_3\exists x_4(D(x_3)\wedge\neg F(x_4))) \quad .$$

Process implication consequent:

$$\forall x_1\exists x_2\forall x_3\exists x_4(B(x_1)\vee C(x_2)\rightarrow D(x_3)\wedge\neg F(x_4)) \quad .$$

The last two steps could be performed in the reverse order as well.

Exercise 5.1.4. Transform each of the following formulas into a prenex normal form. Write down every single step of the process. (Hint: the algorithm is explained in the proof of Theorem 5.1.4.)

a) $\exists xB(x)\rightarrow(\exists xC(x)\rightarrow\exists xD(x))$,

b) $\forall x\exists yB(x,y)\wedge\exists xC(x)\rightarrow\forall y\exists xD(x,y)$,

c) $\exists xB(x,y,z)\rightarrow\forall xC(x,y)\vee\exists yD(y,z)$,

d) $\forall xB(x)\to(\forall xC(x)\to(\forall xD(x)\to\forall xF(x)))$,

e) $((\exists xB(x)\to\exists xC(x))\to\exists xD(x))\to\exists xF(x)$.

Note. From a programmer's point of view, prenex normal forms are, in a sense, a crazy invention. In computer programming, you always try to reduce loop bodies, not to extend them as much as possible!

Exercise 5.1.5 (optional). We may use reduction to prenex normal forms in proofs. More precisely, let us try extending the classical logic by introducing of the following additional inference rule (let us call it **PNF-rule**): given a formula F, replace it by some its prenex normal form F'. Verify, that, in fact, this rule does not extend the classical logic, i.e. if there is a proof of F_1, F_2, ..., $F_n \vdash G$ in [L_1-L_{15}, MP, Gen, PNF-rule], then there is a proof of the same in [L_1-L_{15}, MP, Gen]. (In some other texts, such rules are called **admissible rules**. Thus, the PNF-rule is an admissible rule in the classical logic.)

The notion of prenex normal forms and a version of Theorem 5.1.4 were known to Charles S. Peirce in 1885:

C. S. Peirce. On the algebra of logic: A contribution to the philosophy of notation. *American Journal of Mathematics*, 1885, vol.7, pp.180-202.

As noted by **Alasdair Urquhart** at http://www.cs.nyu.edu/pipermail/fom/2007-July/011720.html: "On page 196 of that article, he gives a brief sketch of conversion to prenex normal form, remarking that it "can evidently be done."".

5.2. Skolem Normal Form

This normal form was first introduced by Thoralf Skolem (1887-1963) in 1928:

Th.Skolem. Über die mathematische Logik. "Norsk matematisk tidsskrift", 1928, vol.10, pp.125-142.

Warning! The principal results of this Section are valid only for the **classical logic**!

The first very important idea was proposed by Skolem already in 1920:

Th. Skolem. Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit und Beweisbarkeit mathematischen Sätze nebst einem Theoreme über dichte Mengen. *Videnskabsakademiet i Kristiania, Skrifter* I, No. 4, 1920, pp. 1-36.

Namely, according to Skolem's idea, further "normalization" becomes possible, if we drop the requirement that the "normal form" must be equivalent to the initial formula, and replace it by the requirement: **"normal form" must**

be logically valid; if and only if the initial formula is logically valid. It appears, that in this way we can "reduce" any closed formula to a closed formula containing only one kind of quantifiers:

$$\exists x_1 \exists x_2 ... \exists x_n H(x_1, x_2, ..., x_n),$$

where H does not contain quantifiers at all (see Theorem 5.2.4 below).

Still, in his original formulation, instead of logical validity, Skolem was interested in a more technical notion – **satisfiability**. Let us remind that, in a predicate language L, a formula F is called satisfiable, if and only if there is an interpretation of the language L such that F is true for some values of its free variables. For our current purpose – **refutation proofs** (to prove that $F_1, .., F_n \vdash G$, we assume $\neg G$ and try to derive a contradiction) satisfiability works as well as does logical validity. Indeed (verify, see Exercise 4.1.1), a set of formulas is inconsistent, if and only if it is unsatisfiable. Thus, if, in a refutation proof, we replace some formula H by an "equally satisfiable" formula H' (i.e. H' is satisfiable, if and only if so is H), then the refutation proof remains valid. I.e. if, this way, we derive a contradiction from $F_1, .., F_n, \neg G$, then this set of fomulas is, indeed, unsatisfiable, i.e. G logically follows from $F_1, .., F_n$ (for a more precise version of this argument see Exercises 5.2.4).

Skolem's second main idea (proposed in his 1928 paper): **allow introduction of new object constants and function constants**. It can be demonstrated on the following example: how could we "simplify" the formula $\forall x \exists y F(x, y)$? It asserts that for each x there is y such that F(x, y) is true. Thus, it asserts, that **there is a function** g, which selects for each value of x a value of y such that F(x, y) is true. Thus, in a sense, $\forall x \exists y F(x, y)$ is "equivalent" to $\forall x F(x, g(x))$. In which sense? In the sense that

$\forall x \exists y F(x, y)$ is satisfiable, if and only if $\forall x F(x, g(x))$ is satisfiable.

Indeed,

1. If $\forall x \exists y F(x, y)$ is satisfiable, then there is an interpretation J where it is true, i.e. for each value of x there is a value of y such that F(x, y) is true. This allows us to define the following interpretation of the function constant g: g(x) is one of y-s such that F(x, y) is true in J. If we extend J by adding this interpretation of the function constant g, we obtain an interpretation J', where $\forall x F(x, g(x))$ is true, i.e. this formula is satisfiable.

2. If $\forall x F(x, g(x))$ is satisfiable, then there is an interpretation J where it is true, i.e. for each value of x the formula F(x, g(x)) is true. Hence, in this interpretation, for each value of x there is a value of y (namely, g(x)) such that F(x, y) is true in J. Thus, $\forall x \exists y F(x, y)$ is true in J, i.e. this formula is

satisfiable.

Note. In the first part of this proof, to define the function g, we need, in general, the Axiom of Choice. Indeed, if there is a non-empty set Y_x of y-s such that F(x, y) is true, to define g(x), we must choose a single element of Y_x. If we know nothing else about the interpretation J, we are forced to use the Axiom of Choice. But, if we know that the interpretation J has a countable domain, then we can define g(x) as the "least" y from the set Y_x. In this way we can avoid the Axiom of Choice.

The third idea is even simpler: the formula $\exists x\, F(x)$ asserts that there is x such that F(x) is true, so, let us denote by (an object constant) c one of these x-s, thus obtaining F(c) as a "normal form" of $\exists x\, F(x)$. Of course (verify),

$\exists x\, F(x)$ is satisfiable, if and only if F(c) is satisfiable.

These two ideas allow "reducing" of any quantifier prefix $Qx_1Qx_2...Qx_n$ to a sequence of universal quantifiers only:

Theorem 5.2.1 (Th. Skolem). Let L be a predicate language. There is an algorithm allowing to construct, for each *closed* formula F of this language, a *closed* formula F' (in a language L' obtained from L by adding a finite set of new object constants and new function constants – depending on F) such that:

a) F' is satisfiable, if and only if F is satisfiable,

b) F' is in form $\forall x_1 \forall x_2...\forall x_n G$, where $n \geq 0$, and G does not contain quantifiers.

If a formula is in form $\forall x_1 \forall x_2...\forall x_n G$, where $n \geq 0$, and G does not contain quantifiers, let us call it **Skolem normal form**. Thus, each closed formula can be reduced to a Skolem normal form in the following sense: for each closed formula F of a language L there is a Skolem normal form $|F|_{Sk}$ (in the language L extended by a finite set of Skolem constants and Skolem functions), which is satisfiable, if and only if so is F.

Note. In computer science slang, the reduction procedure leading to Skolem normal form is called "**skolemization**".

Note. Theorem 5.2.1 **does not** assert that a formula and its Skolem normal form are equivalent. It asserts only that the **satisfiability problem** of the first formula is equivalent to the satisfiability problem of the second formula. As already mentioned above, this is enough to allow using of Skolem reduction in refutation proofs.

Thus, if we are interested in determining the satisfiability of formulas, then reducing to Skolem normal forms is a promising method. Indeed, formulas $\forall x_1 \forall x_2...\forall x_n G$ (where G does not contain quantifiers) are, perhaps, easier to

analyze than more complicated combinations of quantifiers.

Proof of Theorem 5.2.1 First, let us obtain a prenex normal form F_1 of the formula F (see Section 5.1). Indeed, by Theorem 5.1.4, there is a simple algorithm, allowing to construct a closed formula F_1 such that F_1 is a prenex normal form, and, in the classical logic, $\vdash F \leftrightarrow F_1$. Of course, F_1 is satisfiable; if and only if so is F.

If the quantifier prefix of F_1 starts with a sequence of existential quantifiers ($\exists\exists...\exists\forall...$), we will need the following lemma to "reduce" these quantifiers:

Lemma 5.2.2 . A closed formula $\exists x_1 \exists x_2 ... \exists x_n \, H(x_1, x_2, ..., x_n)$ is satisfiable, if and only if $H(c_1, c_2, ..., c_n)$ is satisfiable, where $c_1, c_2, ..., c_n$ are new object constants that do not appear in H.

After this operation, we have a closed prenex formula $H(c_1, c_2, ..., c_n)$ (in a language obtained from L by adding a finite set of new object constants, called **Skolem constants**), which is satisfiable, if and only if so is F_1 (and F). The the quantifier prefix of $H(c_1, c_2, ..., c_n)$ (if any) starts with a sequence of universal quantifiers ($\forall\forall...\forall\exists...$).

To proceed, we will need the following

Lemma 5.2.3. A closed formula $\forall x_1 \forall x_2 ... \forall x_n \exists y K(x_1, x_2, ..., x_n, y)$ is satisfiable, if and only if $\forall x_1 \forall x_2 ... \forall x_n K(x_1, x_2, ..., x_n, g(x_1, x_2, ..., x_n))$ is satisfiable, where g is a new n-ary function constant (called **Skolem function**), which does not appear in K.

By iterating this lemma, we can "reduce" the entire quantifier prefix of $H(c_1, c_2, ..., c_n)$ to a sequence of universal quantifiers only ($\forall\forall...\forall$).

For example, the formula $\exists t \forall x \forall y \exists z \forall u \exists w \, F(t, x, y, z, u, w)$ is satisfiable, if and only if so is

$$\forall x \forall y \forall u \exists w \, F(c, x, y, g(x, y), u, w)$$

(where *c* is a Skolem constant that does not appear in F), and, if and only if so is

$$\forall x \forall y \forall u \exists w \, F(c, x, y, g(x, y), u, w),$$

and, if and only if so is the Skolem normal form:

$$\forall x \forall y \forall u \, F(c, x, y, g(x, y), u, h(x, y, u)),$$

where g and h are Skolem functions that do not appear in F.

Exercise 5.2.1. a) Prove Lemma 5.2.2. b) Prove Lemma 5.2.3.

How many new object constants and new function constants (Skolem constants and functions) do we need to obtain the final formula F'? The number of new symbols is determined by the number of existential quantifiers in the quantifier prefix of the prenex formula F_1. Indeed, a) the number of new object constants is determined by the number of existential quantifiers in front of the prefix, and b) the number of new function constants is determined by the number of existential quantifiers that follow after the universal ones.

This completes the proof of Theorem 5.2.1.

Exercise 5.2.2. Obtain Skolem normal forms of the formulas mentioned in Exercise 5.1.4.

See also:
"Skolemization" from The Wolfram Demonstrations Project. Contributed by: Hector Zenil.

Still, if we are interested in determining the logical validity of formulas, then we should apply the result of Exercise 4.1.1 together with Theorem 5.2.1:

F is logically valid, if and only if ¬F is not satisfiable, if and only if a Skolem normal form of ¬F is not satisfiable, if and only if $\forall x_1 \forall x_2 ... \forall x_n G$ (where $n \geq 0$, and G does not contain quantifiers) is not satisfiable, if and only if $\neg \forall x_1 \forall x_2 ... \forall x_n G$ is logically valid, if and only if $\exists x_1 \exists x_2 ... \exists x_n \neg G$ is logically valid.

Thus we have proved the following

Theorem 5.2.4. Let L be a first order language. There is an algorithm allowing to construct, for each closed formula F of this language, a closed formula F' (in a language L' obtained from L by adding a finite set of new object constants and new function constants – depending on F) such that:

a) F' is logically valid (or, provable in the classical logic), if and only if F is logically valid (or, provable in the classical logic),

b) F' is in form $\exists x_1 \exists x_2 ... \exists x_n G$, where $n \geq 0$, and G does not contain quantifiers.

Skolem Normal Form of a Set of Formulas

Knowledge bases are, as a rule, large sets of *closed* formulas $F_1, F_2, ..., F_n$, i.e., in fact, large conjunctions $F_1 \wedge F_2 \wedge ... \wedge F_n$ of closed formulas. Could we obtain a Skolem normal form of this conjunction simply by reducing to Skolem normal form each formula separately?

Assume that during the entire process of reducing the formulas $F_1, F_2, ..., F_n$ to

their Skolem normal forms F'_1, F'_2, ..., F'_n , these formulas are “kept separated”, i.e. the name of each new Skolem constant and new Skolem function is chosen as “completely new” *with respect to the entire process*.

By examining carefully the proof of Theorem 5.2.1, one can see that this is enough to guarantee that the conjunction $F'_1 \wedge F'_2 \wedge ... \wedge F'_n$ is satisfiable, if and only if so is $F_1 \wedge F_2 \wedge ... \wedge F_n$.

Exercise 5.2.3 (optional, for smart students). In his above-mentioned 1920 paper, for quantifier elimination, Skolem proposed **introduction of new predicate constants** (to the idea that function constants will do better, he arrived only in the 1928 paper). Do not read neither Skolem's papers, nor the above-mentioned online comments, and prove yourself that by introduction of new predicate constants, the satisfiability problem of any closed formula can be reduced to the satisfiability problem of a formula having the form $\forall x_1 \forall x_2 ... \forall x_m \exists y_1 \exists y_2 ... \exists y_n G$, where m, n≥0, and G does not contain quantifiers. Thus, function constants "will do better" – see Theorem 5.2.1.

Exercise 5.2.4 (optional, compare with Exercise 5.1.5). Since, in general, Skolem normal form is not equivalent to the initial formula, we cannot use reduction to Skolem normal forms in the usual ("positive", or affirmative) proofs. But we may use it in "negative" (or, refutation) proofs, i.e. in proofs aimed at deriving a contradiction! More precisely, let us try extending the classical logic by introducing of the following additional inference rule (let us call it **SNF-rule**): given a formula F, replace it by some its Skolem normal form F' (such that the newly introduced object constants and function constants do not appear in the proof before F'). Verify, that, in fact, this rule does not extend the classical logic for refutation proofs, i.e. if, from a set of formulas F_1, F_2, ..., F_n, one can derive a contradiction by using [L_1-L_{15}, MP, Gen, SNF-rule], then one can do the same by using [L_1-L_{15}, MP, Gen]. (Thus, the SNF-rule is **admissible for refutation proofs** in the classical logic.)

5.3. Conjunctive and Disjunctive Normal Forms

Warning! The principal results of this Section are valid only for the **classical logic**!

Let us continue the "normalization" process that we started in Section 5.1 by reducing formulas to their prenex normal forms, where all quantifiers are gathered in front of a formula that does not contain quantifiers. How could we

further "normalize" this "formula that does not contain quantifiers"?

Step 1: eliminate equivalence

First of all, we can eliminate all equivalence connectives because B↔C is only a shortcut for $(B\to C)\wedge(C\to B)$. Why should we? Because, proving of B↔C consists of proving of B→C and proving of C→B. Using the shortcut simplifies the appearance of the formula, not its proof.

Step 2: eliminate implication

After this, our formula will contain only implication, conjunction, disjunction and negation connectives. As the next step, we could try to eliminate one (or two?) of these connectives. The classical logic allows to do that. For example, by Theorem 2.6.4(b),

[L_1-L_{11}, MP]: $\vdash (A\to B)\leftrightarrow \neg A\vee B$.

By using this equivalence, we can eliminate implication connectives. For example, the formula B→(C→D) is equivalent (in the classical logic only!) to $\neg B\vee(\neg C\vee D)$.

But, instead of implications, we could try eliminating disjunctions or conjunctions as well. Indeed,

Exercise 5.3.1. In the classical logic [L_1-L_{11}, MP], prove the following:

a) $\vdash (A\to B)\leftrightarrow \neg(A\wedge\neg B)$.

b) $\vdash (A\vee B)\leftrightarrow(\neg A\to B)$.

c) $\vdash (A\vee B)\leftrightarrow \neg(\neg A\vee\neg B)$.

d) $\vdash (A\wedge B)\leftrightarrow \neg(A\to\neg B)$.

e) $\vdash (A\wedge B)\leftrightarrow \neg(\neg A\vee\neg B)$.

(For smart students) Determine, which parts of these equivalences can be proved in the constructive logic [L_1-L_{10}, MP]. End of Exercise 5.3.1.

By using these results, we could eliminate from our formulas any one (or any two) of the three connectives – implication, conjunction, or disjunction.

However, the best decision would be eliminating only implications. Why? Because conjunction and disjunction are associative and commutative operations – and very much like addition (disjunction) and multiplication (conjunction)! For example, after reducing the formula B→(C→B) to $\neg B\vee(\neg C\vee B)$, we can further transform it to $\neg B\vee\neg C\vee B$ and $(\neg B\vee B)\vee C$ – and conclude that it is "true and provable" (no surprise, it is Axiom L_1).

Step 3: move negations down to atoms

Thus, after Step 2, our formula contains only conjunction, disjunction and negation connectives. Now, let us remind the two de Morgan Laws:

Theorem 2.6.3, [L_1-L_{11}, MP]: $\vdash \neg(A \wedge B) \leftrightarrow \neg A \vee \neg B$.

Theorem 2.4.10(b), [L_1-L_9, MP] $\vdash \neg(A \vee B) \leftrightarrow \neg A \wedge \neg B$.

By using these equivalencies, we can shift negations down – until the atoms of the formula. For example, let us transform the formula

$$((A \rightarrow B) \rightarrow C) \rightarrow B \wedge C \quad .$$

First, eliminate implications:

$\neg((A \rightarrow B) \rightarrow C) \vee (B \wedge C)$,
$\neg(\neg(A \rightarrow B) \vee C) \vee (B \wedge C)$,
$\neg(\neg(\neg A \vee B) \vee C) \vee (B \wedge C)$.

Apply de Morgan Laws:

$$(\neg\neg(\neg A \vee B) \wedge \neg C) \vee (B \wedge C) \quad ,$$
$$(\neg(\neg\neg A \wedge \neg B) \wedge \neg C) \vee (B \wedge C) \quad ,$$
$$((\neg\neg\neg A \vee \neg\neg B) \wedge \neg C) \vee (B \wedge C) \quad .$$

Now, let us remind the Double Negation Law:

Theorem 2.6.1, [L_1-L_{11}, MP]: $\vdash \neg\neg A \leftrightarrow A$.

It allows dropping the excessive negations – we can replace $\neg\neg\neg A$ by $\neg A$ and $\neg\neg B$ – by B:

$$((\neg A \vee B) \wedge \neg C) \vee (B \wedge C) \quad .$$

Note. This form of formulas is called **negation normal form**. Namely, a formula is in negation normal form, if it is built of atoms with or without negations by using conjunctions and disjunctions only. I.e. a formula in negation normal form contains only conjunctions, disjunctions and negations, and negations are located at the atoms only. As we see, in the classical logic, any propositional formula can be reduced (is equivalent) to some formula in negation normal form.

Negation normal form is the starting point for an alternative (to the Resolution method described in this Section 5) method of automated theorem-proving – the so-called Method of Analytic Tableaux. One does not use skolemization here, one simply obtains the negation normal form of the formula (with quantifiers inside) and after this, applies a specific tree algorithm of the Tableaux method. (My exposition for students, in Latvian: Tablo algoritms.) **End of Note.**

Step 4: algebra

After Step 3, our formula is built up by using:

a) atoms,

b) atoms preceded by a negation,

c) conjunction and disjunction connectives.

Conjunction and disjunction are associative and commutative operations. By the behavior of "truth values", conjunction is a kind of multiplication:

$$0\wedge 0=0, 0\wedge 1=1\wedge 0=0, 1\wedge 1=1 \quad ,$$

and disjunction – a kind of addition:

$$0\vee 0=0, 0\vee 1=1\vee 0=1, 1\vee 1=1 \quad .$$

However, for these operations **two** distributive laws are valid (Theorem 2.3.1) – conjunction is distributive to disjunction, and disjunction is distributive to conjunction:

[L_1-L_8, MP]: $\vdash \ (A\wedge B)\vee C\leftrightarrow(A\vee C)\wedge(B\vee C)$,
[L_1-L_8, MP]: $\vdash \ (A\vee B)\wedge C\leftrightarrow(A\wedge C)\vee(B\wedge C)$.

Thus, both of the two decisions could be justified:

1) (Our first "algebra") Let us treat conjunction as multiplication and disjunction – as addition (+). Then the above formula

$((\neg A\vee B)\wedge\neg C)\vee(B\wedge C)$ takes the form ((A'+B)C')+BC (let us replace ¬A by the "more algebraic" A'). After this, the usual algebraic transformations yield the formula A'C'+BC'+BC.

2) (Our second "algebra") Let us treat conjunction as addition (+) and disjunction – as multiplication. Then the above formula

$((\neg A\vee B)\wedge\neg C)\vee(B\wedge C)$ takes the form (A'B+C')(B+C). After this, the usual algebraic transformations yield the formula A'BB+A'BC+C'B+C'C.

Additional rules can be applied in these "algebras".

First rule – conjunction and disjunction are idempotent operations:

[L_1- L_5, MP]: $\vdash \ A\wedge A\leftrightarrow A$ (see Section 2.2).
[L_1, L_2, L_5, L_6-L_8, MP]: $\vdash \ A\vee A\leftrightarrow A$ (Exercise 2.3.1(c)).

Thus, in both of our "algebras": **A+A = AA = A**.

Second rule – $A\wedge\neg A$ (i.e. "false") is a kind of "zero" in the first "algebra", and a kind of "one" – in the second "algebra":

[L_1-L_{10}, MP]: $\vdash B \vee (A \wedge \neg A) \leftrightarrow B$ (Exercise 2.5.1(a)),

[L_1-L_{10}, MP]: $\vdash ((A \wedge \neg A) \wedge B) \vee C \leftrightarrow C$ (Exercise 2.5.1(b)).

Indeed, in the first "algebra", these formulas mean B+AA' = B and AA'B+C = C, i.e. we may think that AA'=0, B0=0, C+0=C. In the second "algebra", these formulas mean B(A+A') = B and (A+A'+B)C = C, i.e. we may think that A+A'=1, B1=B, C+1=1.

Third rule – $A \vee \neg A$ (i.e. "true") is a kind of "one" in the first "algebra", and a kind of "zero" – in the second "algebra":

[L_1-L_{11}, MP]: $\vdash B \wedge (A \vee \neg A) \leftrightarrow B$ (Exercise 2.6.2(a)),

[L_1-L_{11}, MP]: $\vdash ((A \vee \neg A) \vee B) \wedge C \leftrightarrow C$ (Exercise 2.6.2(b)).

Indeed, in the first "algebra", these formulas mean B(A+A') = B and (A+A'+B)C = C, i.e. we may think that A+A'=1, B1=1, C+1=1. In the second "algebra". these formulas mean B+AA' = B and AA'B+C = C, i.e. we may think that AA'=0, B0=0, C+0=C.

Thus, **in both algebras,**

AA'=0, B0=0, C+0=C, A+A'=1, B1=B, C+1=1.

So, let us continue our example

1) (The first "algebra") The formula A'C'+BC'+BC is equivalent to A'C'+B(C'+C) = A'C'+B, or, if we return to logic: $(\neg A \wedge \neg C) \vee B$. Such disjunctions consisting of conjunctions are called **disjunctive normal forms** (DNFs). In a DNF, each conjunction contains each atom no more than once – either without negation, or with it. Indeed, if it contains some atom X twice, then: a) replace XX by X, or b) replace X'X' by X', or c) replace XX' by 0 (in the latter case – drop the entire conjunction from the expression). In this way, for some formulas, we may obtain "zero", i.e. an **empty DNF**. Of course, such formulas take only false values ("false" is "zero" in the first "algebra"). And for some formulas, we may obtain "one", i.e. a kind of **"full" DNF**. Such formulas take only true values ("true" is "one" in the first "algebra").

2) (The second "algebra") The formula A'BB+A'BC+C'B+C'C is equivalent to A'B+A'BC+BC' = A'B(1+C)+BC' = A'B+BC', or, if we return to logic: $(\neg A \vee B) \wedge (B \vee \neg C)$. Such conjunctions consisting of disjunctions are called **conjunctive normal forms** (CNFs). In a CNF, each disjunction contains each atom no more than once – either without negation, or with it. Indeed, if it contains some atom X twice, then: a) replace XX by X, or b) replace X'X' by X', or c) replace XX' by 0 (in the latter case – drop the entire disjunction from the expression). In this way, for some formulas, we may obtain "zero", i.e. an **empty CNF**. Of course, such formulas take only true values ("true" is "zero" in the second "algebra"). And for some formulas, we

may obtain "one", i.e. a kind of **"full" CNF**. Such formulas take only false values ("false" is "one" in the second "algebra").

Thus, we have proved the following

Theorem 5.3.1. In the classical logic, every propositional formula can be reduced to DNF and to CNF. More precisely, assume, the formula F has been built of formulas B_1, B_2, ..., B_n by using propositional connectives only. Then:

a) There is a formula F_1, which is in a (possibly empty or full) disjunctive normal form over B_1, B_2, ..., B_n such that [L_1-L_{11}, MP]: $\vdash F \leftrightarrow F_1$.

b) There is a formula F_2, which is in a (possibly empty or full) conjunctive normal form over B_1, B_2, ..., B_n such that [L_1-L_{11}, MP]: $\vdash F \leftrightarrow F_2$.

Exercise 5.3.2. a) Build DNFs and CNFs of the following formulas. (Hint: the algorithm is explained in the above Steps 1-4.)

$$\neg(A \wedge B \rightarrow C),$$
$$(A \rightarrow B) \leftrightarrow (C \rightarrow D),$$
$$A \vee B \leftrightarrow C \vee D,$$
$$A \wedge B \leftrightarrow C \wedge D.$$

b) Build DNFs and CNFs of the following formulas:

$$\neg(A \vee \neg A),$$
$$((A \rightarrow B) \rightarrow A) \rightarrow A,$$
$$(A \rightarrow B) \rightarrow ((\neg A \rightarrow B) \rightarrow B).$$

The notion of disjunctive normal form was known in 1883 to Oscar Howard Mitchell (1851-1889):

Oscar Howard Mitchell. On a New Algebra of Logic. In: *Studies in Logic by Members of the Johns Hopkins University,* 1885, pp. 72-106.

5.4. Clause Form

Warning! The principal results of this Section are valid only for the **classical logic**!

Clause Forms of Propositional Formulas

Which form is more "natural" – DNF, or CNF? Of course, CNF is more natural. Indeed, a DNF $D_1 \vee D_2 \vee ... \vee D_m$ asserts that one (or more) of the

formulas D_i is true. This is a very complicated assertion – sometimes D_1 is true, sometimes D_2 is true, etc. But, if we have a CNF instead – $C_1 \wedge C_2 \wedge ... \wedge C_n$? It asserts that **all** the formulas C_i are true, i.e. we can replace the long formula $C_1 \wedge C_2 \wedge ... \wedge C_n$ by a set of shorter formulas C_1, C_2, ..., C_n. For human reading and for computer processing, a set of shorter formulas is much more convenient than a single long formula.

Let us return to our example formula $((A \to B) \to C) \to B \wedge C$ of Section 5.3, for which we obtained a DNF $(\neg A \wedge \neg C) \vee B$ and a CNF:

$$(\neg A \vee B) \wedge (B \vee \neg C) \quad .$$

Without a transformation, DNF may be hard for reading and understanding. The CNF is more convenient – it says simply that $\neg A \vee B$ is true and $B \vee \neg C$ is true.

As another step, making the formulas easier to understand, we could apply the following equivalences:

[L_1-L_{11}, MP]: $\vdash \neg A \vee B \leftrightarrow A \to B$,

[L_1-L_{11}, MP]: $\vdash \neg A \vee \neg B \vee C \leftrightarrow A \wedge B \to C$,

[L_1-L_{11}, MP]: $\vdash \neg A \vee B \vee C \leftrightarrow A \to B \vee C$,

[L_1-L_{11}, MP]: $\vdash \neg A \vee \neg B \vee C \vee D \leftrightarrow A \wedge B \to C \vee D$,

etc.

Exercise 5.4.1. Verify these equivalences by proving that, generally (in the classical logic),

[L_1-L_{11}, MP]: $\vdash \neg A_1 \vee \neg A_2 \vee ... \vee \neg A_m \vee B_1 \vee B_2 \vee ... \vee B_n \leftrightarrow$

$$(A_1 \wedge A_2 \wedge ... \wedge A_m \to B_1 \vee B_2 \vee ... \vee B_n) \quad .$$

Thus, we can replace our set of two formulas $\neg A \vee B, B \vee \neg C$ by the set $A \to B, C \to B$. The conjunction of these two formulas is equivalent to the initial formula $((A \to B) \to C) \to B \wedge C$.

Formulas having the form

$$A_1 \wedge A_2 \wedge ... \wedge A_m \to B_1 \vee B_2 \vee ... \vee B_n \quad ,$$

or, alternatively,

$$\neg A_1 \vee \neg A_2 \vee ... \vee \neg A_m \vee B_1 \vee B_2 \vee ... \vee B_n \quad ,$$

where $A_1, A_2, ..., A_m, B_1, B_2, ..., B_n$ are atoms, are called **clauses**. Clauses are

well suited for computer processing. Indeed, in the computer memory, we can represent the above formula simply as a pair of sets of atoms – the negative set $\{A_1, A_2, \ldots, A_m\}$ and the positive set $\{B_1, B_2, \ldots, B_n\}$.

What, if one (or both) of these sets is (are) empty?

If, in the formula $\neg A_1 \vee \neg A_2 \vee \ldots \vee \neg A_m \vee B_1 \vee B_2 \vee \ldots \vee B_n$, we have m = 0 and n > 0, then, of course, this formula asserts simply that $B_1 \vee B_2 \vee \ldots \vee B_n$, i.e. "converting" it into an implication with empty premise

$$\rightarrow B_1 \vee B_2 \vee \ldots \vee B_n$$

leads us to the following definition: the clause $\rightarrow B_1 \vee B_2 \vee \ldots \vee B_n$ means the same as $B_1 \vee B_2 \vee \ldots \vee B_n$.

If, in the formula $\neg A_1 \vee \neg A_2 \vee \ldots \vee \neg A_m \vee B_1 \vee B_2 \vee \ldots \vee B_n$, we have m > 0 and n = 0, then, of course, this formula asserts simply that $\neg A_1 \vee \neg A_2 \vee \ldots \vee \neg A_m$, i.e. "converting" it into an implication with empty consequence

$$A_1 \wedge A_2 \wedge \ldots \wedge A_m \rightarrow$$

leads us to the following definition: the clause $A_1 \wedge A_2 \wedge \ldots \wedge A_m \rightarrow$ means the same as $\neg(A_1 \wedge A_2 \wedge \ldots \wedge A_m)$.

If m=n=0, then, as an empty disjunction, the clause must be qualified as false.

Note. Clauses are similar to sequents – pairs of sets of formulas (S_1, S_2), used in the proof of Theorem 4.4.5 (completeness of the constructive propositional logic) in Section 4.4. In a sequent (S_1, S_2), the sets S_1, S_2 could contain arbitrary formulas, but, in a clause, S_1, S_2 are sets of atoms.

Sets (i.e. conjunctions) of clauses are called **clause forms** (in some texts – *clausal forms*). By Theorem 5.3.1, every propositional formula can be reduced to a (possibly empty, i.e. true) CNF. Since every conjuction member of a CNF represents, in fact, a clause, we have established the following

Theorem 5.4.1. In the classical logic, every propositional formula can be reduced to a clause form. More precisely, assume, the formula F is built of formulas $B_1, B_2, \ldots, B_n$ by using propositional connectives only. Then there is a (possibly empty) clause form F' (i.e. a set of clauses) over $B_1, B_2, \ldots, B_n$ such that $[L_1\text{-}L_{11}, MP]$: $F \leftrightarrow conj(F')$, where conj(F') denotes the conjunction of the clauses contained in the set F'.

For example, as we established above, the set $\neg A \vee B, B \vee \neg C$ (or, alternatively, $A \rightarrow B, C \rightarrow B$) is a clause form of the formula $((A \rightarrow B) \rightarrow C) \rightarrow B \wedge C$.

Exercise 5.4.2. Obtain clause forms of the formulas mentioned in the Exercise 5.3.2.

Clause forms (in a sense, “clouds of simple disjunctions”) are well suited for computer processing. In the computer memory, every clause

$$\neg A_1 \vee \neg A_2 \vee ... \vee \neg A_m \vee B_1 \vee B_2 \vee ... \vee B_n$$

can be represented as a pair of sets of atoms:

$$(-\{A_1, A_2, ..., A_m\}, +\{B_1, B_2, ..., B_n\}),$$

and every clause form – as a set of such pairs – i.e. it means less character string processing and less expression parsing!

Clause Form of a Set of Formulas

In the knowledge base, the set of formulas F_1, F_2, ..., F_k is asserting the conjuction $F_1 \wedge F_2 \wedge ... \wedge F_k$. Hence, the clause form of this set can be obtained simply as the **union** of clause forms of separate formulas F_i.

Clause Forms of Predicate Formulas

Of course (unfortunately), if we would insist that the clause form must be equivalent to the initial formula, then nothing comparable to clause forms could be obtained for predicate formulas. Still, reducing of predicate formulas to "clause forms" becomes possible, if we drop this requirement, and replace it by the requirement that the "clause form" must be satisfiable, if and only if the initial formula is satisfiable. And – if we allow Skolem style extending of the language by adding new object constants and new function constants.

Then, by Skolem's Theorem (Theorem 5.2.1), for each **closed** formula F, we can obtain a **Skolem normal form** $\forall x_1 \forall x_2 ... \forall x_k$ G, where k≥0, the formula G does not contain quantifiers, and this form is satisfiable, if and only if so is F.

As the next step, by Theorem 5.3.1, let us convert G into a **CNF**, and then – into a clause form G', i.e into a set of clauses (with **atomic** sub-formulas of G playing the role of atoms B_1, B_2, ..., B_n). Since conj(G') is equivalent to G, the formula $\forall x_1 \forall x_2 ... \forall x_k$conj(G') is satisfiable, if and only if so is F.

One more step is necessary to separate clauses completely – renaming of variables in such a way that **no two clauses contain common variables**. For the set of clauses G' = {C_1, C_2, ..., C_k}, the formula $\forall x_1 \forall x_2 ... \forall x_n$conj(G') is

equivalent to the formula

$$(\forall x_1 \forall x_2 ... \forall x_n C_1) \wedge (\forall x_1 \forall x_2 ... \forall x_n C_2) \wedge ... \wedge (\forall x_1 \forall x_2 ... \forall x_n C_k) \quad .$$

According to the Replacement Theorem 3, we will obtain an equivalent formula, if we will rename the variables x_i in such a way that no two clauses contain common variables.

After this separation of clauses via renaming of variables, we can simply drop the quantifiers entirely, and the set G' is then called a **clause form** of the formula F. For predicate formulas, clauses are built as disjunctions of **atomic** formulas (without, or with negation), i.e. the formulas having the form $p(t_1, ..., t_m)$, or $\neg p(t_1, ..., t_m)$, where p is a predicate constant, and t_1, ..., t_m are terms (possibly, containing variables).

Thus, we have proved the following

Theorem 5.4.2. Let L be a predicate language. There is an algorithm allowing to construct, for each *closed* formula F of this language, a clause form S, i.e. a finite set of clauses (in a language L' obtained from L by adding a finite set of new object constants and new function constants – depending on F, and no two closes containing common variables) such that F is satisfiable, if and only if so is the formula $\forall x_1 \forall x_2 ... \forall x_n$conj(S), where conj(S) denotes the conjunction of the clauses contained in S, and x_1, x_2, ..., x_n are all the variables appearing in these clauses.

Note. In most texts, the closed formula $\forall x_1 \forall x_2 ... \forall x_n$conj(S) (i.e. where **all** the variables appearing in conj(S) are universally quantified) is called the **universal closure** of conj(S).

As an example, let us consider the formula asserting that there are infinitely many prime numbers:

$$prime(x) \ : \ x > 1 \wedge \neg \exists y \exists z (y > 1 \wedge z > 1 \wedge x = y * z) \ ,$$

$$\forall u \exists x (x > u \wedge prime(x)) \ ,$$

$$\forall u \exists x (x > u \wedge x > 1 \wedge \neg \exists y \exists z (y > 1 \wedge z > 1 \wedge x = y * z)) \qquad (1)$$

Convert it into a prenex normal form:

$$\forall u \exists x (x > u \wedge x > 1 \wedge \forall y \forall z \neg (y > 1 \wedge z > 1 \wedge x = y * z)) \ ,$$

$$\forall u \exists x \forall y \forall z (x > u \wedge x > 1 \wedge \neg (y > 1 \wedge z > 1 \wedge x = y * z)) \ .$$

Replace $\forall u \exists x$ by $\forall u$ by introducing a Skolem function g:

$$\forall u \forall y \forall z (g(u) > u \wedge g(u) > 1 \wedge \neg (y > 1 \wedge z > 1 \wedge g(u) = y * z)) \ .$$

In this Skolem normal form, convert the quantifier-free part into a conjunctive

normal form:

$$\forall u \forall y \forall z (g(u) > u \wedge g(u) > 1 \wedge (\neg(y > 1) \vee \neg(z > 1) \vee \neg(g(u) = y * z))) \quad .$$

This formula is satisfiable, if and only if so is the initial formula (1).

The last step: since the last formula is equivalent to the conjunction of three formulas:

$$\forall u \forall y \forall z (g(u) > u) \quad , \quad \forall u \forall y \forall z (g(u) > 1) \quad ,$$
$$\forall u \forall y \forall z (\neg(y > 1) \vee \neg(z > 1) \vee \neg(g(u) = y * z)) \quad ,$$

we can rename the variables in such a way that no two clauses contain common variables:

$$\forall u_1 (g(u_1) > u_1) \quad , \quad \forall u_2 (g(u_2) > 1) \quad ,$$
$$\forall u_3 \forall y \forall z (\neg(y > 1) \vee \neg(z > 1) \vee \neg(g(u_3) = y * z)) \quad .$$

Thus, we have obtained a set of 3 clauses:

$$g(u_1) > u_1 \quad ,$$
$$g(u_2) > 1 \quad ,$$
$$\neg(y > 1) \vee \neg(z > 1) \vee \neg(g(u_3) = y * z) \quad .$$

or, alternatively,

$$\rightarrow g(u_1) > u_1 \quad ,$$
$$\rightarrow g(u_2) > 1 \quad ,$$
$$y > 1, z > 1, g(u_3) = y * z \rightarrow \quad .$$

These sets of 3 formulas are clause forms of the formula (1).

Exercise 5.4.3. Obtain clause forms of the formulas mentioned in the Exercise 5.1.4 (assume that B, C, D, F are predicate constants).

Clause Form of a Set of Formulas

Knowledge bases are, as a rule, large *sets of closed formulas* F_1, F_2, ..., F_n, i.e., in fact, large conjunctions $F_1 \wedge F_2 \wedge ... \wedge F_n$ of closed formulas. The clause form of this set can be obtained, simply as the **union** of clause forms of separate formulas F_i. However, each formula must be "kept separated" during the entire process:

a) when reducing to Skolem normal forms, the name of each new Skolem constant and Skolem function must be chosen as "completely new" with respect to the entire process (for details, see the end of Section 5.2);

b) when renaming clause variables, one must guarantee that no two clauses of

the entire process contain common variables.

Horn Clauses

Alfred Horn (1918-2001).

In, in a clause

$$A_1 \wedge A_2 \wedge ... \wedge A_m \rightarrow B_1 \vee B_2 \vee ... \vee B_n \quad ,$$

or, alternatively,

$$\neg A_1 \vee \neg A_2 \vee ... \vee \neg A_m \vee B_1 \vee B_2 \vee ... \vee B_n \quad ,$$

we have n=1 or n=0, then it is called Horn clause. I.e.,

$$A_1 \wedge A_2 \wedge ... \wedge A_m \rightarrow B \quad ,$$

or, alternatively,

$$\neg A_1 \vee \neg A_2 \vee ... \vee \neg A_m \vee B \quad .$$

There are formulas that cannot be reduced to Horn clauses (verify).

http://wombat.doc.ic.ac.uk/foldoc/foldoc.cgi?Horn+clause

The name "Horn Clause" comes from the logician Alfred Horn, who first pointed out the significance of such clauses in 1951, in the article "On sentences which are true of direct unions of algebras", Journal of Symbolic Logic, 16, 14-21.

http://www.cs.ucsd.edu/users/goguen/courses/230/s11.html

As a footnote, Alfred Horn, for whom Horn clauses are named, had nothing to do with logic programming; he was a professor of logic at UCLA who in 1951 wrote paper using the sentences that now bear his name for reasons having little to do with computer science. As a second footnote, it seems to me rather misleading to call Prolog a "logic programming" language, since it departs rather far from logic; I would rather have had it called a "relational programming" language, because it is the use and manipulation of relations that is most characteristic of its programming style.

http://www.cs.fit.edu/~ryan/study/bibliography.html

Horn, Alfred. ``On sentences which are true of direct unions of algebras." Journal of Symbolic Logic, volume 16, number 1, March 1951, pages 14-21.

This paper has very little to do with Horn clauses.

To be continued.

5.5. Resolution Method for Propositional Formulas

Warning! The principal results of this Section are valid only for the **classical logic**!

Remember, that we are solving the problem of determining, does the formula G follow from the formulas $F_1, F_2, ..., F_n$. If does so, if and only if the set of formulas $F_1, F_2, ..., F_n, \neg G$ is unsatisfiable. Assume, we have obtained a clause form S of the formula $F_1 \wedge F_2 \wedge ... \wedge F_n \wedge \neg G$. Then S is unsatisfiable, if and only if so is the set $F_1, F_2, ..., F_n, \neg G$. How to determine, is S unsatisfiable, or not? In a sense, S represents "a cloud of simple disjunctions". How to work with such a cloud effectively?

History

J. A. Robinson. Theorem-proving on the computer. "Jour. Assoc. Comput. Mach.", vol.10, N2, 1963, pp.163-174

J. A. Robinson. A machine-oriented logic based on the resolution principle, "Jour. Assoc. Comput. Mach.", vol.12, N1, January 1965, pp.23-41 (available online, Russian translation available: "Kib. sbornik (novaya seriya)", 7, 1970, pp.194-218)

John Alan Robinson: "Born in Yorkshire in 1930, Robinson came to the United States in 1952 with a classics degree from Cambridge University. He studied philosophy at the University of Oregon before moving to Princeton where he received his PhD in philosophy in 1956. Temporarily ``disillusioned with philosophy,`` he went to work as an operations research analyst for Du Pont, where he learnt programming and taught himself mathematics. Robinson moved to Rice University in 1961, spending his summers as a visiting researcher at the Argonne National Laboratory's Applied Mathematics Division. Its then Director, William F. Miller, pointed Robinson in the direction of theorem proving...

Miller showed Robinson a 1960 paper by Martin Davis and Hilary Putnam (coincidentally, the latter had been Robinson's PhD supervisor) proposing a predicate-calculus proof procedure that seemed potentially superior to Gilmore's, but which they had not yet turned into a practical computer program. Miller suggested that Robinson use his programming skills to implement Davis and Putnam's procedure on the Argonne IBM 704. Robinson quickly found that their procedure remained very inefficient. However, while implementing a different procedure also suggested in 1960 by Dag Prawitz, Robinson came to see how the two sets of ideas could be combined into a new, far more efficient, automated proof procedure for first-order predicate logic: "resolution"..." (According to Donald MacKenzie, The Automation of Proof: A Historical and Sociological Exploration, "IEEE Annals of the History of Computing", vol.17, N3, 1995, pp. 7-29).

"In retrospect, unification and resolution seem rather obvious ideas, which arise inevitably when one asks what must be syntactically true of a set of clauses which possesses the semantic property of having no Herbrand models."

(J.A.Robinson, "Unification and Resolution in Retrospect", 1997, see at http://www.univ-orleans.fr/SCIENCES/LIFO/Manifestations/Jfplc_Unif_97/jfplc/invite-francais.html).

Note. Almost at the same time when J.A.Robinson invented the resolution

method, Sergei Maslov invented his **inverse method**, which has a similar range of applications:

S. Yu. Maslov. An inverse method of establishing deducibilities in the classical predicate calculus, "Soviet Mathematics, Doklady", 1964, N5, pp.1420-1423.

See also: Maslov S. Y. (1939-1982), human rights activist in ENCYCLOPAEDIA OF SAINT PETERSBURG.

About the history of the problem see:

J. A. Robinson. Computational Logic: Memories of the Past and Challenges for the Future. *Computational Logic – CL 2000, First International Conference, London, UK, 24-28 July, 2000, Proceedings*, Springer, Lecture Notes in Computer Science, 2000, Vol. 1861, pp. 1-24 (online copy).

M. Davis. The Early History of Automated Deduction.In: *Handbook of Automated Reasoning*, ed. by A. Robinson and A. Voronkov, Elsevier Science, 2001, vol. I, pp. 3-15 (online postscript)

The Method

Again, how to work with “a cloud of simple disjunctions” effectively?

Assume that, in a set of clauses, two clauses are contained such that an atom C appears as a positive member in the first clause, and as a negative member in the second one:

$$\neg A_1 \vee \neg A_2 \vee \ldots \vee \neg A_m \vee B_1 \vee B_2 \vee \ldots \vee B_n \vee \boldsymbol{C} \quad , \qquad (1)$$

$$\neg \boldsymbol{C} \vee \neg D_1 \vee \neg D_2 \vee \ldots \vee \neg D_p \vee E_1 \vee E_2 \vee \ldots \vee E_q \quad , \qquad (2)$$

or, simply,

$$F \vee \boldsymbol{C} \quad , \qquad (1a)$$

$$\neg \boldsymbol{C} \vee G \quad . \qquad (2a)$$

If C is false, then (1a) yields F, and, if C is true, then (2a) yields G. Thus, from (1a) and (2a) we have derived $F \vee G$. I.e. deriving of $F \vee G$ from $F \vee \boldsymbol{C}$ and $\neg \boldsymbol{C} \vee G$ is "logically correct", and it is called **Robinson's Resolution rule** (J.A.Robinson proposed it in the above 1963 paper):

$$\frac{F \vee C \, , \neg C \vee G}{F \vee G} \quad .$$

Taking into account the rule (of the classical logic) $\neg A \vee B \leftrightarrow (A \rightarrow B)$, we can obtain an alternative form of the Resolution rule:

$$\frac{\neg F \rightarrow C \, , C \rightarrow G}{\neg F \rightarrow G} \quad .$$

In the classical logic, this form is equivalent to the **Law of Syllogism**

(transitivity of implication).

If F is empty, then this form derives G from C, C→G, i.e. **Resolution rule includes Modus Ponens** as a special case.

If G is empty, then from $\neg F \vee C, \neg C$ (i.e. F→C, ¬C), the **Resolution rule** derives ¬F, i.e. it **includes Modus Tollens** as a special case.

Exercise 5.5.1. Derive the Resolution rule in the constructive logic, i.e. prove that [L_1-L_{10}, MP]: $C \vee F, \neg C \vee G \vdash F \vee G$. Verify that it cannot be proved in the minimal logic [L_1-L_9, MP]. (Hint: in the positive part – use Theorem 2.5.1(b) [L_1, L_2, L_8, L_{10}, MP]: $F \vee C, \neg C \vdash F$. In the negative part – verify that in the minimal logic, the Resolution rule allows proving of L_{10}, see Section 2.5).

Thus, from the clauses (1) and (2), Robinson's Resolution rule allows deriving of the following clause:

$$\neg A_1 \vee \neg A_2 \vee \ldots \vee \neg A_m \vee \neg D_1 \vee \neg D_2 \vee \ldots \vee \neg D_p \vee B_1 \vee B_2 \vee \ldots \vee B_n \vee E_1 \vee E_2 \vee \ldots \vee E_q$$

At first glance, this approach leads to nothing, because this formula seems to be much longer than (1), and than (2). Still, this is not 100% true, because, additionally, we can reduce the repeating atoms, and, finally, the set of different atoms, used in a clause form, is fixed! If, in our set of clauses, there are N different atoms, then none of the clauses (initial, or generated by resolution) will contain more than N atoms (each with or without negation). And the total number of different clauses will never exceed 3^N (*missing, without negation, with negation*). Thus, repeated applications of the Resolution rule will "rotate" within this restricted "search space".

The smart idea behind Robinson's Resolution rule is as follows: it is a universal tool for **deriving contradictions from inconsistent sets of clauses**! No other axioms and rules of inference are necessary! More precisely, it is universal, if used together with the following trivial rules of inference:

$$\frac{F \vee \boldsymbol{C} \vee \boldsymbol{D} \vee G}{F \vee \boldsymbol{D} \vee \boldsymbol{C} \vee G} \quad \text{(permutation)},$$

$$\frac{F \vee \boldsymbol{C} \vee \boldsymbol{C} \vee G}{F \vee \boldsymbol{C} \vee G} \quad \text{(reduction)}.$$

The permutation rule allows arbitrary reordering of atoms in a clause (for example, moving C to right, and moving ¬C to left). The reduction rule allows reduction of repeating identical atoms.

Exercise 5.5.2. Derive these inference rules in the minimal logic [L_1-L_9, MP].

Theorem 5.5.1 (J. A. Robinson). In the classical propositional logic [L_1-L_{11},

MP], a finite set of propositional clauses is inconsistent, if and only if Robinson's Resolution rule (together with permutation and reduction rules) allows for deriving of a contradiction from it.

Note. In some other texts, this fact is called "the refutation-completeness of the Resolution rule" for the propositional logic.

Proof. 1. As you have proved in the Exercises 5.5.1 and 5.5.2, all the formulas, derived from a set of formulas K_1, K_2, ... , K_s by using the Permutation, Resolution and Reduction rules are consequences of K_1, K_2, ... , K_s. Hence, if these rules allow deriving a contradiction from this set of formulas, then it (the set) is inconsistent.

2. Now, let us assume that a set of propositional clauses K_1, K_2, ... , K_s is inconsistent, i.e. a contradiction $A \wedge \neg A$ can be derived from it:

$$[L_1\text{-}L_{11}, MP]: K_1, K_2, \dots, K_s \vdash A \wedge \neg A \quad .$$

Then, under the classical truth tables, the conjunction $K_1 \wedge K_2 \wedge \dots \wedge K_s$ takes only false values (verify!). Let us mark one of the atoms (the atom C) in it. Let us denote:

- by $C \vee F_i$ – the clauses containing C without negation,

- by $\neg C \vee G_j$ – the clauses containing C with negation,

- by H_k – the clauses that do not contain C.

All the formulas F_i, G_j, H_k are disjunctions of atoms (with or without negations) that do not contain the atom C.

Thus $K_1 \wedge K_2 \wedge \dots \wedge K_s$ is equivalent to

$$conj(C \vee F_i) \wedge conj(\neg C \vee G_j) \wedge conj(H_k) \quad . \qquad (4)$$

Let us apply (the strange) one of the distribution rules (Theorem 2.3.1):

$$[L_1\text{-}L_8, MP] \vdash (A \wedge B) \vee C \leftrightarrow (A \vee C) \wedge (B \vee C) \quad .$$

Hence, $K_1 \wedge K_2 \wedge \dots \wedge K_s$ is equivalent to

$$(C \vee conj(F_i)) \wedge (\neg C \vee conj(G_j)) \wedge conj(H_k) \quad .$$

If C is false, then this formula is equivalent to $conj(F_i) \wedge conj(H_k)$, i.e. $conj(F_i) \wedge conj(H_k)$ takes only false values. If C is true, then it is equivalent to $conj(G_j) \wedge conj(H_k)$, i.e. $conj(G_j) \wedge conj(H_k)$ takes only false values. Thus the disjunction

$$(conj(F_i) \wedge conj(H_k)) \vee (conj(G_j) \wedge conj(H_k)) \qquad (5)$$

also takes only false values. Now, let us, apply (the "normal") one of the distribution rules (Theorem 2.3.1):

$$[L_1\text{-}L_8, MP] \vdash (A \vee B) \wedge C \leftrightarrow (A \wedge C) \vee (B \wedge C) \quad ,$$

obtaining that (5) is equivalent to

$$(conj(F_i) \vee conj(G_j)) \wedge conj(H_k) \quad . \qquad (6)$$

I.e. this formula also takes only false values. And – important note! – it does not contain the atom C.

Finally, by applying, again, (the strange) one of the distribution rules (Theorem 2.3.1) we can conclude that (6) is equivalent to $conj(conj(F_i \vee G_j)) \wedge conj(H_k)$, i.e. to the set of clauses $F_i \vee G_j$ and H_k (where i, j, k run over their initial ranges).

What does this achievement mean? If the set of propositional clauses K_1, K_2, ... , K_s is inconsistent, then there is a set of clauses $F_i \vee G_j$ and H_k (where i, j, k run over their initial ranges), which is inconsistent as well, but which contains one atom less than K_1, K_2, ... , K_s.

Now, imagine, that, in the clause form (4), we have applied the Resolution rule for the atom C **in all the possible ways** (before applying, apply the permutation rule to reorder atoms moving C to right, and ¬C – to left):

$$\frac{F_i \vee C \, , \neg C \vee G_j}{F_i \vee G_j} \quad .$$

After this, apply the permutation and reduction rules to reduce identical atoms. In this way we have obtained exactly the above-mentioned inconsistent set of clauses $F_i \vee G_j$ and H_k (where i, j, k run over their initial ranges).

Thus, if some set of propositional formulas K_1, K_2, ... , K_s is inconsistent, then the Resolution rule (togeher with the permutation and reduction rules) allows to derive from it another inconsistent set of propositional formulas, which contains one atom less.

By iterating this process, at the end of it, we will have an inconsistent set of propositional formulas built of a single atom B. In a clause form, there can be only one such set – the set B, ¬B. This set represents a contradiction.

Q.E.D.

As an example, let us use Robinson's Resolution rule to prove that

$$B \vee C \, , C \rightarrow B \, , B \rightarrow D \quad \vdash \quad B \wedge D \quad .$$

Let us add $\neg(B \wedge D)$ to the premises $B \vee C \, , C \rightarrow B \, , B \rightarrow D$. We must

prove that this set of 4 formulas is inconsistent. First, let us obtain clause forms:

$B \vee C$ in clause form is $B \vee C$,

$C \rightarrow B$ in clause form is $\neg C \vee B$,

$B \rightarrow D$ in clause form is $\neg B \vee D$,

$\neg (B \wedge D)$ is equivalent to $\neg B \vee \neg D$.

Now, let us apply resolution to derive a contradiction from this set of 4 clauses: BvC, ¬CvB, ¬BvD, ¬Bv¬D:

From $B \vee \boldsymbol{C}, \neg \boldsymbol{C} \vee B$ we derive B, and have now 5 clauses:

$$B \vee C, \neg C \vee B, \neg B \vee D, \neg B \vee \neg D, B \quad .$$

From $\neg B \vee \boldsymbol{D}, \neg B \vee \neg \boldsymbol{D}$ we derive ¬B, and have now 6 clauses:

$$B \vee C, \neg C \vee B, \neg B \vee D, \neg B \vee \neg D, B, \neg B \quad .$$

We have derived a contradiction: B, ¬B. This proves that the formula $B \wedge D$ follows from $B \vee C, C \rightarrow B, B \rightarrow D$. Q.E.D.

Exercise 5.5.3. Use the Resolution rule to prove the following:

a) A→B, ¬A→B ⊢B.
b) (A→B)→A ⊢A (Peirce's Law).
c) B→(C→D), B→C ⊢B→D (Axiom L_2).
d) B→D, C→D ⊢ $B \vee C \rightarrow D$. (Axiom L_8).
e) $A \vee B \vee C, B \rightarrow A \vee C, A \rightarrow C$ ⊢C.

From a Programmer's Point of View

Of course, when implementing the Resolution rule in a computer program, we do not need decorations like the permutation and reduction rules. In a program, we will represent each clause $\neg A_1 \vee \neg A_2 \vee ... \vee \neg A_m \vee B_1 \vee B_2 \vee ... \vee B_n$ as a pair of sets: negative atoms, N = {A_1, A_2, ... , A_m}, and positive atoms, P = {B_1, B_2, ... , B_n}. Of course, the sets N, P do not intersect (if they do, then this clause contains $\neg C \vee C \vee ...$, i.e. it can be dropped as "non-informative").

Resolution rule (non-refined version). If there are two clauses N_1, P_1 and N_2, P_2 such that P_1 and N_2 (or N_1 and P_2) contain a common atom C, then we can derive the clause $N_1 \cup N_2 - \{C\}, P_1 \cup P_2 - \{C\}$.

Of course, the set union operation includes reduction of identical members automatically.

The condition "P_1 and N_2 (or N_1 and P_2) contain a common atom C" can be expressed as $C \in (P_1 \cap N_2) \cup (P_2 \cap N_1)$.

If, in the resulting clause, the sets $N_1 \cup N_2 - \{C\}, P_1 \cup P_2 - \{C\}$ intersect, then we should ignore such result. Fortunately, this can be detected in advance. Indeed,

$$(N_1 \cup N_2) \cap (P_1 \cup P_2) = (N_1 \cap P_1) \cup (N_1 \cap P_2) \cup (N_2 \cap P_1) \cup (N_2 \cap P_2) = (P_1 \cap N_2) \cup (P_2 \cap N_1) ,$$

because $N_1 \cap P_1, N_2 \cap P_2$ are empty sets. The set $(P_1 \cap N_2) \cup (P_2 \cap N_1)$ is exactly the set of all atoms C allowing application of the Resolution rule to clauses N_1, P_1 and N_2, P_2. Hence, the sets $N_1 \cup N_2 - \{C\}, P_1 \cup P_2 - \{C\}$ will not intersect, if and only if the set $(P_1 \cap N_2) \cup (P_2 \cap N_1)$ **contains exactly one atom** C, i.e., if and only if there is exactly one atom allowing application of the Resolution rule.

Resolution rule (refined version). If there are two clauses N_1, P_1 and N_2, P_2 such that the set $(P_1 \cap N_2) \cup (P_2 \cap N_1)$ contains exactly one atom C, then we can derive the clause $N_1 \cup N_2 - \{C\}, P_1 \cup P_2 - \{C\}$.

Now, let us try to design a program implementing the last step of "proving by resolution" – suppose, we have already the **initial list** of clauses, and we wish to apply the Resolution rule trying to derive a contradiction.

The main data storage will be a growing list of clauses (the **main list**):

$$(N_1, P_1), (N_2, P_2), ..., (N_k, P_k), ...$$

It will start as the initial list, and each application of the Resolution rule will append a new clause to it.

To guarantee a success, we must apply the Resolution rule **in all the possible ways**, i.e. we must scan all pairs of clauses $(N_i, P_i)(N_j, P_j)$, where i = 1, 2, ...; j = i+1, i+2, ... To achieve this, let us use the following pair enumeration process:

$(N_1, P_1)(N_2, P_2)$ – first, scan all pairs (i, j) with **j=2**, i<j.

$(N_1, P_1)(N_3, P_3)$, $(N_2, P_2)(N_3, P_3)$ – after this, scan all pairs (i, j) with **j=3**, i<j.

$(N_1, P_1)(N_4, P_4)$, $(N_2, P_2)(N_4, P_4)$, $(N_3, P_3)(N_4, P_4)$ – after this, scan all pairs (i, j) with **j=4**, i<j.

Etc.

The process will stop, when we will arrive at the level j, and the main list will

contain less than j (in fact, j-1) clauses. For a set of n atoms, there are only 3^n different clauses. For example, for two atoms A, B, there are 9 different clauses: $\neg A \vee \neg B, \neg A \vee B, A \vee \neg B, A \vee B, \neg A, A, \neg B, B$, and the empty clause (representing contradiction). I.e., if we will prohibit duplicate clauses in the main list, then our process will always stop.

Thus, the following pseudo-code will do (no string processing, no expression parsing necessary!):

function propositional resolution (initial list) { of clauses }
begin
if initial list contains contradiction **then return** TRUE { contradiction found }
main list = eliminate duplicates (initial list)
for j = 2 **by** 1
begin
- **if** count (main list) < j **then return** FALSE { no contradiction derived }
- **else**
- **for** i = 1 **to** j−1 **by** 1
-- { consider i-th and j-th clauses in the main list: (N_i, P_i), (N_j, P_j) }
-- **if** $(N_i \cap P_j) \cup (P_i \cap N_j)$ contains exactly one element C **then**
-- **begin**
--- {apply resolution}
--- **if** ($N_i \cup N_j - \{C\}, P_i \cup P_j - \{C\}$) **not in** main list **then**
--- **begin**
---- **add** it **to** main list
---- **if** main list contains contradiction **then return** TRUE { contradiction derived }
-- **end**
--- **end**
end
end

Exercise 5.5.4 (optional). Develop a computer program implementing the above pseudocode.

Note. See my version of such a program in C++: header file, implementation).

Warning!

Despite its beauty, resolution method cannot overcome the general complexity problem, mentioned at the end of Section 4.2: in the classical propositional logic, the task of reasoning is "co-NP-complete". And a closer analysis shows that, in the **worst possible case**, given a set of formulas of total length *n,* the time required by resolution method will be exponentional – about 2^{Cn} seconds. But in many **practical situations**, experience shows that resolution method solves its task, and – in acceptable time. In particular, Prolog

interpreters are using resolution, and are solving many practical tasks in acceptable time!

5.6. Herbrand's Theorem

Warning! The principal results of this Section are valid only for the **classical logic**!

Jacques Herbrand (1908-1931) "... After leaving Göttingen, Herbrand decided on a holiday in the Alps before his intended return to France. However he was never to complete his plans for he died in a mountaineering accident in the Alps only a few days after his holiday began. His death at the age of 23 in one of the tragic losses to mathematics." (according to MacTutor History of Mathematics archive).

Herbrand proved his famous theorem in 1929:

J.Herbrand. Recherches sur la théorie de la démonstration. Ph.D. Thesis, University of Paris, 1930 (approved in April 1929).

Unlike the proof presented below, the original proof of Herbrand's Theorem does not depend on Gödel's Completeness Theorem (or Model Existence Theorem). Herbrand completed his Ph.D. thesis in 1929. In the same 1929 Gödel completed his doctoral dissertation about completeness (see Section 4.3). In fact, Herbrand's method allows proving of Gödel's Completeness Theorem, but he (Herbrand) "did not notice it". Why? See

Samuel R. Buss. On Herbrand's Theorem. "Lecture Notes in Computer Science", Vol.960, 1995, Springer-Verlag, pp.195-209 (available online).

The flavour of this famous theorem can be best presented in its simplest version. In this version, F(x) is a quantifier-free formula containing only one variable x. Then, Herbrand's Theorem says:

The formula $\exists$xF(x) is logically valid, if and only if there is a finite set of constant terms t_1, ..., t_n such that the disjunction $F(t_1)\vee...\vee F(t_n)$ is logically valid.

Or, equivalently (via Gödel's Completeness Theorem),

The formula $\exists$xF(x) is provable in the classical logic, if and only if there is a finite set of constant terms t_1, ..., t_n such that the disjunction $F(t_1)\vee...\vee F(t_n)$ is provable in the classical logic.

As we will see in the proof, Herbrand's Theorem is "caused" by the simple

"fact" that in any proof of ∃xF(x) only a finite set of terms could be used.

Now, more precisely.

Let L be a predicate language, containing at least one object constant, and let F be a quantifier-free formula.

Idea #1 (author?). The formula $p(c_1) \wedge q(c_2, f(x))$ is quantifie-free (c_1, c_2 are object constants, f – a function constant, p, q – predicate constants). In a sense, any "closed" interpretation domain for this formula must contain objects denoted by the terms c_1, c_2, $f(c_1)$, $f(c_2)$, $f(f(c_1))$, $f(f(c_2))$,...

So, let us define the so-called **Herbrand's universe** of the formula F (let us denote it by HU_F) as the minimum set of all constant terms such that:

a) If c is an object constant occurring in F, then c is in HU_F.

b) If F does not contain object constants, then one of the constants of the language L is in HU_F.

c) If terms t_1, ..., t_k are in HU_F, and f is a k-ary function constant occurring in F, then the term $f(t_1, ..., t_k)$ is in HU_F.

Exercise 5.6.1. Verify that HU_F is a non-empty finite or countable set (provide an algorithm generating the members of HU_F).

Theorem 5.6.1 (Herbrand's Theorem – the simplest case). Let L be a predicate language, containing at least one object constant, and let F(x) be a quantifier-free formula containing only one free variable x. Then the formula ∃xF(x) is **logically valid (i.e. provable in the classical predicate logic)**, if and only if there is a finite set of terms t_1, ..., t_n from HU_F such that the disjunction $F(t_1) \vee ... \vee F(t_n)$ is **logically valid (i.e. provable in the classical predicate logic)**.

Proof. Let us assume the contrary – that none of the disjunctions $F(t_1) \vee ... \vee F(t_n)$ is logically valid (t_i-s are terms from HU_F). Idea #2 – then the following theory T is consistent:

$$T = \{ \neg F(t) \mid t \text{ is a term from } HU_F \}.$$

Indeed, if T would be inconsistent, then there would be a T-proof of some formula B&¬B. In this proof, only a finite set of the axioms ¬F(t) would be used, i.e. for some terms t_1, ..., t_n from HU_F:

$$[L_1\text{-}L_{15}, \text{MP, Gen}]: \neg F(t_1), ..., \neg F(t_n) \vdash B \wedge \neg B \quad .$$

Hence, by Deduction Theorem 2 (it is applicable here, because F(x) contains

only one free variable, and t_i-s are constant terms, i.e. every $\neg F(t_i)$ is a closed formula):

$$[L_1\text{-}L_{15}, MP, Gen]: \vdash \neg F(t_1) \wedge ... \wedge \neg F(t_n) \rightarrow B \wedge \neg B \ ,$$

$$[L_1\text{-}L_{15}, MP, Gen]: \vdash \neg(F(t_1) \vee ... \vee F(t_n)) \rightarrow B \wedge \neg B \ ,$$

and thus,

$$[L_1\text{-}L_{15}, MP, Gen]: \vdash F(t_1) \vee ... \vee F(t_n) \ .$$

I.e., $F(t_1) \vee ... \vee F(t_n)$ is logically valid. This contradicts our assumption, that none of the disjunctions $F(t_1) \vee ... \vee F(t_n)$ is logically valid. Hence, T is a consistent theory.

Idea #3 – if T is consistent, then, by the Model Existence Theorem, there is a model J of T. In this model, all the axioms of T are true, i.e. so are all the formulas $\neg F(t)$ with t from the set HU_F.

Idea #4 – let us restrict the domain of the model J to those elements of it, which are interpretations of terms from the set HU_F, and let us restrict the entire interpretation correspondingly. Let us denote this new interpretation by J_1. Then,

a) All the formulas $\neg F(t)$ (with t from the set HU_F) are true in J_1. Indeed, $\neg F(t)$ contains only constant terms from HU_F (idea #1 working!), and all of them have the same interpretations in J_1 that they had in J. Thus, if $\neg F(t)$ was true in J, it remains true in J_1.

b) Hence, the formula $\forall x \neg F(x)$ is true in J_1 (because the domain of J_1 consists only of those elements, which are interpretations of terms from the set HU_F).

c) Hence, the formula $\exists x F(x)$ is false in J_1.

This contradicts the logical validity of $\exists x F(x)$.

Q.E.D.

Exercise 5.6.2. Repeat the above proof, proving a more general form of Herbrand's Theorem:

Theorem 5.6.2 (Herbrand's Theorem – the simplest case). Let L be a predicate language, containing at least one object constant, and let $F(x_1, ..., x_m)$ be a quantifier-free formula containing only m free variables $x_1, ..., x_m$. The formula $\exists x_1 ... \exists x_m F(x_1, ..., x_m)$ is **logically valid**, if and only if there is a finite

set of m-tuples tt_1, ..., tt_n of terms from HU_F such that the disjunction $F(tt_1)\vee...\vee F(tt_n)$ is **logically valid**.

As you verified it in the Exercise 4.1.1, any formula G is logically valid, if and only if ¬G is unsatisfiable. Thus, $\exists x_1...\exists x_m F(x_1, ..., x_m)$ is logically valid, if and only if $\forall x_1...\forall x_m \neg F(x_1, ..., x_m)$ is unsatisfiable. On the other hand, $F(tt_1)\vee...\vee F(tt_n)$ is logically valid, if and only if $\neg F(tt_1)\wedge...\wedge\neg F(tt_n)$ is unsatisfiable. Now, let us replace F by ¬F, and we have proved

Theorem 5.6.3 (Herbrand's Theorem – a more useful alternative form). Let L be a predicate language, containing at least one object constant, and let $F(x_1, ..., x_m)$ be a quantifier-free formula containing only m free variables x_1, ..., x_m. The formula $\forall x_1...\forall x_m F(x_1, ..., x_m)$ is **unsatisfiable** (i.e. **inconsistent** in the classical logic**)**, if and only if there is a finite set of m-tuples tt_1, ..., tt_n of terms from HU_F such that the conjunction $\neg F(tt_1)\wedge...\wedge\neg F(tt_n)$ is **unsatisfiable** (i.e. **inconsistent** in the classical logic**)**.

Note. As you verified it in the Exercise 4.3.6, a set of formulas is inconsistent in the classical logic, if and only if it is unsatisfiable.

Why is this form "more useful"? Let us try applying this form of Herbrand's Theorem to sets of formulas in clause form.

1) The "meaning"of any set of closed formulas F_1, ... , F_k is represented by their conjunction $F_1\wedge...\wedge F_k$.

2) A clause is any disjunction of atomic formulas or their negations. For example, $\neg p(c_1)\vee p(c_2)\vee q(x, f(y))$, or $p(x)\vee\neg q(y, f(z))$. The "meaning" of a set of clauses is represented by their universally quantified conjunction. For example,

$$\forall x\,\forall y\,\forall z([\neg p(c_1)\vee p(c_2)\vee q(x, f(y))]\wedge[p(x)\vee\neg q(y, f(z))]) \quad .$$

3) As we know from the previous Section 5.4, the set F_1, ... F_k can be reduced to a clause from, i.e. there is a set of clauses S such that F_1, ... , F_k is unsatisfiable, if and only if S is unsatisfiable.

Now, let us apply the above form of Herbrand's Theorem (Theorem 5.6.3). If S contains m variables (of course, all of them are universally quantified), then S is unsatisfiable, if and only if there is a finite set of m-tuples tt_1, ..., tt_n of terms from HU_S such that the conjunction $S(tt_1)\wedge...\wedge S(tt_n)$ is unsatisfiable.

If we take a clause from S, and substitute some terms from HU_S for all its variables, then we obtain a (so-called) **ground clause** of S. For example, if

$$S = \{ \quad \neg p(c_1) \vee p(c_2) \vee q(x, f(y)) \quad ; \quad p(x) \vee \neg q(y, f(z)) \quad \},$$

then the substitution { c_1 / x; c_2 / y; f(c_2) / z } yields the following two ground clauses:

$$\neg p(c_1) \vee p(c_2) \vee q(f(c_1), f(c_2)) \quad ,$$
$$p(c_1) \vee \neg q(c_2, f(f(c_2))) \quad .$$

Of course, the conjunction $S(tt_1) \wedge ... \wedge S(tt_n)$ is a set of ground clauses. Thus, **if S is unsatisfiable, then there is an unsatisfiable finite set of ground clauses of S.** And conversely?

If there is an unsatisfiable finite set C = {C_1, ..., C_n} of ground clauses of S, then each C_i is generated by some substitution, which can be represented as an m-tuple tt_i of terms from HU_S. If {C_1, ..., C_n} is unsatisfiable, then {S(tt_1), ..., S(tt_n)} – as a super-set of the former, is unsatisfiable, too ("even more unsatisfiable").

Now, if S would be satisfiable, then (because all the variables of S are meant universally quantified) so would be the formula $S(tt_1) \wedge ... \wedge S(tt_n)$. Contradiction.

Thus, we have proved another form of Herbrand's Theorem.

Theorem 5.6.4 (Herbrand's Theorem – the most useful form. Author – Herbert B.Enderton?). Let L be a predicate language, containing at least one object constant, and let F_1, ..., F_k be a set of closed formulas in L. Then this set is **unsatisfiable**, if and only if its clause form allows an **unsatisfiable** finite set of ground clauses.

Why is this form "the most useful"? Because (let us ignore performance problems),

a) The clause form of F_1, ..., F_k is a finite set S, generated by a simple (but a very slow) algorithm (see Sections 5.1-5.4).

b) Herbrand's universe HU_S is a finite or infinite set of constant terms, generated by a simple algorithm (see Exercise 5.6.1).

c) Thus, all the possible finite sets of ground clauses of S can be generated by a simple combination of the above two algorithms.

d) Unsatisfiability of each finite set of ground clauses can be detected by a simple (but a very slow) algorithm (see Lemma 5.6.5 below).

Thus, we have here a **simple algorithm** (but a very slow one) **for checking provability in the classical predicate logic.**

Lemma 5.6.5. A finite set of ground clauses is unsatisfiable, if and only if the conjunction of these clauses is unsatisfiable under the classical truth tables.

Proof. In the above example of ground clauses:

$$\neg p(c_1) \vee p(c_2) \vee q(f(c_1), f(c_2)) \quad ,$$
$$p(c_1) \vee \neg q(c_2, f(f(c_2))) \quad ,$$

we have 5 different atoms: $p(c_1)$, $p(c_2)$, $q(f(c_1), f(c_2))$, $q(c_2, f(f(c_2)))$. Let us denote these atoms by Q_1, Q_2, Q_3, Q_4. Thus we obtain the following propositional formula

$$(\neg Q_1 \vee Q_2 \vee Q_3) \wedge (Q_1 \vee \neg Q_4) \quad .$$

1. If this formula **cannot** be satisfied under the classical truth tables, then we cannot assign truth values to predicates p, q in a way making all the corresponding clauses true. I.e. then the corresponding set of ground clauses also cannot be satisfied. Q.E.D.

2. If this formula **can** be satisfied under the classical truth tables, then we can find a truth value assignement making it true, for example:

Q_1=false (this makes the first disjunction true),

Q_4=false (this makes the second disjunction true).

Now, we can define the following interpretation J making the corresponding ground clauses true:

D_J = { c_1, c_2, $f(c_1)$, $f(c_2)$, $f(f(c_2)$ } (the set of all terms appearing in the clauses, i.e. a subset of the Herbrand universe);

$p(c_1)$=false, $q(c_2, f(f(c_2))$=false (these assignements make both ground clauses true).

All the other truth values are irrelevant, so, we can define them, for example, as follows:

$p(c_2)$=true, $p(f(c_1))$=true, $p(f(c_2))$=true, $p(f(f(c_2))$=true;

q(x, y)=true, if x is not c_2, or y is not $f(f(c_2)$.

Q.E.D.

...

To be continued.

...

Further reading:

Michael Genesereth. Computational Logic (see at http://logic.stanford.edu/classes/cs157/2005fall/cs157.html).

5.7. Resolution Method for Predicate Formulas

Warning! The principal results of this Section are valid only for the **classical logic**!

If we are interested only in deriving contradictions from inconsistent sets of formulas, then we can note that a set of closed predicate formulas is inconsistent (i.e. allows deriving a contradiction in the classical logic), if and only if the conjunction of these formulas is unsatisfiable (Exercise 4.3.6). Thus, instead of the initial set, we can analyze the set of clause forms of these formulas. Indeed, if we derive a contradiction from the set of clause forms, then this set is unsatisfiable, i.e., by Theorem 5.4.2, so is the initial set, and hence, the initial set is inconsistent. And conversely, if the initial set of formulas is consistent, then it is satisfiable, i.e. so is the set of clause forms, i.e. we will be not able to derive a contradiction from it.

The next step forward – in clause forms, we can **drop all the universal quantifiers**. Indeed, if we derive a contradiction from a set universally quantified clause forms, then we can derive it from the corresponding non-quantified set (we can apply the Gen inference rule $F(x) \vdash \forall xF(x)$ to obtain the quantified forms from the non-quantified ones). And conversely, if we derive a contradiction from a set of non-quantified clause forms, then we can derive it from the corresponding universally quantified set (apply the Axiom L_{12}: $\forall xF(x) \to F(x)$ to obtain non-quantified forms from the quantified ones).

After dropping quantifiers, sets of clause forms become simply sets of clauses (conjunction of conjunctions is equivalent to a "joint" conjunction).

Thus, we can concentrate on sets of clauses that do not contain quantifiers, like as the one obtained in Section 5.4:

$$\begin{gathered} \to g(u_1) > u_1 \quad , \\ \to g(u_2) > 1 \quad , \\ y > 1, z > 1, g(u_3) = y * z \to \quad . \end{gathered}$$

Note that clauses consist of **atomic** formulas only, and **no two clauses**

contain common variables. Thus, clauses are completely separated, and this separation will greatly simplify processing of clauses by means of substitution (see below).

Will the Robinson's Resolution rule remain a universal tool for deriving contradictions also from inconsistent sets of predicate formulas (i.e. sets of non-quantified clauses, consisting of atomic formulas)?

Let us imagine, we have derived the following two formulas (p is a unary predicate constant, 0 – an object constant):

$$p(x_1) \vee F(x_1, y_1) \quad , \quad \neg p(0) \vee G(x_2, y_2) \quad .$$

To apply the Robinson's Resolution rule, we must first, in $p(x_1)$, substitute 0 for x_1:

$$p(0) \vee F(0, y_1) \quad , \quad \neg p(0) \vee G(x_2, y_2) \quad .$$

Now, we can apply the Resolution rule, obtaining the formula

$$F(0, y_1) \vee G(x_2, y_2) \quad .$$

Surprisingly, this simple idea of "**unification by substitution**" is sufficient to make Robinson's Resolution rule a universal tool for deriving contradictions also from inconsistent sets of predicate formulas! And, in general, the necessary substitutions are not much more complicated than in the above simplest example.

Note. In fact, unification is a very general phenomenon in human and computer reasoning – it appears as one of the main components in deductive, inductive and analogical reasoning as well. More – at the end of this Section.

The **substitution rule** allows, in some clause C, replacing of all occurrences of some variable x by any term t.

Theorem 5.7.1 (J. A. Robinson). In the classical predicate logic [L_1-L_{11}, L_{12}-L_{15}, MP, Gen], a set of predicate clauses is inconsistent, if and only if Robinson's **Resolution rule** (together with permutation, reduction and **substitution** rules) allows deriving a contradiction from it.

Note. In some other texts, this fact is called "the refutation-completeness of the Resolution rule".

Proof. 1. All the formulas, derived from a set of clauses K_1, K_2, ... , K_s by using permutation, reduction, substitution and Resolution rules, are consequences of K_1, K_2, ... , K_s. Hence, if these rules allow deriving a contradiction from this set of clauses, then it (the set) is inconsistent.

2. Now, let us assume that the set of clauses S = {K_1, K_2, ... , K_s} is

inconsistent. Then it is unsatisfiable (Exercise 4.3.6). And then, by Herbrand's Theorem, it allows a finite unsatisfiable set of ground clauses C_1, ..., C_n. Each C_i of these ground clauses is obtained from some clause in S by means of some substitution sub_i (of terms from the Herbrand universe HU_S), i.e. by applying the substitution rule.

By Lemma 5.6.5, the set C_1, ..., C_n is unsatisfiable, if and only if the conjunction $C_1 \wedge ... \wedge C_n$ is unsatisfiable under the classical truth tables, i.e., **if and only if** the set C_1, ..., C_n is inconsistent. And, by Theorem 5.5.1, a finite set of propositional clauses is inconsistent; if and only if Robinson's Resolution rule (together with permutation and reduction rules) allows deriving a contradiction from it.

Q.E.D.

Refinements – Step 1 (First of the Two Smart Ideas)

Let us examine once more the part two of the proof of Theorem 5.7.1, where a specific (hopeless!) "proof strategy" is used.

First, since two clauses K_i do not contain common variables, we can think that each of the substitutions sub_j is applied to a single clause, i.e. we can think, in fact, of a (finite) set of substitutions sub_{ij}, where each sub_{ij} is applied only to the clause K_i. Let us denote by F.sub the result of application of the substitution sub to the formula F.

Second, to derive a contradiction from $\{K_1, K_2, ... , K_s\}$, we may apply, **first**, all the necessary substitutions (stage 1 – substitutions only!), and, **after this**, all the necessary permutations, reductions and resolutions (stage 2 – no more substitutions!). This is exactly the above-mentioned specific (hopeless!) "proof strategy". Why hopeless? Because, before applying the substitutions sub_{ij}, we must **find them among all the possible substitutions** of terms from the infinite set HU_S. This is a performance problem that does not affect our above theoretical considerations, but could make their result useless. The smart ideas #1 and #2 introduced below, allow to restrict the substitution search space considerably.

Imagine one of the resolutions of stage 2, where C_1 is an atomic formula:

$$\frac{F_1 \vee C_1, \neg C_1 \vee G_1}{F_1 \vee G_1} \quad .$$

If both premises $F_1 \vee C_1, \neg C_1 \vee G_1$ are coming directly from stage 1, then they have been obtained from some initial clauses $F \vee C, \neg D \vee G$ by two

substitutions sub_1 and sub_2 such that:

$$F_1 \text{ is } F.sub_1,\ C_1 \text{ is } C.sub_1,\ \neg C_1 \text{ is } \neg D.sub_2,\ G_1 \text{ is } G.sub_2.$$

We can call such pair of substitutions a **unifier**, because $C.sub_1$ and $D.sub_2$ represent the same atomic formula (compare the example before the text of Theorem 5.7.1).

If one (or both) of the premises does not come directly from stage 1, then it is either an initial clause, or the result of a previous resolution. By putting an empty substitution (which does no change formulas) instead of sub_1 or sub_2 (or both) we can still think of the premises as obtained by a unification.

And, finally, if, to derive a contradiction B, ¬B from $K_1, K_2, \ldots, K_s$, we do not need resolution at all, then we need, nevertheless, unifying substitutions, converting two clauses B_1 and $\neg B_2$ into B and ¬B.

Thus (smart idea #1), **to derive contradictions, we can do with one specific kind of the substitution rule – the unification rule**:

a) Take two clauses, mark a positive atom C in the first clause, and a negative atom ¬D in the second one. Thus, we are considering two clauses: $F \vee C$ and $\neg D \vee G$.

b) Try to find two substitutions sub_1 and sub_2 such that $C.sub_1$ and $D.sub_2$ represent the same atom C_1. And you do not need to introduce variables of the other clauses! If you succeed, you have obtained two clauses: $F_1 \vee C_1, \neg C_1 \vee G_1$, where C_1 is $C.sub_1$ (=$D.sub_2$), F_1 is $F.sub_1$ and G_1 is $G.sub_2$. Since clauses do not contain common variables, the union $sub_1 \cup sub_2$ is a substitution (a unifier of C and D).

c) Apply resolution, obtaining the clause $F_1 \vee G_1$.

We have proved the following refined version of Theorem 5.7.1:

Theorem 5.7.2 (J.A.Robinson). In the classical predicate logic [L_1-L_{11}, L_{12}-L_{15}, MP, Gen], a set of predicate clauses is inconsistent; if and only if Robinson's **Resolution rule** (together with permutation, reduction and **unification** rules) allows deriving a contradiction from it.

Why is this refinement important? Because now, instead of trying out all the possible substitutions (of terms from HU_S for clause variables), we can concentrate on substitutions that unify two clauses. This allows to restrict the substitution search space considerably.

Refinements – Step 2 (Second of the Two Smart Ideas)

Substitution "Algebra"

In general, each substitution involves a list of distinct variables x_1, ..., x_k and a list of terms t_1, ...,t_k. All occurrences of the variable x_i are replaced by the term t_i. Thus, this operation can be most naturally represented by the set of pairs { t_1 / x_1, ..., t_k / x_k }. The order of pairs t_i / x_i is irrelevant because of the following "anti-cascading" condition: the new occurrences of the variables x_1, ..., x_k created by the substitution, are not replaced. The result of application of some substitution sub to some expression (term or formula) F, is usually denoted by F.sub.

For example, if F is p(x, f(y)) and sub = { f(z) / x, z / y }, then F.sub is p(f(z), f(z)).

The empty set of pairs {} represents the so-called empty substitution. Of course, F.{} = F, for any expression F.

If the variable sets of two substitutions sub_1 and sub_2 do not intersect, and the terms of sub_1 do not contain the variables of sub_2, and the terms of sub_2 do not contain the variables of sub_1, then the **union** $sub_1 \cup sub_2$ (of two sets of pairs) defines a substitution.

Still, the most important operation on substitutions is **composition**. If sub_1 and sub_2 are two substitutions, then $sub_1.sub_2$ denotes the composed substitution "apply first sub_1, and after this, apply sub_2". For example, if sub_1 = { f(z) / x, z / y } and sub_2 = { f(w) / z }, then

$$sub_1.sub_2 = \{ f(f(w)) / x, f(w) / y, f(w) / z \}.$$

Exercise 5.7.2. a) Verify that the substitution composition is associative and non-commutative (provide a counter-example), and that the empty substitution is the only "unit element" (i.e. {}.sub = sub.{} = sub for any substitution sub). b) Is there any algebraic correlation between composition and union of substitutions?

Most General Unifiers

How do behave unifiers in the substitution "algebra"? Assume, sub_1 and sub_2 are two different unifiers of the same pair of expressions F and G. I.e.

$$F.sub_1 = G.sub_1, F.sub_2 = G.sub_2.$$

If there would be a substitution sub such that sub_2=sub_1.sub, then we could say that sub_1 is a **no less general unifier** than sub_2. For example, let us try to

unify the first members of the following two formulas:

$$p(x_1) \vee F(x_1, y_1) \quad , \quad \neg p(f(x_2)) \vee G(x_2, y_2) \quad .$$

It would be natural to use the substitution sub_1 = { f(z) / x_1, z / x_2 }, obtaining

$$p(f(z)) \vee F(f(z), y_1) \quad , \quad \neg p(f(z)) \vee G(z, y_2) \quad .$$

But, in principle, one could use also the substitution sub_2 = { f(f(z)) / x_1, f(z) / x_2 }, obtaining

$$p(f(f(z))) \vee F(f(f(z)), y_1) \quad , \quad \neg p(f(f(z))) \vee G(f(z), y_2) \quad .$$

Of course, sub_1 is "better", because sub_2 = sub_1.{ f(z) / z }. Why? If our purpose was unifying p(x_1) with p(f(x_2)), then sub_1 performs this (as well as sub_2), but it "leaves more space" for subsequent substitutions (than sub_2). Indeed, to continue after sub_1, instead of sub_2 = sub_1.{ f(z) / z }, we can choose also sub_3 = sub_1.{ g(z) / z } etc. Thus, using a more general unifier is preferable.

So, let us call a unifier sub of two expressions F and G a **most general unifier (mgu)** of F and G, if and only if it is no less general than any other unifier of F and G (i.e. if and only if, for any other unifier sub' of F and G, there is a substitution sub" such that sub' = sub.sub").

Lemma 5.7.3. If two expressions lists FF and GG are unifiable, then there exists an mgu of FF and GG.

Proof (long, but easy). Let us define the **total length** of an expression list as follows: a) (atomic expressions) the total length of a constant or of a variable is 1, b) the total length of the expression list e_1, ..., e_n is the sum of the total lengths of the members e_1, ..., e_n, c) (composite expressions) the total length of the expression f(t_1, ..., t_n) (where f is function constant or predicate constant), is the total length of the expression list t_1, ..., t_n plus 1.

Let us prove our Lemma by induction using

min(total_length(FF), total_length(GG))

as the induction parameter.

1) **Induction base.** The total length of FF or GG is 1. Let us assume total_length(FF)=1.

a) FF is a constant c. Then FF and GG are unifiable, if and only if GG is the same constant c. Then, empty substitution is the only possible mgu (verify).

b) FF is a variable x. Then, FF and GG are not unifiable, if: b_1) GG is a list of more than one expression, or, b_2) GG is a composite expression that contains x (then any substitution of t for x makes GG longer than t). And, FF and GG are unifiable, if and only if GG is a variable, or GG is a composite expression that does not contain x.

If GG is the variable x, then the empty substitution is the only possible mgu (verify).

If GG is a variable y (other than x), then all unifications of FF and GG have the form { t / x, t / y, ... }, where t is any term. Among them, mgu-s are { z / x, z / y }, where z is any variable (verify).

If GG is a composite expression that does not contain x, then all unifications of FF and GG have the form { GG.sub / x, ... } U sub, where sub is any substitution that does not substitute for x (verify). Among them, mgu-s are { GG.sub / x}U sub, where sub substitutes distinct variables for distinct variables (verify).

This completes the induction base.

2) **Induction step.** Assume, min(total_length(FF), total_length(GG))=n, where n>1. If FF and GG are unifiable, then, as lists, they contain the same number of members.

2a) FF and GG contain are single expressions. Since min(total_length(FF), total_length(GG))>1, both are composite expressions – suppose, FF is f(s_1, ..., s_m) (where f is function constant or predicate constant, and s_1, ..., s_m are terms), and GG is g(t_1, ..., t_n) (where g is function constant or predicate constant, and t_1, ..., t_n are terms). FF and GG are unifiable, if and only if a) f and g represent the same constant, and b) the lists s_1, ..., s_m and t_1, ..., t_n are unifiable. Thus, the unifiers of FF and GG coincide with the unifiers of lists. Since min(total_length(s_1, ..., s_m), total_length(t_1, ..., t_n))<n, by the induction assumption, Lemma 5.7.3 holds for the lists, i.e. it holds also for FF and GG.

2b) FF and GG contain two or more members. If FF and GG are unifiable, then so are their first members ("heads") F_1 and G_1. Let us denote by FF_2 and GG_2 the rests of lists ("tails"). Since min(total_length(F_1), total_length(G_1))<n, by the induction assumption, there exists at least one mgu of F_1 and G_1. The same is true also for FF_2 and GG_2.

Let us denote by mgu_1 an arbitrary mgu of F_1 and G_1

Now, let us consider an arbitrary unifier u of FF and GG. It must unify also F_1

with G_1, and FF_2 with GG_2. Hence, u = $mgu_1.sub_1$, where sub_1 is some substitution. We know that $F_1.mgu_1 = G_1.mgu_1$.

But what about $FF_2.mgu_1$ and $GG_2.mgu_1$? Let us apply sub_1 to both:

$$FF_2.mgu_1.sub_1 = FF_2.u$$

$$GG_2.mgu_1.sub_1 = GG_2.u$$

Since u unifies FF_2 with GG_2,

$$FF_2.mgu_1.sub_1 = GG_2.mgu_1.sub_1,$$

i.e. sub_1 unifies $FF_2.mgu_1$ with $GG_2.mgu_1$. Let us denote by mgu_{12} an arbitrary mgu of $FF_2.mgu_1$ and $GG_2.mgu_1$. Then, $sub_1 = mgu_{12}.sub_{12}$, where sub_{12} is some substitution, and

$$mgu_1.mgu_{12}.sub_{12} = mgu_1.sub_1=u.$$

Thus, we have established that for an arbitrary unifier u of FF and GG there is a substitution sub_{12} such that $mgu_1.mgu_{12}.sub_{12}$ = u. Of course, the composition $mgu_1.mgu_{12}$ unifies FF with GG (since it unifies F_1 with G_1, and FF_2 with GG_2). Hence, $mgu_1.mgu_{12}$ is an mgu of FF and GG.

Q.E.D.

Unification Algorithm

How could we determine, can two atomic formulas C and D be unified, or not? This problem can be solved by the following simple pseudo-code *GetMostGeneralUnifier*, which follows the above proof of Lemma 5.7.3, and where **expression lists** are defined in the LISP style:

1) Each variable, constant, function constant and predicate constant is an expression list (consisting of a single member).

2) If s_1, ..., s_n are expression lists, then the list of s_1, ..., s_n is an expression list (consisting of members s_1, ..., s_n). The first member s_1 is called the **head** of the list, and the list of s_2, ..., s_n – the **tail** of the list.

Thus, instead of, for example, f(t_1, .., t_n), we use simply the (LISP style) list f, t_1, .., t_n. This simplifies the recursion interface.

This program detects, are two expression lists unifiable, or not, and, if they are, it returns one of their most general unifiers.

```
function GetMostGeneralUnifier (expression_list1, expression_list2)
begin
if length(expression_list1) > 1 and length(expression_list2) > 1 then
begin
--- h1 = head(expression_list1);
--- h2 = head(expression_list2);
--- subH = GetMostGeneralUnifier(h1, h2);
--- if subH = false then return false; {unification impossible}
--- t1 = tail(expression_list1).subH;
--- t2 = tail(expression_list2).subH;
--- subT = GetMostGeneralUnifier(t1, t2);
--- if subT = false then return false; {unification impossible, note that subH is
a mgu!}
--- return subH.SubT; {this composition unifies expression_list1 and
expression_list2}
end
{now, expression_list1, or expression_list2 consists of a single member: m1 or
m2}
if length(expression_list1) = 1 and m1 is variable then
begin
--- if m1 = expression_list2 then return {}; {empty substitution}
--- if m1 occurs in expression_list2 then return false; {unification impossible
– verify!}
--- return {expression_list2 / m1}; {substitute expression_list2 for m1}
end
if length(expression_list2) = 1 and m2 is variable then
begin
--- if m2 = expression_list1 then return {}; {empty substitution}
--- if m2 occurs in expression_list1 then return false; {unification impossible
– verify!}
--- return {expression_list1 / m2}; {substitute expression_list1 for m2}
end
{now, expression_list1, or expression_list2 consists of a single member that is
not variable}
if expression_list1 = expression_list2 then return {}; {empty substitution}
return false; {unification impossible – verify!}
end
```

Exercise 5.7.3. Verify that this program detects, are two expression lists unifiable, or not, and, if they are, it returns one of their mgu-s. (Hint: repeat the proof of Lemma 5.7.3.)

Smart idea #2:

To derive contradictions, we can do with even more specific kind of the

unification rule – the mgu-rule:

a) Take two clauses, mark a positive atom C in the first clause, and a negative atom ¬D in the second one. Thus, we are considering two clauses: FvC and ¬DvG.

b) Try to find any mgu of C and D. If you succeed, you have obtained two clauses: $F.mgu \vee C_1, \neg C_1 \vee G.mgu$, where C_1 is C.mgu (=D.mgu).

c) Apply resolution, obtaining the clause $F.mgu \vee G.mgu$.

Theorem 5.7.4 (J. A. Robinson). In the classical predicate logic [L_1-L_{11}, L_{12}-L_{15}, MP, Gen], a set of predicate clauses is inconsistent; if and only if Robinson's **Resolution rule** (together with permutation, reduction and **mgu**-rules) allows deriving a contradiction from it.

Why is this (second!) refinement important? Because now, instead of trying out all the possible unifications, we can concentrate on mgu-s that unify two clauses. This allows to further restrict the substitution search space (when compared with Theorem 5.7.2).

The hard part of the proof is inventing of the following

Lemma 5.7.5. Any proof $K_1, K_2, \ldots, K_s \vdash K$ (all K-s are clauses), where only permutation, reduction, **substitution** and Resolution rules are used, can be converted into a proof $K_1, K_2, \ldots, K_s \vdash K'$ such that: a) in the proof, only permutation, reduction, **mgu** and Resolution rules are used; b) K can be obtained from K' by a single (possibly empty) substitution, followed by a chain of permutations and reductions.

Proof of Theorem 5.7.4. Assume, the set of clauses $K_1, K_2, \ldots, K_s$ is inconsistent. Then, by Theorem 5.7.1, there are two proofs $K_1, K_2, \ldots, K_s \vdash B$, $K_1, K_2, \ldots, K_s \vdash \neg B$, where where only permutation, reduction, **substitution** and Resolution rules are used. From clauses, these rules allow deriving only of clauses. Hence, B is an atomic formula.

By Lemma 5.7.5, both proofs can be converted into proofs $K_1, K_2, \ldots, K_s \vdash B_1$, $K_1, K_2, \ldots, K_s \vdash \neg B_2$ such that: a) in the proofs, only permutation, reduction, **mgu** and Resolution rules are used; b_1) B can be obtained from B_1 by a single (possibly empty) substitution (permutations and reductions do not apply to atomic formulas), b_2) B can be obtained from B_2 by a single (possibly empty) substitution.

Thus, B_1 and B_2 are unifiable. Let us take their mgu, and apply it. As the

result, we obtain a contradiction B', ¬B', where B' is B_1.mgu (= B_2.mgu). And we have obtained this contradiction from the clauses K_1, K_2, ... , K_s by using only permutation, reduction, **mgu-** and Resolution rules. Q.E.D.

Proof Lemma 5.7.5.. Induction by the "height of the resolution tree" (see below).

1. Induction base – no resolutions applied in the proof K_1, K_2, ... , $K_s \vdash K$. Then K is obtained from some K_i by a chain of permutations, reductions and substitutions. Add to this fact an "empty" proof K_1, K_2, ... , $K_s \vdash K_i$. And let us compose all the substitutions into a single substitution. Q.E.D.

2. Induction step. Assume, we have the proof K_1, K_2, ... , $K_s \vdash K$, containing at least one resolution. Imagine the last resolution in this proof (C is an atomic formula):

$$\frac{F \vee C, \neg C \vee G}{F \vee G} .$$

Then K is obtained from the formula $F \vee G$ by a chain of permutations, reductions and substitutions.

The proofs of the formulas $F \vee C, \neg C \vee G$ possess a "height of the resolution tree" less than the one of the proof K_1, K_2, ... , $K_s \vdash K$. Thus, by induction assumption, we can convert these proofs into permutation-reduction-**mgu-**resolution proofs of some formulas $F_1 \vee C_1 \vee F_2, G_1 \vee \neg C_2 \vee G_2$ such that:

a) $F \vee C$ can be obtained from $F_1 \vee C_1 \vee F_2$ by a single (possibly empty) substitution sub_1, followed by a chain of permutations and reductions. Under sub_1, the atomic formula C_1 is converted into C.

b) $\neg C \vee G$ can be obtained from $G_1 \vee \neg C_2 \vee G_2$ by a single (possibly empty) substitution sub_2, followed by a chain of permutations and reductions. Under sub_2, the atomic formula C_2 is converted into C.

Since the clauses $F_1 \vee C_1 \vee F_2, G_1 \vee \neg C_2 \vee G_2$ do not contain common variables, the substitutions sub_1 and sub_2 do not intersect, hence, their union $sub_1 U sub_2$ is a substitution sub (a unifier of C_1 and C_2) such that:

a_1) F can be obtained from $(F_1 \vee F_2).sub$ by a chain of permutations and reductions.

b_1) G can be obtained from $(G_1 \vee G_2).sub$ by a chain of permutations and

reductions.

As we know from the above, the atomic formulas C_1 and C_2 are unifiable. Let us take their mgu, and apply it to the formulas $F_1 \vee C_1 \vee F_2, G_1 \vee \neg C_2 \vee G_2$. Let us denote by C' the formula C_1.mgu (it is equal to C_2.mgu). Thus, we have two formulas F_1.mgu v C' v F_2.mgu and G_1.mgu v ¬C' v G_2.mgu, and, by permutation and resolution, we can obtain the formula

$$(F_1 \vee F_2).mgu \vee (G_1 \vee G_2).mgu \quad .$$

Thus, for the formula $(F_1 \vee F_2).mgu \vee (G_1 \vee G_2).mgu$, we have a permutation-reduction-**mgu-**resolution proof. It remains to show that, from this formula, $F \vee G$ can be obtained by a single substitution, followed by a chain of permutations and reductions.

Since the substitution sub is a unifier of C_1 and C_2, then, by the definition of mgu, sub=mgu.sub', where sub' is some substitution. Hence,

a_2) F can be obtained from $(F_1 \vee F_2).mgu$ by the substitution sub', followed by a chain of permutations and reductions.

b_2) G can be obtained from $(G_1 \vee G_2).mgu$ by the substitution sub', followed by a chain of permutations and reductions.

Thus, $F \vee G$ can be obtained from $(F_1 \vee F_2).mgu \vee (G_1 \vee G_2).mgu$ by the substitution sub', followed by a chain of permutations and reductions. Q.E.D.

Warning!

Despite its beauty, the resolution method cannot overcome the general complexity problem, mentioned at the end of Section 4.3: by Church-Kalmar Theorem, in the classical predicate logic, the task of reasoning is not algorithmically solvable. And a closer analysis shows that all computer programs implementing resolution method run into loop in many situations, when the formula to be proved is, in fact, unprovable. But in many **practical situations**, experience shows that resolution method solves its task, and – in acceptable time. In particular, Prolog interpreters are using resolution, and are solving many practical tasks in acceptable time!

Further reading:

Logic. Part 2 by **Giorgio Ingargiola**

Rajjan Shinghal. Formal Concepts in Artificial Intelligence. Fundamentals. Chapman&Hall, 1992, 666 pp.

Handbook of Automated Reasoning, ed. by J. A. Robinson and A. Voronkov,

Elsevier and MIT Press, 2001, vol. I, II.

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About the ubiquity of the above-mentioned unification operation in human and computer reasoning:

John F. Sowa, Arun K. Majumdar. Analogical Reasoning. In: *Conceptual Structures for Knowledge Creation and Communication*, Proceedings of ICCS 2003, LNAI 2746, Springer-Verlag, Berlin, 2003, pp. 16-36. (available online).

6. Miscellaneous

6.1. Negation as Contradiction or Absurdity

The idea behind this approach is as follows: let us define ¬B (i.e. "B is false") as "B implies absurdity". So, let us add to our first order language a predicate constant f (meaning "false", or "absurdity"), and let us replace all negation expressions ¬F by F→f. Then, the three negation axioms will take the following forms:

L_9: (B→C)→((B→¬C)→¬B),

L_9': (B→C)→((B→(C→f))→(B→f)),

L_{10}: ¬B→(B→C),

L_{10}': (B→f)→(B→C),

L_{11}: $B \vee \neg B$,

L_{11}': $B \vee (B \to f)$.

After this, surprisingly, the axiom L_9' becomes derivable from L_1-L_2! Indeed,

(1)	B→C	Hypothesis.
(2)	B→(C→f)	Hypothesis.
(3)	B	Hypothesis.
(4)	C→f	By MP, from (2) and (3)
(5)	C	By MP, from (1) and (3)
(6)	f	By MP, from (4) and (5)

Hence, by Deduction Theorem 1,

$$[L_1, L_2, MP] \vdash (B \to C) \to ((B \to (C \to f)) \to (B \to f)).$$

Second observation. The axiom L_{10}': (B→f)→(B→C) can be replaced simply by f→C. Indeed, if we assume f→C, then L_{10}' becomes derivable:

(1)	B→f	Hypothesis.
(2)	B	Hypothesis.
(3)	f	By MP, from (1) and (2)
(4)	f→C	f→C
(5)	C	By MP, from (3) and (4)

Hence, by Deduction Theorem 1, $[L_1, L_2, f\to C, MP] \vdash (B\to f)\to(B\to C)$.

Third observation. As we know from Theorem 2.4.9: $[L_1, L_2, L_9, MP] \vdash \neg B\to(B\to\neg C)$, in the minimal logic we can prove 50% of L_{10}: "Contradiction implies that all is wrong". After our replacing negations by B→f the formula (B→f)→(B→(C→f) becomes derivable from L_1-L_2. Indeed,

(1)	B→f	Hypothesis.
(2)	B	Hypothesis.
(3)	f	By MP, from (1) and (2)
(4)	f→(C→f)	Axiom L_1
(5)	C→f	By MP, from (3) and (4)

Hence, by Deduction Theorem 1, $[L_1, L_2, MP] \vdash (B\to f)\to(B\to(C\to f))$.

Thus, we see that L_1 (and not L_9!) is responsible for the provability of the 50% "crazy" formula ¬B→(B→¬C). Is L_1 50% as "crazy" as L_{10}? Yes! Let us compare:

L_{10}: ¬B→(B→C) states that "Contradiction implies anything".

L_1: B→(C→B) states that "If B is true, then B follows from anything".

Let us remind our "argument" in favour of L_{10} in Section 1.3: "...we do not need to know, were C "true" or not, if ¬B and B were "true" simultaneously. By assuming that "if ¬B and B were true simultaneously, then anything were true" we greatly simplify our logical apparatus."

Now, similarly: if B is (unconditionally) true, then we do not need to know, follows B from C or not. By assuming that "if B is true, then B follows from anything" we greatly simplify our logical apparatus.

In a sense, the axiom L_9 "defines" the negation of the minimal logic, the axioms L_9 and L_{10} "define" the negation of the constructive logic, and L_9-L_{11} "define" the negation of the classical logic. Is our definition of ¬B as B→f equivalent to these "definitions"? Yes!

Theorem 6.1.1. For any formula F, let us denote by F' the formula obtained from F by replacing all sub-formulas ¬G by G→f. Then, for any formulas B_1, ..., B_n, C:

$[L_1\text{-}L_9, MP]: B_1, ..., B_n \vdash C$, if and only if $[L_1\text{-}L_8, MP]: B'_1, ..., B'_n \vdash C'$.

Proof.

1) →.

Let us consider a proof of $[L_1\text{-}L_9, MP]: B_1, ..., B_n \vdash C$. In this proof:

− let us replace each formula G by its "translation" G',

− before each instance of L_9, let us insert a proof of the corresponding instance of L'_9 in $[L_1, L_2, MP]$ (see above).

In this way we obtain a proof of $[L_1\text{-}L_8, MP]: B'_1, ..., B'_n \vdash C'$. Indeed,

a) If some formula B is an instance of L_1-L_8, then B' is an instance of the same axiom (verify!).

b) (B→D)' is B'→D', hence, if the initial proof contains a conclusion by MP from B and B→D to D, then, in the derived proof, it is converted into a conclusion by MP from B' and B'→D' to D'.

c) If the initial proof contains an instance of L_9, then the derived proof contains the corresponding instance of L'_9 preceded by its proof in $[L_1, L_2, MP]$.

Q.E.D.

2) ←.

Let us remind the above translation operation: for any formula F, we denoted by F' the formula obtained from F by replacing all sub-formulas ¬G by G→f. Now, let us introduce a kind of a converse operation – the re-translation operation: for any formula F, let us denote by F" the formula obtained from F: a) by replacing all sub-formulas G→f by ¬G, and after this, b) by replacing all the remaining f's (f means "false"!) by ¬(a→a), where a is some closed formula of the language considered.

Of course, for any formula F, (F')" is F (verify).

Note. Replacing f by a formula preceded by negation, is crucial – it will allow applying of Theorem 2.4.9: [L_1-L_9, MP]: ¬B→(B→¬C) instead of the Axiom L_{10}: ¬B→(B→C).

Now, let us consider a proof of [L_1-L_8, MP]: B'_1, ..., B'_n ⊢C'. In this proof, let us replace each formula G by its re-translation G". Then C' becomes C, and B'_1, ..., B'_n become B_1, ..., B_n, but what about the remaining formulas contained in the proof?

a) Instances of the axioms L_1-L_8.

L_1: B→(C→B)

If B is not f, then (B→(C→B))" is B"→(C"→B"), i.e. re-translation yields again an instance of L_1.

If B is f, then (f→(C→f))" is ¬(a→a)→¬C". This formula is provable in [L_1-L_9, MP]. Indeed,

(1)	¬(a→a)	Hypothesis.
(2)	⊢¬(a→a)→((a→a)→¬C")	Theorem 2.4.9, [L_1-L_9, MP].
(3)	⊢a→a	Theorem 1.4.1 [L_1-L_2, MP].
(4)	¬C"	By MP, from (1), (2) and (3).

Thus, re-translation of any instance of L_1 is provable in [L_1-L_9, MP].

L_2: (B→(C→D))→((B→C)→(B→D))

If C and D are not f, then re-translation yields again an instance of L_2.

If C is f, and D is not, then re-translation yields

(B"→(¬(a→a)→D"))→(¬B"→(B"→D")).

This formula is provable in [L_1-L_9, MP]. Indeed,

(1)	B"→(¬(a→a)→D")	Hypothesis.
(2)	¬B"	Hypothesis.
(3)	B"	Hypothesis.
(4)	¬(a→a)→D"	By MP, from (1) and (3).

(5) $\vdash \neg B'' \rightarrow (B'' \rightarrow \neg(a \rightarrow a))$ Theorem 2.4.9 [L_1-L_9, MP].

(6) $\neg(a \rightarrow a)$ By MP, from (2), (3) and (5).

(7) D'' By MP, from (4) and (6).

Hence, by Deduction Theorem 1,

$$[L_1\text{-}L_9, \text{MP}] \vdash (B'' \rightarrow (\neg(a \rightarrow a) \rightarrow D'')) \rightarrow (\neg B'' \rightarrow (B'' \rightarrow D'')).$$

If D is f, and C is not, then re-translation yields

$$(B'' \rightarrow \neg C'') \rightarrow ((B'' \rightarrow C'') \rightarrow \neg B'').$$

This formula is provable in [L_1-L_9, MP]. Indeed,

(1) $B'' \rightarrow \neg C''$ Hypothesis.

(2) $B'' \rightarrow C''$ Hypothesis.

(3) $\neg B''$ By MP, from Axiom L_9.

Hence, by Deduction Theorem 1,

$$[L_1\text{-}L_9, \text{MP}] \vdash (B'' \rightarrow \neg C'') \rightarrow ((B'' \rightarrow C'') \rightarrow \neg B'').$$

If C and D both are f, then re-translation yields

$$(B'' \rightarrow \neg\neg(a \rightarrow a)) \rightarrow (\neg B'' \rightarrow \neg B'').$$

This formula is provable in [L_1-L_9, MP]. Indeed,

(1) $\vdash \neg B'' \rightarrow \neg B''$ Theorem 1.4.1 [L_1-L_2, MP].

(2) $\vdash (\neg B'' \rightarrow \neg B'') \rightarrow (X \rightarrow (\neg B'' \rightarrow \neg B''))$ Axiom L_1, X is $B'' \rightarrow \neg\neg(a \rightarrow a)$.

(3) $\vdash X \rightarrow (\neg B'' \rightarrow \neg B'')$ By MP, X is $B'' \rightarrow \neg\neg(a \rightarrow a)$.

Thus, re-translation of any instance of L_2 is provable in [L_1-L_9, MP].

L_3: $B \wedge C \rightarrow B$

If B is not f, then re-translation yields again an instance of L_3.

If B is f, then re-translation yields via $\neg(f \wedge C)$ the formula $\neg(\neg(a \rightarrow a) \wedge C)$. This formula is provable in [L_1-L_9, MP]. Indeed,

(1)	$\neg(a \to a) \wedge C \to \neg(a \to a)$	Axiom L_3.
(2)	$\neg\neg(a \to a) \to \neg(\neg(a \to a) \wedge C)$	From (1), by the Contraposition Law.
(3)	(a→a)→¬¬(a→a)	Theorem 2.4.4: $[L_1, L_2, L_9, MP] \vdash A \to \neg\neg A$
(4)	a→a	Theorem 1.4.1 $[L_1\text{-}L_2, MP]$.
(5)	$\neg(\neg(a \to a) \wedge C)$	By MP, from (3), (4) and (2).

Thus, re-translation of any instance of L_3 is provable in $[L_1\text{-}L_9, MP]$.

L_4: $B \wedge C \to C$

Similarly to L_3 – re-translation of any instance of L_4 is provable in $[L_1\text{-}L_9, MP]$.

L_5: $B \to (C \to B \wedge C)$

Re-translation yields again an instance of L_5.

L_6: $B \to B \vee C$

Re-translation yields again an instance of L_6.

L_7: $C \to BvC$

Re-translation yields again an instance of L_7.

L_8: $(B \to D) \to ((C \to D) \to (B \vee C \to D))$

If D is not f, then re-translation yields again an instance of L_8.

If D is f, then re-translation yields $\neg B \to (\neg C \to \neg(B \vee C))$. By Theorem 2.4.10(b), this formula is provable in $[L_1\text{-}L_9, MP]$.

Thus, re-translation of any instance of L_8 is provable in $[L_1\text{-}L_9, MP]$.

Hence, re-translations of all (i.e. L_1-L_8) axiom instances are provable in $[L_1\text{-}L_9, MP]$. What about applications of MP in the initial proof? If the initial proof contains a conclusion by MP from B and B→D to D, then the following situations are possible:

a) If B and D are not f, then, in the derived proof, this conclusion is converted into a conclusion by MP from B" and B"→D" to D".

b) If B is f, and D is not, then, in the derived proof, this conclusion is converted into a conclusion by MP from ¬(a→a) and ¬(a→a)→D" to D".

c) If D is f, and B is not, then, in the derived proof, this conclusion is converted into three formulas: B", ¬B", ¬(a→a). To derive ¬(a→a) from B" and ¬B", we can use MP and Theorem 2.4.9:

$$[L_1\text{-}L_9, MP] \vdash \neg B" \rightarrow (B" \rightarrow \neg(a \rightarrow a)).$$

d) If B and D are both f, then, in the derived proof, this conclusion is converted into three formulas: ¬(a→a), ¬¬(a→a), ¬(a→a). Simply drop the third formula from the proof.

Thus, the re-translation operation, when applied to all formulas of a proof of $[L_1\text{-}L_8, MP]$: $B'_1, ..., B'_n \vdash C'$, yields a sequence of formulas that are provable in $[L_1\text{-}L_9, MP]$ from hypotheses $B_1, ..., B_n$. Hence, so is C.

Q.E.D.

This completes the proof of Theorem 6.1.1.

Corollary 6.1.2. a) A formula C is provable in the minimal propositional logic $[L_1\text{-}L_9, MP]$, if and only if $[L_1\text{-}L_8, MP] \vdash C'$.

b) A formula C is provable in the constructive propositional logic $[L_1\text{-}L_{10}, MP]$, if and only if $[L_1\text{-}L_8, f \rightarrow B, MP] \vdash C'$.

c) A formula C is provable in the classical propositional logic $[L_1\text{-}L_{11}, MP]$, if and only if $[L_1\text{-}L_8, f \rightarrow B, L'_{11}, MP] \vdash C'$.

Proof. a) Consider an empty set of hypotheses in Theorem 6.1.1.

b) If $[L_1\text{-}L_{10}, MP] \vdash C$, then $[L_1\text{-}L_9, MP]: B_1, ..., B_n \vdash C$, where hypotheses are instances of the axiom L_{10}. By Theorem 6.1.1,

$$[L_1\text{-}L_8, MP]: B'_1, ..., B'_n \vdash C'.$$

As established above, $B'_1, ..., B'_n$ can be proved by using the axiom schema $f \rightarrow B$, i.e. $[L_1\text{-}L_8, f \rightarrow B, MP] \vdash C'$. Q.E.D.

Now, if $[L_1\text{-}L_8, f \rightarrow B, MP] \vdash C'$, then,

c) If $[L_1\text{-}L_{11}, MP] \vdash C$, then $[L_1\text{-}L_9, MP]: B_1, ..., B_n \vdash C$, where hypotheses are instances of the axioms L_{10} and L_{11}. Return to case (b). Q.E.D.

Corollary 6.1.3. a) A formula C is provable in the minimal predicate logic $[L_1\text{-}L_9, L_{12}\text{-}L_{15}, MP, Gen]$, if and only if $[L_1\text{-}L_8, L_{12}\text{-}L_{15}, MP, Gen] \vdash C'$.

b) A formula C is provable in the constructive predicate logic [L_1-L_{10}, L_{12}-L_{15}, MP, Gen], if and only if [L_1-L_8, f→B, L_{12}-L_{15}, MP, Gen] ⊢C'.

c) A formula C is provable in the classical predicate logic [L_1-L_{11}, L_{12}-L_{15}, MP, Gen], if and only if [L_1-L_8, f→B, L_{11}', L_{12}-L_{15}, MP, Gen] ⊢C'.

Exercise 6.1.1. Prove the Corollary 6.1.3.