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Towards the theory of $\mathbb{M}$-approximate systems: Fundamentals and examples

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Abstract

The concept of an $\mathbb{M}$-approximate system is introduced. Basic properties of the category of $\mathbb{M}$-approximate systems and in a natural way defined morphisms between them are studied. It is shown that categories related to fuzzy topology as well as categories related to rough sets can be described as special subcategories of the category of $\mathbb{M}$-approximate systems.

Keywords: $\mathbb{M}$-approximate operator; $\mathbb{M}$-approximate system; cl-Monoid; Fuzzy topology; Fuzzy bitopology; L-topology; (L, M)-fuzzy topology; Rough set; L-rough set

1. Introduction and motivation

In 1968, that is only three years after Zadeh has published his famous work "Fuzzy Sets" [56], thus laying down the principles of what can be called *Mathematics of Fuzzy Sets*, his student Chang [5] introduced the concept of a fuzzy topological space thus marking the beginning of Fuzzy Topology, the counterpart of General Topology in the context of fuzzy sets. Now Fuzzy Topology is one of the most well developed fields of Mathematics of Fuzzy Sets, and there are published dozens of fundamental works on this subject.

In 1982, Pawlak [34] has introduced the concept of a rough set which can be viewed as a certain alternative for the concept of a fuzzy set for the study of mathematical problems of applied nature. Pawlak's work was followed by many other publications where rough sets and mathematical structures on the basis of rough sets were introduced, studied, and applied.

Although at the first glance it may seem that the concepts of a fuzzy set, of a (fuzzy) topological space and of a rough set are of an essentially different nature and "have nothing in common", this is not the case. Probably, the first one to start studying the intermediate relations between topologies, fuzzy sets and rough sets was Kortelainen [26,27], see also paper by Kortelainen and Järvinan [28]. Further the study of different relations between fuzzy sets, rough sets and some other related concepts was done in a series of papers by Yao (see e.g. [49,50]), Järvinen [20], Eklund and Galan [8]

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and other researchers, see also the monograph written by a group of Polish mathematicians [25]. Our short introduction into the topic of relations between rough sets, fuzzy sets and fuzzy topology would be incomplete if we do not mention the critical paper by Gutiérrez García and Rodabaugh, see [12].

The aim of this work is to present an alternative view on the relations between fuzzy sets, fuzzy topological spaces and rough sets and to develop a framework allowing to generalize these concepts and corresponding theories. The tool allowing to realize this aim is the concept of an $\mathbb{M}$-approximate system. The concept of an $\mathbb{M}$-approximate system was first introduced in [46]; basic properties of $\mathbb{M}$-approximate systems and their relations to some categories related to fuzzy topology and rough sets were discussed in [47–49].

The structure of this work is as follows. After the Introduction, in Section 2 we discuss concepts which make the context for our work. In Section 3 we define the basic concepts studied and used in this work: upper and lower $\mathbb{M}$-approximate operators, $\mathbb{M}$-approximate systems, $\mathbb{M}$-approximate spaces and some related notions. In Section 4 the lattice of $\mathbb{M}$-approximate systems on a fixed lattice $\mathbb{L}$ is studied. In Section 5 we define the morphisms between $\mathbb{M}$-approximate systems thus coming to the category $\mathbf{AS}^{\mathbb{M}}$ of $\mathbb{M}$-approximate systems. Properties of this category and some of its subcategories are studied. In particular, it is shown that $\mathbf{AS}^{\mathbb{M}}$ is a topological category over the category $\mathbf{IDL}^{op}$, where $\mathbf{IDL}$ of complete infinitely distributive lattices as objects and mappings preserving *arbitrary sups and infs* as morphisms, with respect to the forgetful functor $\mathcal{F} : \mathbf{AS}^{\mathbb{M}} \to \mathbf{IDL}^{op}$.

In Section 6 some important general subcategories of the category $\mathbf{AS}^{\mathbb{M}}$ are studied. In Section 7 different known categories related to fuzzy topology are characterized as subcategories of $\mathbf{AS}^{\mathbb{M}}$. In Section 8 the category of rough sets is described in the terms of approximate systems. Also the concept of an L-rough set is introduced and characterized by means of L-approximate system. In Section 9 the defuzzification L-approximate operators on the lattice L^X of L-subsets of a set X are introduced.

2. The context

In our work two lattices will play the fundamental role. The first one is an infinitely distributive lattice, that is a complete lattice

$$\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee),$$

satisfying the infinite distributivity laws

$$a \wedge \left(\bigvee_{i\in\mathcal{I}} b_i\right) = \bigvee_{i\in\mathcal{I}}(a \wedge b_i) \quad \text{and} \quad a \vee \left(\bigwedge_{i\in\mathcal{I}} b_i\right) = \bigwedge_{i\in\mathcal{I}}(a \vee b_i)$$

for all $a \in \mathbb{L}$ and for all $\{b_i | i \in \mathcal{I}\} \subseteq \mathbb{L}$. The top and the bottom elements of $\mathbb{L}$ are denoted by $1_{\mathbb{L}}$ and $0_{\mathbb{L}}$, respectively.

Sometimes we will assume that the lattice $\mathbb{L}$ is equipped with one of the following operations: a monotone mapping $^c : \mathbb{L} \to \mathbb{L}$ or a binary operation $* : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$.

A lattice $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, ^c)$ will be called *adjunctive* if the pair $(^c, ^c)$ is an adjunction

$$(^c, ^c) : \mathbb{L} \vdash \mathbb{L}^{op},$$

that is

$$a \leq b^c \iff b \leq a^c, \quad \forall a, b \in \mathbb{L},$$

cf. e.g. [9]. A lattice $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, ^c)$ will be called *involutive* if $^c : \mathbb{L} \to \mathbb{L}$ is an involution, that is if

$$(a^c)^c = a, \quad \forall a \in \mathbb{L}.$$

One can easily see that in an adjunctive involutive lattice involution $^c : \mathbb{L} \to \mathbb{L}$ is order reversing:

$$a \leq b \Longrightarrow b^c \leq a^c, \quad \forall a, b \in \mathbb{L},$$

and conversely, if $^c : \mathbb{L} \to \mathbb{L}$ is order reversing involution, then $(^c, ^c) : \mathbb{L} \vdash \mathbb{L}^{op}$ is an adjunction.

Remark. Note that, since adjunctive involution is order reversing, complete *completely* distributive adjunctive involutive lattices made the context for the approach to fuzzy topology developed by Hutton, see [16–18]. Further such lattices called *Hutton lattices* or *Hutton algebras* where used by many researches. Note that in this work when speaking about adjunctive involutive infinitely distributive lattices we do not assume that they are completely distributive.

Concerning the second, binary operation $* : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ (conjunction) it will be assumed that $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, *)$ is a commutative cl-monoid (see e.g. [2]), that is

- $*$ is commutative: $a * b = b * a$ for all $a, b \in \mathbb{L}$;
- $*$ is associative: $(a * b) * c = a * (b * c)$ for all $a, b, c \in \mathbb{L}$;
- $*$ distributes over arbitrary joins: $a * (\bigvee_{i \in \mathcal{I}} b_i) = \bigvee_{i \in \mathcal{I}}(a * b_i), \forall a \in \mathbb{L}, \forall \{b_i | i \in \mathcal{I}\} \subseteq \mathbb{L}$;
- $a * 1_{\mathbb{L}} = a,\ a * 0_{\mathbb{L}} = 0_{\mathbb{L}}$ for all $a \in \mathbb{L}$.

It is well-known (see e.g. [2]) that in a cl-monoid there is a further binary operation $\mapsto : \mathbb{L} \to \mathbb{L}$ (residuation) which is related to conjunction $*$ by Galois connection:

$$a * b \leq c \iff a \leq b \mapsto c, \quad \forall a, b, c \in \mathbb{L}.$$

Explicitly residuation is given by

$$a \mapsto b = \bigvee \{c \in \mathbb{L} | a * c \leq b\}.$$

One can easily see that residuation is non-increasing by the first argument and non-decreasing by the second argument, and that $b * (b \mapsto a) \leq a\ \forall a, b \in \mathbb{L}$. In particular $b * (b \mapsto 0) \leq 0$, and hence

$$b \leq (b \mapsto 0) \mapsto 0, \quad \forall b \in \mathbb{L}.$$

This allows to conclude that by setting $a^c = a \mapsto 0$ we obtain an adjunction $(^c, {}^c) : \mathbb{L} \vdash \mathbb{L}^{op}$. Indeed, if $a \leq b \mapsto 0$, then

$$b \leq (b \mapsto 0) \mapsto 0 \leq a \mapsto 0.$$

A cl-monoid is called a Girard monoid [22] if

$$(a \mapsto 0) \mapsto 0 = a, \quad \forall a \in \mathbb{L}.$$

Hence in case $\mathbb{L}$ is a Girard monoid, residuation $\mapsto$ induces an order reversing involution $^c : \mathbb{L} \to \mathbb{L}$.

An important situation in our research will be the following. Let $\mathrm{L} = (\mathrm{L}, \leq, \wedge, \vee)$ be a lattice and X be a set. Then the L-powerset $\mathrm{L}^X =: \mathbb{L}$ becomes a lattice $(\mathbb{L}, \leq, \wedge, \vee)$ by pointwise extending the lattice structure from L to $\mathbb{L}$. Besides $\mathbb{L}$ is infinitely distributive whenever L was infinitely distributive. Moreover, if $\mathrm{L} = (\mathrm{L}, \leq, \wedge, \vee, ^c)$ is an adjunctive (involutive) lattice then by pointwise extending operation ^{c} from L to $\mathbb{L}$, an adjunctive (resp. involutive) lattice $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, ^c)$ is obtained. In case $\mathrm{L} = (\mathrm{L}, \leq, \wedge, \vee, *)$ is a cl-monoid, by pointwise extension of $* : \mathrm{L} \times \mathrm{L} \to \mathrm{L}$ to $* : \mathbb{L} \times \mathbb{L} \to \mathbb{L}$ we obtain a cl-monoid $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee, *)$. Besides, if L is a Girard monoid, then $\mathbb{L}$ is a Girard monoid as well.

The second lattice belonging to the context of our work is denoted by $\mathbb{M}$. At the moment we assume only its completeness, however, sometimes it will be requested that $\mathbb{M}$ is completely distributive. The bottom and the top elements of $\mathbb{M}$ are $0_{\mathbb{M}}$ and $1_{\mathbb{M}}$, respectively. An important case is when $\mathbb{M}$ is a two-point lattice **2**. Note that as different from the lattice $\mathbb{L}$ we do not exclude the case when $\mathbb{M}$ is a one-point lattice

$$\mathbb{M} = \{\cdot_{\mathbb{M}}\}$$

and hence in this case $0_{\mathbb{M}} = 1_{\mathbb{M}}$. A one-point lattice $\mathbb{M}$ will be denoted by $\bullet$.

The reader is referred to monographs [2] or [9] for the terms from Lattice Theory which are not defined here. Concerning the terms related to Category Theory we follow [1]. If **Cat** is a category, then $\mathcal{O}b(\mathbf{Cat})$ is the class of objects of **Cat** and $\mathcal{M}or(\mathbf{Cat})$ is the class of morphism of **Cat**. If **Cat** is a category, then by $\mathbf{Cat}^{op}$ the opposite category is denoted.

3. Basic definitions

Definition 3.1. An upper $\mathbb{M}$-approximate operator on $\mathbb{L}$ is a mapping $u : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ such that

(1u) $u(0_{\mathbb{L}}, \alpha) = 0_{\mathbb{L}},\ \forall \alpha \in \mathbb{M}$;
(2u) $a \le u(a, \alpha),\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(3u) $u(a \vee b, \alpha) = u(a, \alpha) \vee u(b, \alpha),\ \forall a, b \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(4u) $u(u(a, \alpha), \alpha) = u(a, \alpha),\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(5u) $\alpha \le \beta, \alpha, \beta \in \mathbb{M} \Longrightarrow u(a, \alpha) \le u(a, \beta),\ \forall a \in \mathbb{L}$;
(6u) if $0_{\mathbb{M}} \ne 1_{\mathbb{M}}$ (that is $\mathbb{M}$ is not a one-point lattice), then $u(a, 0_{\mathbb{M}}) = a,\ \forall a \in \mathbb{L}$.

Operator u is called (upper) semicontinuous from above (usca) if

(usca) $u(a, \bigwedge_{i\in\mathcal{I}} \alpha_i) = \bigwedge_{i\in\mathcal{I}} u(a, \alpha_i), \quad \forall a \in \mathbb{L},\ \forall \{\alpha_i i \in \mathcal{I}\} \subseteq \mathbb{M}$.

In case all elements of the lattice $\mathbb{M}$ are isolated from above, every upper $\mathbb{M}$-approximate operator is semicontinuous from above. In particular, every upper $\bullet$-approximate operator and every upper $\mathbf{2}$-approximate operator are semicontinuous from above.

Definition 3.2. A lower $\mathbb{M}$-approximate operator on $\mathbb{L}$ is a mapping $l : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ such that

(1l) $l(1, \alpha) = 1,\ \forall \alpha \in \mathbb{M}$;
(2l) $a \ge l(a, \alpha),\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(3l) $l(a \wedge b, \alpha) = l(a, \alpha) \wedge l(b, \alpha),\ \forall a, b \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(4l) $l(l(a, \alpha), \alpha) = l(a, \alpha),\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(5l) $\alpha \le \beta, \alpha, \beta \in \mathbb{M} \Longrightarrow l(a, \alpha) \ge l(a, \beta),\ \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}$;
(6l) if $0_{\mathbb{M}} \ne 1_{\mathbb{M}}$ (that is $\mathbb{M}$ is not a one-point lattice), then $l(a, 0_{\mathbb{M}}) = a,\ \forall a \in \mathbb{L}$.

Operator l is called (lower) semicontinuous from above (lsca) if

(lsca) $l(a, \bigwedge_{i\in\mathcal{I}} \alpha_i) = \bigvee_{i\in\mathcal{I}} l(a, \alpha_i),\ \forall a \in \mathbb{L},\ \forall \{\alpha_i\ i \in \mathcal{I}\} \subseteq \mathbb{M}$.

In case all elements of the lattice $\mathbb{M}$ are isolated from above, every lower $\mathbb{M}$-approximate operator is semicontinuous from above. In particular, every lower $\bullet$-approximate operator and every lower $\mathbf{2}$-approximate operator are semicontinuous from above.

Definition 3.3. A triple $(\mathbb{L}, u, l)$, where $u : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ and $l : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ are upper and lower $\mathbb{M}$-approximate operators on $\mathbb{L}$, is called an $\mathbb{M}$-approximate system. In case when X is a set, L is a lattice, $\mathbb{L} = \mathrm{L}^X$ and $(\mathbb{L}, u, l)$ is an $\mathbb{M}$-approximate system, the quadruple (X, L, u, l) is called an $\mathbb{M}$-approximate space.

Definition 3.4. An $\mathbb{M}$-approximate system $(\mathbb{L}, u, l)$ is called semicontinuous from above if both u and l are semicontinuous from above.

Remark 3.5. Although in this work we are mainly interested in semicontinuity of upper and lower $\mathbb{M}$-approximate operators only from above, one can consider also the following alternative types of semicontinuity:

(uscb) Operator u is called semicontinuous from below if

$$u\left(a, \bigvee_{i\in\mathcal{I}} \alpha_i\right) = \bigvee_{i\in\mathcal{I}} u(a, \alpha_i), \quad \forall a \in \mathbb{L},\ \forall \{\alpha_i\ i \in \mathcal{I}\} \subseteq \mathbb{M}.$$

(lscb) Operator l is called semicontinuous from below if

$$l\left(a, \bigvee_{i\in\mathcal{I}} \alpha_i\right) = \bigwedge_{i\in\mathcal{I}} l(a, \alpha_i), \quad \forall a \in \mathbb{L},\ \forall \{\alpha_i\ i \in \mathcal{I}\} \subseteq \mathbb{M}.$$

(wuscb) Operator u is called weakly semicontinuous from below if

$$u(a, \alpha_i) = \bar{a},\ \forall \alpha_i, i \in \mathcal{I} \quad \text{and} \quad \alpha = \bigvee_{i \in \mathcal{I}} \alpha_i, \Longrightarrow u(a, \alpha) = \bar{a}.$$

(wlscb) Operator l is called weakly semicontinuous from below if

$$l(a, \alpha_i) = a^0,\ \forall \alpha_i, i \in \mathcal{I} \quad \text{and} \quad \alpha = \bigvee_{i \in \mathcal{I}} \alpha_i \Longrightarrow l(a, \alpha) = a^0.$$

Such properties of $\mathbb{M}$-approximate are useful when considering some concrete $\mathbb{M}$-approximate operators, see e.g. Section 7.1 in this work. However, we do not intend to study in this paper $\mathbb{M}$-approximate systems possessing such properties.

In case $\mathbb{L}$ is equipped with a unary operation ${}^c : \mathbb{L} \to \mathbb{L}$, an $\mathbb{M}$-approximate system $(\mathbb{L}, u, l)$ is called *self-dual* if

$$u(a^c, \alpha) = (l(a, \alpha)t)^c \quad \text{and} \quad l(a^c, \alpha) = (u(a, \alpha))^c, \quad \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}.$$

Note that in case when $(\mathbb{L}, \leq, \wedge, \vee, {}^c)$ is an involutive infinitely distributive lattice, the system $(\mathbb{L}, u, l)$ is self-dual if and only if

$$(u(a^c, \alpha))^c = l(a, \alpha) \quad \text{iff} \quad (l(a^c, \alpha))^c = u(a, \alpha), \quad \forall a \in \mathbb{L},\ \forall \alpha \in \mathbb{M}.$$

Remark 3.6. Sometimes we consider $\mathbb{M}$-approximate systems in case of a one-point lattice $\mathbb{M} = \bullet = \{\cdot\}$. Obviously, in this case the use of the second argument in the notation of approximate systems is redundant and we write just $u(a)$ and $l(a)$ instead of $u(a, \cdot)$ and $l(a, \cdot)$, respectively. Besides, in this case we use the terms *upper and lower approximate operators*, *approximate system*, etc., omitting the prefix $\mathbb{M}$. Note also that in case $\mathbb{M} = \mathbf{2}$ is a two-point lattice then, taking into account conditions (6u) and (6l) in Definitions 3.1 and 3.2, an $\mathbb{M}$-approximate system $(\mathbb{L}, u, l)$ in an obvious sense is equivalent to the $\bullet$-approximate system $(\bullet, u', l')$ where $u' : \mathbb{L} \to \mathbb{L}$ and $l' : \mathbb{L} \to \mathbb{L}$ are defined by $u'(a) = u(a, 1_{\mathbb{M}})$ and $l'(a) = l(a, 1_{\mathbb{M}})$, respectively. (Compare with the usual practice of interpreting subsets A of a set X as characteristic functions $\chi_A : X \to \{0, 1\}$ or just identifying A and χ_A.)

4. Lattice of $\mathbb{M}$-approximate systems on a lattice $\mathbb{L}$

Let $\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$ stand for the family of $\mathbb{M}$-approximate systems $(\mathbb{L}, u, l)$ where $\mathbb{L}$ and $\mathbb{M}$ are fixed. We introduce an order $\preceq$ on $\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$ by setting

$$(\mathbb{L}, u_1, l_1) \preceq (\mathbb{L}, u_2, l_2) \quad \text{iff} \quad u_1 \geq u_2 \text{ and } l_1 \leq l_2.$$

To study the lattice structure of $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$ we first need to relate with every upper $\mathbb{M}$-approximation operator $u : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ a subset $C_u \subseteq \mathbb{L} \times \mathbb{M}$:

$$C_u = \{(a, \alpha) | (a, \alpha) \in \mathbb{L} \times \mathbb{M}, u(a, \alpha) = a\}$$

and to relate with every lower $\mathbb{M}$-approximate operator $l : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ a subset $T_l \subseteq \mathbb{L} \times \mathbb{M}$:

$$T_l = \{(a, \alpha) | (a, \alpha) \in \mathbb{L} \times \mathbb{M}, l(a, \alpha) = a\}.$$

One can establish the properties of families C_u and T_l collected in the following five lemmas. We omit the proofs of the first four lemmas since they can be easily done patterned after the proofs of well known results about the relations between closure operators and closed L-sets in L-topological spaces (in case of Lemmas 4.1 and 4.2 establishing relations between an upper $\mathbb{M}$-approximate operator u and the family C_u) and the relations between interior operators and open L-sets in L-topological spaces (in case of Lemmas 4.3 and 4.4 establishing relations between an lower $\mathbb{M}$-approximate operator l and the family T_l).

Lemma 4.1. *For every upper $\mathbb{M}$-approximate operator $u : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$:*

(1C) $(0_{\mathbb{L}}, \alpha) \in C_u,\ \forall \alpha \in \mathbb{M};$

(2C) $(1_{\mathbb{L}}, \alpha) \in C_u, \ \forall \alpha \in \mathbb{M}$;
(3C) $(a, \alpha), (b, \alpha) \in C_u \Longrightarrow (a, \alpha) \vee (b, \alpha) \in C_u, \ \forall \alpha \in \mathbb{M}, \ \forall a, b \in \mathbb{L}$;
(4C) $(a_\lambda, \alpha) \in C_u, \ \forall \lambda \in \Lambda \Longrightarrow (\bigwedge_\lambda a_\lambda, \alpha) \in C_u, \ \forall \alpha \in \mathbb{M}$;
(5C) $(a, \alpha) \in C_u$ *and* $\beta < \alpha, \alpha, \beta \in \mathbb{M} \Longrightarrow (a, \beta) \in C_u$.

Conversely, if a family $C \subseteq \mathbb{L} \times \mathbb{M}$ *satisfies properties* (1C)–(5C), *then by setting*

$$u_C(a, \alpha) = \bigwedge \{b | (b, \alpha) \in C, b \geq a\}$$

an upper $\mathbb{M}$*-approximate operator* u_C *is defined. Besides* $u_{C_u} = u$ *and* $C_{u_C} = C$.

Lemma 4.2. *Given two upper* $\mathbb{M}$*-approximate operators* u_1 *and* u_2:

$$u_1 \geq u_2 \Longrightarrow C_{u_1} \subseteq C_{u_2}.$$

Conversely, given two systems $C_1, C_2 \subseteq \mathbb{L} \times \mathbb{M}$ *satisfying properties* (1C)–(5C) *from Lemma* 4.1 *we have*

$$C_1 \subseteq C_2 \Longrightarrow u_{C_1} \geq u_{C_2}.$$

Lemma 4.3. *For every lower* $\mathbb{M}$*-approximate operator* $l : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$:

(1T) $(1_{\mathbb{L}}, \alpha) \in T_l, \ \forall \alpha \in \mathbb{M}$;
(2T) $(0_{\mathbb{L}}, 1_{\mathbb{M}}) \in T_l, \ \forall \alpha \in \mathbb{M}$;
(3T) $(a, \alpha), (b, \alpha) \in T_l \Longrightarrow (a, \alpha) \wedge (b, \alpha) \in T_l$;
(4T) $(a_\lambda, \alpha) \in T_l, \ \forall \lambda \in \Lambda \Longrightarrow (\bigvee_\lambda a_\lambda, \alpha) \in \mathcal{T}_l$.
(5T) $(a, \alpha) \in T_l$ *and* $\beta < \alpha \Longrightarrow (a, \beta) \in T_l$.

Conversely, if we have a family $T \subseteq \mathbb{L} \times \mathbb{M}$ *satisfying properties* (1T)–(5T), *then by setting*

$$l_T(a, \alpha) = \bigvee \{b | (b, \alpha) \in T, b \leq a\}$$

a lower $\mathbb{M}$*-approximate operator* $l_T : \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ *is defined. Besides* $l_{T_l} = l$ *and* $T_{l_T} = T$.

Lemma 4.4. *Given two lower* $\mathbb{M}$*-approximate operators* l_1 *and* l_2:

$$l_1 \leq l_2 \Longrightarrow T_{l_1} \subseteq T_{l_2}.$$

Conversely, given two systems $T_1, T_2 \subseteq \mathbb{L} \times \mathbb{M}$ *satisfying properties* (1T)–(5T) *from Lemma* 4.3

$$T_1 \subseteq T_2 \Longrightarrow l_{T_1} \leq l_{T_2}.$$

Lemma 4.5. *An* $\mathbb{M}$*-approximate system* $(\mathbb{L}, u, l)$ *is self-dual iff*

$$C_u = \{(a, \alpha) | (a^c, \alpha) \in T_l\}$$

and

$$T_l = \{(a, \alpha) | (a^c, \alpha) \in C_u\}.$$

The proof can be done patterned after the proof of the well-known result about the relations between closed and open L-sets on one hand and the relations between the corresponding closure and interior operators in an L-topological space.

Now we can come to the main result of this section:

Theorem 4.6. $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$ *is a complete lattice*.

Proof. Let

$$u_\top(a, \alpha) = l_\top(a, \alpha) = a, \quad \forall a \in \mathbb{L}, \ \forall \alpha \in \mathbb{M}.$$

It is easy to see that $u_\top : \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ and $l_\top : \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ are, respectively, the upper and the lower $\mathbb{M}$-approximate operators. Besides $u_\top \le u$ for every upper $\mathbb{M}$-approximate operator and $l_\top \ge l$ for every lower $\mathbb{M}$-approximate operator. Hence $(\mathbb{L}, u_\top, l_\top)$ is the top element in $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$.

Further, let

$$u_\bot(a,\alpha) = \begin{cases} 1_{\mathbb{L}} & \text{if } a \ne 0_{\mathbb{L}} \text{ and } \alpha \ne 0_{\mathbb{M}}, \\ 0_{\mathbb{L}} & \text{if } a = 0_{\mathbb{L}}, \\ a & \text{if } \alpha = 0_{\mathbb{M}} \end{cases}$$

and

$$l_\bot(a,\alpha) = \begin{cases} 0_{\mathbb{L}} & \text{if } a \ne 1_{\mathbb{L}} \text{ and } \alpha \ne 0_{\mathbb{M}}, \\ 1_{\mathbb{L}} & \text{if } a = 1_{\mathbb{L}}, \\ a & \text{if } \alpha = 0_{\mathbb{M}}. \end{cases}$$

One can easily check that $u_\bot : \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ is an upper $\mathbb{M}$-approximate operator and $l_\bot : \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ is a lower $\mathbb{M}$-approximate operator. Besides for any other upper $\mathbb{M}$-approximate operator $u : \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ and any other lower $\mathbb{M}$-approximate operator $l : \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ it holds $u \le u_\bot$ and $l \ge l_\bot$. Hence the $\mathbb{M}$-approximate system $(\mathbb{L}, u_\bot, l_\bot)$ is the bottom element in the lattice $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$.

Thus to show that $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$ is a complete lattice now it is sufficient to show that every non-empty subset $\mathcal{S} = \{(\mathbb{L}, u_i, l_i) | i \in \mathcal{I}\} \subseteq \mathcal{AS}^{\mathbb{M}}(\mathbb{L})$ has infimum in $\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$ (see e.g. [9, Proposition 0-2.2]). Let

$$\mathcal{S} = \{(\mathbb{L}, u_i, l_i) | i \in \mathcal{I}\} \subseteq \mathcal{AS}^{\mathbb{M}}(\mathbb{L}).$$

For each $i \in \mathcal{I}$ according to our notations

$$C_{u_i} = \{(a,\alpha) | (a,\alpha) \in \mathbb{L}\times\mathbb{M}, u_i(a,\alpha) = a\}.$$

We define $C = \bigcap_{i\in\mathcal{I}} C_{u_i}$. Thus

$$(a,\alpha) \in C \iff \forall i \in \mathcal{I}, \quad u_i(a,\alpha) = a.$$

One can easily notice that C satisfies properties (1C)–(5C) from Lemma 4.1, since all C_{u_i} satisfy these properties. We define $u_0 : \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ by setting

$$u_0(a,\alpha) = \bigwedge\{(b,\alpha) b \ge a, (b,\alpha) \in C\}.$$

Since $C_{u_i} \supseteq C$ for all $i \in \mathcal{I}$, it follows that $u_0 \ge u_i$. Besides, applying Lemmas 4.1 and 4.2, it is easy to notice that u_0 is the smallest ($\le$) one of all upper $\mathbb{M}$-approximate operators u such that $u \ge u_i$ for all $i \in \mathcal{I}$. Thus u_0 is the supremum of the family $\{u_i | i \in \mathcal{I}\}$ in the lattice $\mathbb{L}^{\mathbb{L}\times\mathbb{M}}$.

Further, for each $i \in \mathcal{I}$ let

$$T_{l_i} = \{(a,\alpha) | (a,\alpha) \in \mathbb{L}\times\mathbb{M},\ l_i(a,\alpha) = a\}.$$

We define $T = \bigcap_{i\in\mathcal{I}} T_{u_i}$. Thus

$$(a,\alpha) \in T \iff \forall i \in \mathcal{I}, \quad l_i(a,\alpha) = a.$$

One can easily notice that T satisfies properties (1T)–(5T) from Lemma 4.3, since all T_{l_i} satisfy these properties. We define $l_0 : \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ by setting

$$l_0(a,\alpha) = \bigvee\{(b,\alpha) | b \le a, (b,\alpha) \in T\}.$$

Since $T_{l_i} \supseteq T$ for all $i \in \mathcal{I}$, it follows that $l_0 \le l_i$ for all $i \in \mathcal{I}$. Besides, applying Lemmas 4.3 and 4.4, it is easy to notice that l_0 is the largest ($\ge$) one of all lower $\mathbb{M}$-approximate operators l such that $l \le l_i$ for all $i \in \mathcal{I}$. Thus l_0 is the infimum of the family $\{l_i | i \in \mathcal{I}\}$ in the lattice $\mathbb{L}^{\mathbb{L}\times\mathbb{M}}$.

Summarizing the above received statements we conclude that $(\mathbb{L}, u_0, l_0)$ is the infimum of the system $\mathcal{S} = \{(\mathbb{L}, u_i, l_i) | i \in \mathcal{I}\} \subseteq \mathcal{AS}^{\mathbb{M}}(\mathbb{L})$ in the lattice $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$ and hence $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$ is a complete lattice. □

The construction of the upper $\mathbb{M}$-approximation operator u_0 and the construction of the lower $\mathbb{M}$-approximation operator l_0 for an $\mathbb{M}$-approximate system $(\mathbb{L}, u_i, l_i)$ in the proof of the previous theorem is done levelwise and hence semicontinuity from above will not be destroyed. Besides in case $0_{\mathbb{M}}$ is isolated from above in the lattice $\mathbb{M}$ the $\mathbb{M}$-approximate systems $(\mathbb{L}, u_\top, l_\top)$ and $(\mathbb{L}, u_\perp, l_\perp)$ defined above are semicontinuous from above. Hence we obtain the following corollary from Theorem 4.6:

Theorem 4.7. *In case* $0_{\mathbb{M}}$ *isolated from above, the family* (SCA-$AS^{\mathbb{M}}(\mathbb{L}), \preceq$) *of semicontinuous from above* $\mathbb{M}$-*approximate systems is a complete sublattice of the lattice* $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$.

Further, notice that, in case $\mathbb{L}$ is adjunctive involutive lattice, the $\mathbb{M}$-approximate systems $(\mathbb{L}, u_\top, l_\top)$ and $(\mathbb{L}, u_\perp, l_\perp)$ constructed in the proof of the above theorem are self-dual. Besides, if all $\mathbb{M}$-approximate systems $(\mathbb{L}, u_i, l_i) \in \mathcal{S})$ are self-dual, then applying Lemma 4.5 one can easily conclude that u_0 and l_0 are also self-dual. In the result we get the following corollary from Theorem 4.6:

Theorem 4.8. *If* $\mathbb{L}$ *is an adjunctive involutive infinitely distributive lattice, then the family* (D-$\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq$) *of self-dual* $\mathbb{M}$-*approximate systems is a complete sublattice of the lattice* $(\mathcal{AS}^{\mathbb{M}}(\mathbb{L}), \preceq)$.

5. Category $\mathbf{AS}^{\mathbb{M}}$ of $\mathbb{M}$-approximate systems

Let $\mathbb{M}$ be fixed and let $\mathbf{AS}^{\mathbb{M}}$ be the family of all $\mathbb{M}$-approximate systems $(\mathbb{L}, u, l)$. To consider $\mathbf{AS}^{\mathbb{M}}$ as a category whose class of objects are all $\mathbb{M}$-approximate systems $(\mathbb{L}, u, l)$ we have to specify its morphisms. Since the ground of our construction is an infinitely distributive lattice $\mathbb{L}$ we start with the category **IDL** of infinitely distributive lattices as objects and mappings $f : \mathbb{L}_1 \to \mathbb{L}_2$ between such lattices, preserving *arbitrary infs and arbitrary sups* as morphisms.

Now we are ready to define morphisms in the category $\mathbf{AS}^{\mathbb{M}}$. Given $(\mathbb{L}_1, u_1, l_1), (\mathbb{L}_2, u_2, l_2) \in \mathcal{O}b(\mathbf{AS}^{\mathbb{M}})$ by a morphism

$$f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_2, u_2, l_2)$$

we call a mapping $f : \mathbb{L}_2 \to \mathbb{L}_1$ such that

(1m) $f : \mathbb{L}_1 \to \mathbb{L}_2$ is a morphism in the category $\mathbf{IDL}^{op}$;
(2m) $u_1(f(b), \alpha) \le f(u_2(b, \alpha)),\ \forall b \in \mathbb{L}_2,\ \forall \alpha \in \mathbb{M}$;
(3m) $f(l_2(b, \alpha)) \le l_1(f(b), \alpha),\ \forall b \in \mathbb{L}_2,\ \forall \alpha \in \mathbb{M}$.

A morphism $f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_2, u_2, l_2)$ is also referred to as a continuous mapping between the corresponding $\mathbb{M}$-approximate systems.

Theorem 5.1. $\mathbf{AS}^{\mathbb{M}}$ *thus obtained is indeed a category.*

Proof. Let $f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_2, u_2, l_2)$ and $g : (\mathbb{L}_2, u_2, l_2) \to (\mathbb{L}_3, u_3, l_3)$ be continuous mappings and let $g \circ f : \mathbb{L}_1 \to \mathbb{L}_3$ be their composition in $\mathbf{IDL}^{op}$. We have to verify that $g \circ f$ satisfies conditions (2m) and (3m) above. Since it is sufficient to verify these conditions for a fixed $\alpha \in \mathbb{M}$, to simplify the reasonings we omit the second argument in the notation of the approximate operators. Let $c \in \mathbb{L}_3$. Then

$$u_1(f(g(c))) \le f(u_2(g(c))) \le f(g(u_3(c))).$$

In a similar way we can show that $f(g(l_3(c))) \le l_1(g(f(c)))$. Thus the composition $g \circ f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_3, u_3, l_3)$ is continuous whenever f and g are continuous. We conclude the proof by noticing that the identity mapping $f : (\mathbb{L}, u, l) \to (\mathbb{L}, u, l)$ is obviously continuous. □

Remark 5.2. Often we are interested in the categories $\mathbf{AS}^{\mathbb{M}}$ of $\mathbb{M}$-approximate systems when $\mathbb{M}$ is a two-point or a one-point lattice. We denote the corresponding categories, respectively, by $\mathbf{AS}^{\mathbf{2}}$ and $\mathbf{AS}^{\bullet}$.

In the sequel, when discussing categorical properties of $\mathbf{AS}^{\mathbb{M}}$ and other categories we refer to the monograph [1].

Theorem 5.3. *Every source*

$$f_i : \mathbb{L}_1 \to (\mathbb{L}_i, u_i, l_i), \quad i \in \mathcal{I}$$

has a unique initial lift

$$f_i : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_i, u_i, l_i), \quad i \in \mathcal{I}.$$

Proof. Taking into account Theorem 4.6 it is sufficient to consider the case when the source contains only one morphism $f : \mathbb{L}_1 \to (\mathbb{L}_2, u_2, l_2)$ in $\mathbf{IDL}^{op}$. Define an upper approximate operator $u_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_1$ by

$$u_1(a, \alpha) = \bigwedge\{f(u_2(b, \alpha)) | f(b) \geq a\}, \quad \forall a \in \mathbb{L}_1, \ \alpha \in \mathbb{M}.$$

Note first that the condition

$$u_1(f(b), \alpha) \leq f(u_2(b, \alpha)), \quad \forall b \in \mathbb{L}_2, \ \forall \alpha \in \mathbb{M}$$

is obviously fulfilled. We verify that u_1 thus defined is indeed an upper $\mathbb{M}$-approximate operator. As in the previous theorem in our reasoning we fix $\alpha \in \mathbb{M}$ and omit it in notation of approximate operators when verifying the properties (1u)–(4u).

The first two properties are obvious from the definition of u_1 and the corresponding properties of u_2:

$$u_1(0_{\mathbb{L}_1}) = 0_{\mathbb{L}_1} \quad \text{and} \quad u_1(a) \geq a \quad \text{for all } a \in \mathbb{L}_1.$$

To verify property (3u) let $a_1, a_2 \in \mathbb{L}_1$, then

$$\begin{aligned} u_1(a_1 \vee a_2) &= \bigwedge\{f(u_2(b)) | f(b) \geq a_1 \vee a_2\} \\ &\leq \bigwedge\{f(u_2(b_1 \vee b_2)) \mid f(b_1) \geq a_1, \ f(b_2) \geq a_2\} \\ &= \bigwedge\{f(u_2(b_1)) \vee f(u_2(b_2)) | f(b_1) \geq a_1, \ f(b_2) \geq a_2\} \\ &= \left(\bigwedge\{f(u_2(b_1)) | f(b_1) \geq a_1\}\right) \vee \left(\bigwedge\{f(u_2(b_2)) | f(b_2) \geq a_2\}\right) = u_1(a_1) \vee u_1(a_2). \end{aligned}$$

The converse inequality is obvious and hence $u_1(a_1 \vee a_2) = u_1(a_1) \vee u_1(a_2)$.

To verify the fourth condition notice that

$$\begin{aligned} u_1(u_1(a)) &= u_1\left(\bigwedge\{f(u_2(b)) | f(b) \geq a\}\right) \\ &\leq \bigwedge\{u_1(f(u_2(b))) | f(b) \geq a\} \leq \bigwedge\{f(u_2(u_2(b))) | f(b) \geq a\} = u_1(a). \end{aligned}$$

The converse inequality is obvious and hence $u_1(u_1(a)) = u_1(a)$.

To verify property (5u) for u_1 note that the implication

$$\alpha \leq \beta, \alpha, \beta \in \mathbb{M} \Longrightarrow u_1(a, \alpha) \leq u_1(a, \beta)$$

is guaranteed by the analogous property of the operator $u_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ and the definition of u_1. The validity of condition (6u) for u_1 also obviously follows from its definition.

Coming to the lower $\mathbb{M}$-approximate operator $l_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_1$, we define it by the equality

$$l_1(a, \alpha) = \bigvee\{f(l_2(b, \alpha)) | f(b) \leq a\}, \quad \forall a \in \mathbb{L}_1, \ \forall \alpha \in \mathbb{M}.$$

Notice first that from the definition it is clear that

$$f(l_2(b, \alpha)) \leq l_1(f(b), \alpha), \quad \forall b \in \mathbb{L}_2, \ \forall \alpha \in \mathbb{M}.$$

We show that $l_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_1$ thus defined is indeed a lower $\mathbb{M}$-approximate operator. Again, we omit in notation α when it can be fixed. The validity of the first two conditions follows from the definition of l_1 and the corresponding properties of l_2:

$$l_1(1_{\mathbb{L}_1}, \alpha) = 1_{\mathbb{L}_1} \quad \text{and} \quad l_1(a, \alpha) \leq a \quad \text{for all } a \in \mathbb{L}_1, \ \forall \alpha \in \mathbb{M}.$$

To verify the third condition let $a_1, a_2 \in \mathbb{L}_1$. Then

$$l_1(a) \wedge l_1(a_2) = \left(\bigvee\{f(l_2(b_1))|f(b_1)\right) \leq a_1\}) \vee \left(\bigvee\{f(l_2(b_2))|f((b_2)) \leq a_2,\}\right)$$

$$= \bigvee\{f(l_2(b_1)) \wedge f(l_2(b_2))|f((b_i)) \leq a_i, i = 1, 2\}$$

$$\leq \bigvee\{f(l_2(b_1 \wedge b_2)|f(b_1) \wedge f(b_2) \leq a_1 \wedge a_2\}$$

$$= \bigvee\{f(l_2(b))|f(b) \leq a_1 \wedge a_2\} = l_1(a_1 \wedge a_2).$$

The converse inequality is obvious and hence $l_1(a_1 \wedge a_2) = l_1(a_1) \wedge l_1(a_2)$.

The idempotence of the operator $l_1 : \mathbb{L}_1 \to \mathbb{L}_1$ is established as follows: Given $a \in \mathbb{L}$ we have

$$l_1(l_1(a)) = l_1\left(\bigvee\{f(l_2(b))|f(b) \leq a\}\right) \geq \bigvee\{l_1 f(l_2(b))|f(b) \leq a\}$$

$$\geq \bigvee\{f(l_2(l_2(b))|f(b) \leq a\} = \bigvee\{f(l_2(b))|f(b) \leq a\} = l_1(a).$$

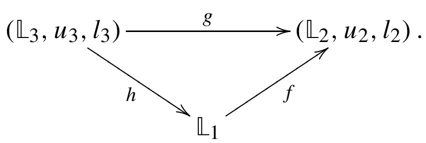
The opposite inequality is obvious and hence $l_1(l_1(a)) = l_1(a)$.

Finally, condition (5l), that is

$$\alpha \leq \beta, \alpha, \beta \in \mathbb{M} \Longrightarrow l_1(a, \alpha) \geq l_1(a, \beta)$$

and condition (6l) for l_1 are guaranteed by the analogous properties of the operator $l_2 : \mathbb{L}_2 \to \mathbb{L}_2$ and the definition of l_1.

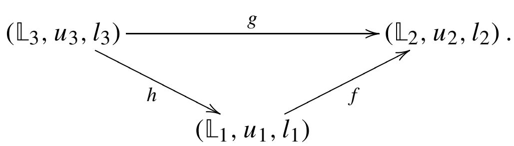
To complete the proof, let $g : (\mathbb{L}_3, u_3, l_3) \to (\mathbb{L}_2, u_2, l_2)$ be a morphism in $\mathbf{AS}^{\mathbb{M}}$ and let $h : \mathbb{L}_3 \to \mathbb{L}_1$ be a morphism in $\mathbf{IDL}^{op}$ such that $f \circ h = g$.



Then from the construction it is clear that

- $u_3(h(a), \alpha) \leq h(u_1(a, \alpha)),\ \forall a \in \mathbb{L}_1,\ \forall \alpha \in \mathbb{M}$;
- $h(l_1(a, \alpha)) \leq l_3(h(a), \alpha),\ \forall a \in \mathbb{L}_1,\ \forall \alpha \in \mathbb{M}$

and hence $h : (\mathbb{L}_3, u_3, l_3) \to (L_1, u_1, l_1)$ is a morphism in $\mathbf{AS}^{\mathbb{M}}$:



Thus $f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_2, u_2, l_2)$ is indeed the initial lift of the source $f : \mathbb{L}_1 \to (\mathbb{L}_2, u_2, l_2)$. The uniqueness of the lift is obvious. □

Let **AIIDL** denote the subcategory of the category **IDL** whose objects are adjunctive involutive lattices and whose morphisms are involution preserving maps.[1]

Theorem 5.4. *Let $\mathbb{L}_1, \mathbb{L}_2$ be adjunctive involutive infinitely distributive lattices, let $(\mathbb{L}_2, u_2, l_2)$ be an $\mathbb{M}$-approximate system and $f : \mathbb{L}_1 \to \mathbb{L}_2$ be a morphism in $\mathbf{AIIDL}^{op}$. If $\mathbb{M}$-approximate operators $l_2, u_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ are self-dual, then the $\mathbb{M}$-approximate operators $l_1, u_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_1$ constructed in the proof of Theorem 5.3 are self-dual as well.*

[1] Recall that we assume that the morphisms in the category **IDL** preserve both arbitrary sups and arbitrary inf.

Proof. Indeed, let $(\mathbb{L}_2, u_2, l_2)$ be a self-dual $\mathbb{M}$-approximate system and $a \in \mathbb{L}_1$. Then

$$l_1(a^c) = \bigvee\{f(l_2(b))|f(b) \le a^c\} = \left(\bigwedge\{(f(l_2(b)))^c|f(b) \le a^c\}\right)^c$$
$$= \left(\bigwedge\{f(u_2(b^c))|f(b) \le a^c\}\right)^c$$
$$= \left(\bigwedge\{f(u_2(b^c))|f(b^c) \ge a\}\right)^c = \left(\bigwedge\{f(u_2(d))|f(d) \ge a\}\right)^c = (u_1(a))^c.$$

(Here in the proof again we omit the second coordinate $\alpha \in \mathbb{M}$ where it does not essentially influences the proof.)

In a similar way the equality $u_1(a^c) = (l_1(a))^c$ can be established. □

Theorem 5.5. *If an* $\mathbb{M}$*-approximate system* $(\mathbb{L}_2, u_2, l_2)$ *is semicontinuous from above, then the* $\mathbb{M}$*-approximate system* $(\mathbb{L}_1, u_1, l_1)$ *constructed in the previous theorem is semicontinuous from above, too.*

Proof. Indeed, if $(\mathbb{L}_2, u_2, l_2)$ is a semicontinuous from above $\mathbb{M}$-approximate system, then

$$u_1\left(a, \bigwedge_i \alpha_i\right) = \bigwedge\left\{f\left(u_2\left(b, \bigwedge_i \alpha_i\right)\right)\middle| f(b) \ge a\right\}$$
$$= \bigwedge\left\{f\left(\bigwedge_i (u_2(b, \alpha_i))\right)\middle| f(b) \ge a\right\} = \bigwedge_i \bigwedge\{f(u_2(b, \alpha_i))|f(b) \ge a\} = \bigwedge_i u_1(a, \alpha_i),$$
$$l_1\left(a, \bigwedge_i \alpha_i\right) = \bigvee\left\{f\left(l_2\left(b, \bigwedge_i \alpha_i\right)\right)\middle| f(b) \le a\right\}$$
$$= \bigvee\left\{f\left(\bigvee_i (l_2(b, \alpha_i))\right)\middle| f(b) \le a\right\} = \bigvee_i \bigvee\{f(l_2(b, \alpha_i))|f(b) \le a\} = \bigvee_i l_1(b, \alpha_i),$$

thus the system $(\mathbb{L}_1, u_1, l_1)$ is semicontinuous from above. □

Theorem 5.6. *Every sink* $f_i : (\mathbb{L}_i, u_i, l_i) \to \mathbb{L}_2,\ i \in \mathcal{I}$ *has a unique final lift*: $f_i : (\mathbb{L}_i, u_i, l_i) \to (\mathbb{L}_2, u_2, l_2),\ i \in \mathcal{I}$.

Proof. Taking into account Theorem 4.6 it is sufficient to consider the case of the sink consisting of a single morphism $f : (\mathbb{L}_1, u_1, l_1) \to \mathbb{L}_2$. We define an upper $\mathbb{M}$-approximate operator $u_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ by

$$u_2(b, \alpha) = \bigwedge\{c \in \mathbb{L}_2|c \ge b,\ f(c) \ge u_1(f(b), \alpha)\}.$$

It is obvious that

$$u_1(f(b), \alpha) \le f(u_2(b, \alpha)), \quad \forall\, b \in \mathbb{L}_2,\ \forall \alpha \in \mathbb{M}.$$

Further, from the definition of u_2 and the corresponding properties of u_1 it is clear that $u_2(0_{\mathbb{L}_2}, \alpha) = 0_{\mathbb{L}_2}$ and $b \le u_2(b, \alpha)$ for all $b \in \mathbb{L}_2, \alpha \in \mathbb{M}$ and hence u_1 satisfies the first two requirements of Definition 3.1. Showing that the rest of conditions of Definition 3.1 hold for $u_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ we omit notation α in case of conditions (3u) and (4u), since it can be fixed.

Let $b_1, b_2 \in \mathbb{L}_2$. Then

$$u_2(b_1) \vee u_2(b_2) = \left(\bigwedge\{c_1 \in \mathbb{L}_2|c_1 \ge b_1,\ f(c_1) \ge u_1(f(b_1))\}\right)$$
$$\vee \left(\bigwedge\{c_2 \in \mathbb{L}_2|c_2 \ge b_2,\ f(c_2) \ge u_1(f(b_2))\}\right)$$
$$= \bigwedge\{c_1 \vee c_2|c_i \ge b_i,\ f(c_i) \ge u_1(f(b_i)),\ i = 1, 2\}$$
$$\ge \bigwedge\{c|c \ge b_1 \vee b_2,\ f(c) \ge u_1(f(b_1)) \vee u_1(f(b_2))\}$$

$$= \bigwedge\{c | c \geq b_1 \vee b_2,\ f(c) \geq u_1(f(b_1 \vee b_2))\}$$
$$= u_2(b_1 \vee b_2).$$

Since the opposite inequality is obvious, we get $u_2(b_1) \vee u_2(b_2) = u_2(b_1 \vee b_2)$.

To verify the fourth condition note first that

$$u_2(u_2(b)) = \bigwedge\{c | c \geq u_2(b),\ u_1(f(u_2(b))) \leq f(c)\}.$$

Since $u_2(b)$ is among the elements c satisfying the conditions on the right-hand side of the above inequality, we conclude that $u_2(u_2(b)) \leq u_2(b)$. The opposite inequality is obvious and hence $u_2(u_2(b)) = u_2(b)$.

The validity of properties (5u) and (6u) for the operator u_2 is guaranteed by the analogous properties of the operator $u_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_1$ and the definition of u_2.

We define the lower $\mathbb{M}$-approximate operator $l_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ by

$$l_2(b, \alpha) = \bigvee\{c \in \mathbb{L}_2 | c \leq b,\ f(c) \leq l_1(f(b), \alpha)\}.$$

Note first that from the definition of l_2 it is clear that

$$f(l_2(b, \alpha)) \leq l_1(f(b), \alpha), \quad \forall b \in \mathbb{L}_2,\ \ \alpha \in \mathbb{M}.$$

Further, from the definition of l_2 and the corresponding properties of l_1 it is obvious that

$$l_2(1_{\mathbb{L}_2}, \alpha) = 1_{\mathbb{L}_2} \quad \text{and} \quad b \geq l_2(b, \alpha) \quad \text{for all } b \in \mathbb{L}_2 \text{ and } \alpha \in \mathbb{M}$$

and hence the first two conditions of Definition 3.2 for $l_2 : \mathbb{L}_2 \times \mathbb{M} \to \mathbb{L}_2$ are valid. To verify the third property let $b_1, b_2 \in \mathbb{L}_2$. Then we have the following chain of inequalities (again, we omit in notations $\alpha \in \mathbb{M}$ since it can be fixed):

$$l_2(b_1) \wedge l_2(b_2)$$
$$= \bigvee\{c_1 \wedge c_2 | c_1 \leq b_1, c_2 \leq b_2,\ f(c_1) \leq l_1(f(b_1)),\ f(c_2) \leq l_1(f(b_2))\}$$
$$\leq \bigvee\{c_1 \wedge c_2 | c_1 \wedge c_2 \leq b_1 \wedge b_2,\ \ f(c_1) \wedge f(c_2) \leq l_1(f(b_1)) \wedge l_1(f(b_2))\}$$
$$\leq \bigvee\{c | c \leq b_1 \wedge b_2,\ f(c) \leq l_1(f(b_1 \wedge b_2))\} = l_2(b_1 \wedge b_2).$$

Since the opposite inequality is obvious we have $l_2(b_1) \wedge l_2(b_2) = l_2(b_1 \wedge b_2)$.

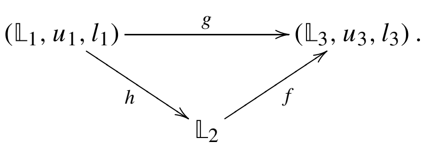
To show the fourth axiom note that

$$l_2(l_2(b)) = \bigvee\{c \in \mathbb{L}_2 | c \leq l_2(b),\ f(c) \leq l_1(f(b))\}$$

and since $l_2(b)$ is one of c appearing in the right-hand side of the above formula, it holds $l_2(l_2(b)) \geq l_2(b)$. Since the converse inequality is obvious we have $l_2(l_2(b)) = l_2(b)$.

The validity of properties (5l) and (6l) for l_2 is guaranteed by the analogous properties of the operator $l_1 : \mathbb{L}_1 \times \mathbb{M} \to \mathbb{L}_2$ and the definition of l_2.

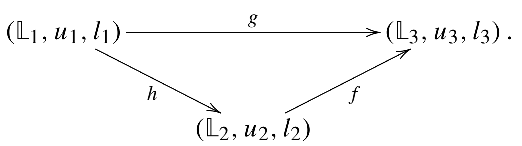
To complete the proof, let $g : (\mathbb{L}_1, u_1, l_1) \to (L_3, u_3, l_3)$ be a morphism in $\mathbf{AS}^{\mathbb{M}}$ and $h : \mathbb{L}_2 \to \mathbb{L}_3$ be a morphism in $\mathbf{IDL}^{op}$ such that $h \circ f = g$.



Then from the construction it is clear that

- $u_2(h(a), \alpha) \leq h(u_3(a, \alpha)),\ \forall a \in \mathbb{L}_1,\ \forall \alpha \in \mathbb{M};$
- $h(l_2(a, \alpha)) \leq l_3(h(a), \alpha),\ \forall a \in \mathbb{L}_1,\ \forall \alpha \in \mathbb{M}$

and hence $h : (\mathbb{L}_2, u_2, l_2) \to (\mathbb{L}_3, u_3, l_3)$ is a morphism in $\mathbf{AS}^{\mathbb{M}}$. Thus $f : (\mathbb{L}_1, u_1, l_1) \to (\mathbb{L}_2, u_2, l_2)$ is indeed the final lift of the sink $f : (\mathbb{L}_1, u_1, l_1) \to \mathbb{L}_2$:



The uniqueness of the lift is obvious. □

Theorem 5.7. *If an $\mathbb{M}$-approximate system $(\mathbb{L}_1, u_1, l_1)$ is self-dual, and $f : \mathbb{L}_1 \to \mathbb{L}_2$ is a morphism in the category* **AIIDL** *of adjunctive involutive infinitely distributive lattices, then the $\mathbb{M}$-approximate system $(\mathbb{L}_2, u_2, l_2)$ constructed in the proof of the previous theorem is self-dual.*

Indeed, let a self-dual $\mathbb{M}$-approximate system $(\mathbb{L}_1, u_1, l_1)$ be given and let $b \in \mathbb{L}_2$. Then we have

$$
\begin{aligned}
(l_2(b))^c &= \left(\bigvee\{d|d \le b,\ f(d) \le l_1(f(b))\}\right)^c \\
&= \bigwedge\{d^c|d \le b,\ f(d) \le l_1(f(b))\} = \bigwedge\{d^c|d^c \ge b^c, (f(d))^c \ge (l_1(f(b)))^c\} \\
&= \bigwedge\{d^c|d^c \ge b^c,\ f(d^c) \ge u_1(f(b^c))\} \\
&= \bigwedge\{e|e \ge b^c,\ f(e) \ge u_1(f(b^c))\} = u_2(b^c),
\end{aligned}
$$

that is $(l_2(b))^c = u_2(b^c)$. In a similar way the equality $(u_2(b))^c = l_2(b^c)$ can be proved.

One can easily establish also the following:

Theorem 5.8. *If an $\mathbb{M}$-approximate system $(\mathbb{L}_1, u_1, l_1)$ is semicontinuous from above, then the $\mathbb{M}$-approximate system $(\mathbb{L}_2, u_2, l_2)$ constructed in the proof of Theorem 5.6 is semicontinuous from above.*

Proof. Indeed, if $(\mathbb{L}_1, u_1, l_1)$ is a semicontinuous from above $\mathbb{M}$-approximate system, then

$$
\begin{aligned}
u_2\left(b, \bigwedge_i \alpha_i\right) &= \bigwedge\left\{c \in \mathbb{L}_2 | c \ge b,\ f(c) \ge u_1\left(f(b), \bigwedge_i \alpha_i\right)\right\} \\
&= \bigwedge\left\{c \in \mathbb{L}_2 | c \ge b,\ f(c) \ge \bigwedge_i u_1(f(b), \alpha_i)\right\} \\
&= \bigwedge \bigwedge_i \{c \in \mathbb{L}_2 | c \ge b,\ f(c) \ge u_1(f(b), \alpha_i)\} = \bigwedge_i u_2(b, \alpha_i), \\
l_2\left(b, \bigwedge_i \alpha_i\right) &= \bigvee\left\{c \in \mathbb{L}_2 | c \le b, l_1\left(f(c), \bigwedge_i \alpha_i\right) \le f(b)\right\} \\
&= \bigvee\left\{c \in \mathbb{L}_2 | c \le b, \bigvee_i l_1(f(c), \alpha_i) \le f(b)\right\} \\
&= \bigvee \bigvee_i \{c \in \mathbb{L}_2 | c \le b, l_1(f(c), \alpha_i) \le f(b)\} = \bigvee_i l_2(b, \alpha_i).
\end{aligned}
$$

Thus the system $(\mathbb{L}_1, u_1, l_1)$ is semicontinuous from above. □

From Theorems 5.3 and 5.6, we obtain the following important:

Corollary 5.9. *Category* $\mathbf{AS}^{\mathbb{M}}$ *is topological over the category* $\mathbf{IDL}^{op}$ *of infinitely distributive lattices with respect to the forgetful functor* $\mathfrak{F} : \mathbf{AS}^{\mathbb{M}} \longrightarrow \mathbf{IDL}^{op}$.

Besides, taking into account Theorems 5.4, 5.7, 5.5 and 5.8 we have:

Corollary 5.10. *The category* D-$\mathbf{AS}^{\mathbb{M}}$ *of self-dual* $\mathbb{M}$*-approximate systems is topological over the category* $\mathbf{AIIDL}^{op}$ *with respect to the forgetful functor* $\mathfrak{F}$: D-$\mathbf{AS}^{\mathbb{M}} \longrightarrow \mathbf{AIIDL}^{op}$.

Let $\mathbf{IDL_0}$ denote the full subcategory of the category $\mathbf{IDL}$ of infinitely distributive lattices $\mathbb{L}$ for which the element $0_{\mathbb{L}}$ is isolated from above and let SCA$_0$-$\mathbf{AS}^{\mathbb{M}}$ denote the full subcategory of the category of semicontinuous from above $\mathbb{M}$-approximate systems SCA-$\mathbf{AS}^{\mathbb{M}}$ with objects $(\mathbb{L}, u, l)$ such that $\mathbb{L} \in \mathbf{IDL_0}$.

Corollary 5.11. *The category* SCA$_0$-$\mathbf{AS}^{\mathbb{M}}$ *is topological over the category* $\mathbf{AIIDL_0}^{op}$ *with respect to the forgetful functor* $\mathfrak{F}$: SCA-$\mathbf{AS}^{\mathbb{M}} \longrightarrow \mathbf{IDL}_{\mathbf{0}}^{op}$.

In the rest of this paper we shall describe several both new and known categories as subcategories of the category $\mathbf{AS}^{\mathbb{M}}$. Here we only define these subcategories and mention some of their properties. The detailed study of these categories as subcategories of the category $\mathbf{AS}^{\mathbb{M}}$ which in the context of our research is viewed upon as the universal one will be the subject of the consequent papers.

6. Some general subcategories of the category $\mathbf{AS}^{\mathbb{M}}$

6.1. Category $\mathbf{AS}^{\mathbb{M}}(\mathbf{IDL}^{\mathbf{SET}})$

An important subcategory of the category $\mathbf{AS}^{\mathbb{M}}$ is the category $\mathbf{AS}^{\mathbb{M}}(\mathbf{IDL}^{\mathbf{SET}})$. Its objects are L-powersets L^X where $X \in \mathcal{O}b(\mathbf{SET})$ is an arbitrary set and $\mathrm{L} \in \mathcal{O}b(\mathbf{IDL})$ is an arbitrary infinitely distributive lattice. The morphisms in $\mathbf{AS}^{\mathbb{M}}(\mathbf{IDL}^{\mathbf{SET}})$ are those morphisms from $\mathbf{AS}^{\mathbb{M}}$ which are induced by mappings of the corresponding sets X.[2] Here are the details: Let the objects of $\mathbf{AS}^{\mathbb{M}}(\mathbf{IDL}^{\mathbf{SET}})$ be $\mathbb{M}$-approximate systems (L^X, u, l) where $\mathrm{L} \in \mathcal{O}b(\mathbf{IDL})$ and $X \in \mathcal{O}b(\mathbf{SET})$. Further, let the morphisms in $\mathbf{AS}^{\mathbb{M}}(\mathbf{IDL}^{\mathbf{SET}})$ be pairs

$$f = (F, \varphi) : (\mathrm{L}_1^{X_1}, u_1, l_1) \to (\mathrm{L}_2^{X_2}, u_2, l_2),$$

where $F : X_1 \to X_2$ is a mapping of sets, $\varphi : \mathrm{L}_2 \to \mathrm{L}_1$ is a morphism in $\mathbf{IDL}$ and the mapping $F_\varphi^{\leftarrow} : \mathrm{L}_2 \to \mathrm{L}_1$ defined by

$$F_\varphi^{\leftarrow}(b) = \varphi \circ b \circ F \; (= F^{-1}(\varphi(b))), \quad \forall b : X_2 \to \mathrm{L}_2$$

is a morphism

$$F_\varphi^{\leftarrow} : (\mathrm{L}_1^{X_1}, u_1, l_1) \to (\mathrm{L}_2^{X_2}, u_2, l_2)$$

in the category $\mathbf{AS}^{\mathbb{M}}$. The last condition means that the following two inequalities must hold for every $b : X_2 \to \mathrm{L}_2$ and every $\alpha \in \mathbb{M}$:

$$u_1(\varphi \circ b \circ F, \alpha) \le \varphi \circ u_2(b, \alpha) \circ F,$$

$$\varphi \circ l_2(b, \alpha) \circ F \le l_1(\varphi \circ b \circ F, \alpha).$$

Remark 6.1. Note that the category thus defined is not the full subcategory of $\mathbf{AS}^{\mathbb{M}}$: to get a full subcategory of $\mathbf{AS}^{\mathbb{M}}$ with objects as in $\mathbf{AS}^{\mathbb{M}}(\mathbf{IDL}^{\mathbf{SET}})$ we had to take *all* morphisms between the corresponding $\mathbb{M}$-approximate systems, and not only those ones which are induced by mappings of the corresponding sets as it was done above.

---

[2] Notice that the idea of considering category of such type can be traced in Rodabaugh's papers, see e.g. [35,38]; cf. also Section 7.5 in this paper.

6.2. *Category* $\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\mathbf{SET}})$

Let L be a fixed complete infinitely distributive lattice. We define the category $\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\mathbf{SET}})$ as the subcategory of $\mathbf{AS}^{\mathbb{M}}(\mathbf{IDL}^{\mathbf{SET}})$, whose objects are $\mathbb{M}$-approximate systems (L^X, u, l), and where for morphisms we take those morphisms from $\mathbf{AS}^{\mathbb{M}}(\mathbf{IDL}^{\mathbf{SET}})$, in which $\varphi : \mathrm{L} \to \mathrm{L}$ is the identity mapping. So again, obviously, $\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\mathbf{SET}})$ is not a full subcategory of $\mathbf{AS}^{\mathbb{M}}(\mathbf{IDL}^{\mathbf{SET}})$.

An important special case is a two point lattice $\mathrm{L} = \mathbf{2}$: in this case we come to the category of $\mathbb{M}$-approximate structures on ordinary sets (of course, for this one has to interpret a subset A of a set X as the characteristic function $\chi_A : X \to \mathbf{2}$). In particular, if $\mathbb{M}$ is a two-point lattice $\mathbf{2}$ (or, equivalently, a one-point lattice $\bullet$, cf. Remark 3.6) we come to the concept of an approximation system as it was considered by some authors, see e.g. [49,50].

By taking different subclasses **Set** of the class **SET** of sets, we obtain corresponding full subcategories $\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\mathbf{Set}})$ of the category $\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\mathbf{SET}})$. In particular, the category $\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\bullet})$, where $\bullet$ is a one point set, is just the lattice $\mathcal{AS}^{\mathbb{M}}(\mathbb{L})$ of $\mathbb{M}$-approximate systems on the lattice $\mathbb{L} = \mathrm{L}^{\bullet}$ viewed as the corresponding category.

6.3. *Categories of* $\mathbf{AS}^{\mathbb{M}}(\mathbf{Lat})$- $\mathbf{AS}^{\mathbb{M}}(\mathbf{Lat}^{\mathbf{Set}})$-, P-$\mathbf{AS}^{\mathbb{M}}(\mathbf{Lat})$- *and* P-$\mathbf{AS}^{\mathbb{M}}(\mathbf{Lat}^{\mathbf{Set}})$-*types*

Let **Lat** be some full subcategory of the category **IDL**. Then by restricting the class of objects of the category $\mathbf{AS}^{M}$ by those ones whose lattice belongs to $\mathcal{O}b(\mathbf{Lat})$ and whose morphisms belong to $\mathcal{M}or(\mathbf{Lat})$ we define a category $\mathbf{AS}^{\mathbb{M}}$ (**Lat**). The category $\mathbf{AS}^{\mathbb{M}}(\mathbf{AIIDL})$ of $\mathbb{M}$-approximate systems on adjunctive involutive infinitely distributive lattices is an example of a category of such type.

By restricting in the category $\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\mathbf{SET}})$ defined in Section 6.1 the class of objects whose lattices are taken from $\mathcal{O}b$ (**Lat**) and whose morphisms are from $\mathcal{M}\, or$ (**Lat**) we naturally come to a subcategory $\mathbf{AS}^{\mathbb{M}}(\mathbf{Lat}^{\mathbf{Set}})$ of the category $\mathbf{AS}^{\mathbb{M}}(\mathbf{IDL}^{\mathbf{SET}})$.

Note that given a fixed lattice L, the category $\mathbf{AS}^{\mathbb{M}}(\{\mathrm{L}\})$ is not the same as the category $\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\bullet})$ considered in the previous subsection: the both categories have the same class of objects, but $\mathbf{AS}^{\mathbb{M}}(\{\mathrm{L}\})$ has a wider class of morphisms than $\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\bullet})$.

Another way allowing to obtain important subcategories of $\mathbf{AS}^{\mathbb{M}}$ is to impose additional conditions P on the approximation operators u and l. Subcategories of such type will be denoted by P-$\mathbf{AS}^{\mathbb{M}}(\mathbf{Lat})$ and P-$\mathbf{AS}^{\mathbb{M}}(\mathbf{Lat}^{\mathbf{Set}})$, respectively. Categories D-$\mathbf{AS}^{\mathbb{M}}(\mathbf{AIIDL})$ and SCA-$\mathbf{AS}^{\mathbb{M}}$ which appeared in the previous sections are examples of such type of subcategories of the category $\mathbf{AS}^{\mathbb{M}}$.

7. Categories of fuzzy topologies as subcategories of $\mathbf{AS}^{\mathbb{M}}$

7.1. *Category of* (L, $\mathbb{M}$)-*fuzzy topological spaces*

We start by interpreting the category **FTOP**(L, $\mathbb{M}$) of (L, $\mathbb{M}$)-fuzzy topological spaces see e.g. [43,29,45,15] as a subcategory of $\mathbf{AS}^{\mathbb{M}}$. In this subsection $\mathbb{M}$ is assumed to be completely distributive and L is a fixed complete infinitely distributive lattice.

Definition 7.1 (*cf. Kubiak [29] and Šostak [43,44]*). A mapping $\mathcal{T} : \mathrm{L}^X \to \mathbb{M}$ is an (L, $\mathbb{M}$)-fuzzy topology on a set X if

(1FT) $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1$;
(2FT) $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V),\ \forall U, V \in \mathrm{L}^X$;
(3FT) $\mathcal{T}\left(\bigvee_{i\in\mathcal{I}} U_i\right) \geq \bigwedge_{i\in\mathcal{I}} \mathcal{T}(U_i),\ \forall \{U_i | i \in \mathcal{I}\} \subseteq \mathrm{L}^X$.

A pair $(X, \mathcal{T})$ is called an (L, $\mathbb{M}$)-fuzzy topological space and the value $\mathcal{T}(U)$ is interpreted as the degree of openness of a fuzzy set $U \in \mathrm{L}^X$. A mapping $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is called continuous if $\mathcal{T}_X(f^{-1}(V)) \geq \mathcal{T}_Y(V)$ for all $V \in \mathrm{L}^Y$.

Following e.g. [30–32] we denote the category of (L, $\mathbb{M}$)-fuzzy topological spaces by **FTOP**(L, $\mathbb{M}$).

Let $(X, \mathcal{T})$ be an (L, $\mathbb{M}$)-fuzzy topological space. By setting

$$\operatorname*{int}_{\mathcal{T}}(A, \alpha) = \bigvee \{U \in \mathrm{L}^X | U \leq A,\ \mathcal{T}(U) \geq \alpha\},$$

we define the interior operator $\mathrm{int}_{\mathcal{T}} : \mathrm{L}^X \times \mathbb{M} \to \mathrm{L}^X$ (see e.g. [45]). The relations between (L, $\mathbb{M}$)-fuzzy topologies and lower $\mathbb{M}$-approximate operators are described in the following theorem[3]:

Theorem 7.2. *The interior operator* $\mathrm{int}_{\mathcal{T}} : \mathrm{L}^X \times \mathbb{M} \to \mathbb{M}$ *is a weakly semicontinuous from below lower* $\mathbb{M}$*-approximate operator on* $\mathbb{L} = \mathrm{L}^X$. *Conversely, if* $l : \mathrm{L}^X \times \mathbb{M} \to \mathrm{L}^X$ *is a weakly semicontinuous from below lower* $\mathbb{M}$*-approximate operator, then by setting*

$$\mathcal{T}_l(U) = \bigvee\{\alpha | l(U, \alpha) \geq \alpha\}$$

we obtain a mapping $\mathcal{T}_l : \mathrm{L}^X \to \mathrm{L}^X$ *satisfying conditions* (1FT) *and* (3FT) *of Definition* 7.1. *Besides* $\mathcal{T}_{\mathrm{int}_{\mathcal{T}}} = \mathcal{T}$ *and* $l_{\mathcal{T}_l} = l$.

Further, assume that L is an adjunctive involutive infinitely distributive lattice and let $^c : \mathrm{L} \to \mathrm{L}$ be the corresponding involution. Then by setting

$$\mathrm{cl}_{\mathcal{T}}(A, \alpha) = \bigwedge\{B | B \geq A, \mathcal{T}(B^c) \geq \alpha\}$$

a closure operator $\mathrm{cl}_{\mathcal{T}} : \mathrm{L}^X \times \mathbb{M} \to \mathrm{L}^X$ [45] is defined. One can easily show that $\mathrm{cl}_{\mathcal{T}}$ is a weakly semicontinuous from below upper $\mathbb{M}$-approximate operator and prove a theorem establishing relations between weakly semicontinuous from below upper $\mathbb{M}$-approximate operators and (L, $\mathbb{M}$)-fuzzy topologies via closure operators, analogous to Theorem 7.2:

Theorem 7.3 (*see footnote 3*). *Let* L *be an adjunctive involutive infinitely distributive lattice. Then the closure operator* $\mathrm{cl}_{\mathcal{T}} : \mathrm{L}^X \times \mathbb{M} \to \mathrm{L}^X$ *is a weakly semicontinuous from below upper* $\mathbb{M}$*-approximate operator on* $\mathbb{L} = \mathrm{L}^X$. *Conversely, if* $u : \mathrm{L}^X \times \mathbb{M} \to \mathrm{L}^X$ *is a weakly semicontinuous from below upper* $\mathbb{M}$*-approximate operator, then by setting*

$$\mathcal{T}_u(A) = \bigvee\{\alpha | u(A^c, \alpha) \geq \alpha\}$$

we obtain a mapping $\mathcal{T}_u : \mathrm{L}^X \to \mathrm{L}^X$ *satisfying conditions* (1FT) *and* (3FT) *of Definition* 7.1. *Besides* $\mathcal{T}_{u_{\mathcal{T}}} = \mathcal{T}$ *and* $u_{\mathcal{T}_u} = u$.

Theorem 7.4. *The* $\mathbb{M}$*-approximation system* $(\mathrm{L}^X, \mathrm{cl}_{\mathcal{T}}, int_{\mathcal{T}})$ *constructed in Theorems* 7.1 and 7.3 *is self-dual*.

Thus in case of an adjunctive involutive infinitely distributive lattice L an (L, $\mathbb{M}$)-fuzzy topological space $(X, \mathcal{T})$ can be interpreted as a weakly semicontinuous from below $\mathbb{M}$-approximate self-dual system $(\mathbb{L}, \mathrm{cl}, \mathrm{int})$ where $\mathbb{L} = \mathrm{L}^X$. This allows to identify the category **FTOP**(L, $\mathbb{M}$) of (L, $\mathbb{M}$)-fuzzy topological spaces with the full subcategory DWSCB-$\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\mathbf{SET}})$ of the category $\mathbf{AS}^{(\mathbb{M})}(\mathrm{L}^{\mathbf{SET}})$ whose objects are self-dual weakly semicontinuous from below $\mathbb{M}$-approximate systems (L^X, u, l). In this case we write the corresponding $\mathbb{M}$-approximate system also in the form $(\mathrm{L}^X, \mathrm{cl}, \mathrm{int})$.

7.2. *Category of* (L, $\mathbb{M}$)*-fuzzy bitopological spaces*

Generalizing the previous situation let $\mathcal{T}^1 : \mathrm{L}^X \to \mathbb{M}$ and $\mathcal{T}^2 : \mathrm{L}^X \to \mathbb{M}$ be two (L, $\mathbb{M}$)-fuzzy topologies on the set X where L is an adjunctive involutive lattice. The triple $(X, \mathcal{T}^1, \mathcal{T}^2)$ naturally could be called an (L, $\mathbb{M}$)-fuzzy bitopological space. Further, let **FBTOP**(L, $\mathbb{M}$) be the category whose objects are (L, $\mathbb{M}$)-fuzzy bitopological spaces and whose morphisms are continuous mappings, that is mappings

$$f : (X, \mathcal{T}_X^1, \mathcal{T}_X^2) \to (Y, \mathcal{T}_Y^1, \mathcal{T}_Y^2)$$

such that

$$\mathcal{T}_X^i(f^{-1}(V)) \geq \mathcal{T}_Y^i(V), \quad i = 1, 2, \ \forall V \in \mathrm{L}^Y.$$

[3] This result was first stated in [31]. The first statement of this theorem was essentially proved in [45]. The remaining part of the theorem will be proved and discussed in our paper [32].

Further, let $u = \mathrm{cl}_{\mathcal{T}^1}$ and $l = \mathrm{int}_{\mathcal{T}^2}$ be the closure and interior operators induced by (L, $\mathbb{M}$)-fuzzy topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Then the category **FBTOP**(L, $\mathbb{M}$) of (L, $\mathbb{M}$)-fuzzy bitopological spaces can be identified with the category WSCB-$\mathbf{AS}^{\mathbb{M}}(\mathrm{L}^{\mathbf{SET}})$ of weakly semicontinuous from below $\mathbb{M}$-approximate systems.

7.3. Category of Chang–Goguen L-topological spaces

The first definition of a topological structure in the context of fuzzy sets was proposed by Chang [5] (in case L = [0, 1]) and soon extended for the case of an arbitrary infinitely distributive lattice L (and actually for the case of a cl-monoid L) by Goguen [10,11]. The corresponding objects according to the accepted now terminology are called L-topological spaces. The continuity of mappings of L-topological spaces was also considered in these papers.

Starting from the concept of an (L, $\mathbb{M}$)-fuzzy topological space, L-topological spaces as they are defined in the papers by Chang [5] and Goguen [10] can be identified with (L, **2**)-fuzzy topological spaces, that is when the second lattice $\mathbb{M}$ is the two-point lattice. Then the continuity of mappings of L-topological spaces reduces to the continuity of the corresponding (L, **2**)-fuzzy topological spaces. Hence to characterize L-topological spaces by means of $\mathbb{M}$-approximate systems we restrict the construction developed in Section 7.1 by taking the two-point lattice **2** in the role of $\mathbb{M}$. Thus the category of Chang–Goguen L-topological spaces can be identified with the category D-$\mathbf{AS}^{\mathbf{2}}(\mathrm{L}^{\mathbf{SET}})$—recall that in case of a two-point lattice $\mathbb{M}$ every $\mathbb{M}$-approximate system is semicontinuous both from above and from below. In particular D-$\mathbf{AS}^{\mathbf{2}}(2^{\mathbf{SET}})$ can be identified with the category **TOP** of ordinary topological spaces and continuous mappings.

By omitting the condition of self-duality of the **2**-approximate system we obtain the category $\mathbf{AS}^{\mathbf{2}}(\mathrm{L}^{\mathbf{SET}})$ which can be identified with the category of L-bitopological spaces (see e.g. [23] in case L = [0, 1] is the unit interval) and the category $\mathbf{AS}^{\mathbf{2}}(2^{\mathbf{SET}})$ which is essentially the classical category **BTOP** of bitopological spaces [24].

7.4. Category of L-fuzzifying topological spaces

The concept of an L-fuzzifying topological space and the corresponding category was originally defined by Höhle [13]. Later (in case L = [0, 1]) it was independently rediscovered by Ying [52] by means of semantic analysis of the classical topological axioms. Further the category of L-fuzzifying topological spaces which we denote **F-TOP**(L) was studied by Höhle [14] and in a series of papers by Ying and his students, see e.g. [53–55,40,42].

Referring to the definition of an (L, $\mathbb{M}$)-fuzzy topological space given in Section 7.1 the category of **F-TOP**(L) of L-fuzzifying topological spaces can be described as the category **FTOP**(**2**, L) where L is an adjunctive involutive infinitely distributive lattice. Now applying the construction developed in Section 7.1, the category of L-fuzzifying topological spaces can be identified with the category D-$\mathbf{AS}^{\mathrm{L}}(\mathbf{2}^{\mathbf{SET}})$.

7.5. Category of Hutton fuzzy topological spaces

According to Hutton [16,17] a fuzzy topological space is a pair $(\mathbb{L}, \tau)$ where $\mathbb{L}$ is a completely distributive lattice endowed with an order reversing involution $^c : \mathbb{L} \to \mathbb{L}$ (that is an adjunctive involutive lattice in our terminology) and $\tau \subseteq \mathbb{L}$ satisfies the following conditions:

(1HT) $0_{\mathbb{L}}, 1_{\mathbb{L}} \in \tau$;
(2HT) $a, b \in \tau \Longrightarrow a \wedge b \in \tau$;
(3HT) $a_i \in \tau,\ \forall i \in \mathcal{I} \Longrightarrow \bigvee_{i\in\mathcal{I}} a_i \in \tau$.

The morphisms $f : (\mathbb{L}_1, \tau_1) \to (\mathbb{L}_2, \tau_2)$ in the category **H-TOP** of Hutton fuzzy topological spaces are mappings $f : \mathbb{L}_2 \to \mathbb{L}_1$ such that $f(\tau_2) \subseteq \tau_1$.

The category **H-TOP** can be identified with the category of $\mathbb{M}$-approximate systems D-$\mathbf{AS}^{\mathbf{2}}(\mathbf{AIIDL})$. In this case a Hutton fuzzy topological space $(\mathbb{L}, \tau)$ is identified with the **2**-approximate system $(\mathbb{L}, u, l)$ where upper and lower **2**-approximate operators $u : \mathbb{L} \times 2 \to \mathbb{L}$ and $l : \mathbb{L} \times 2 \to \mathbb{L}$ are defined, respectively, by

$$u(a, 1_{\mathbf{2}}) = \bigwedge\{b | b \geq a, b^c \in \tau\}, \quad \forall a \in \mathbb{L},$$

$$u(a, 0_{\mathbf{2}}) = a, \quad \forall a \in \mathbb{L},$$

$$l(a, 1_{\mathbf{2}}) = \bigvee\{b | b \leq a, b \in \tau\}, \quad \forall a \in \mathbb{L},$$

$$l(a, 0_{\mathbf{2}}) = a, \quad \forall a \in \mathbb{L}.$$

7.6. Category of variable-basis fuzzy topological spaces

In [35], Rodabaugh has introduced the category **FUZZ** of variable-basis fuzzy topological spaces. Further the category **FUZZ** of variable-basis fuzzy topological spaces and some other similarly defined categories were studied in a series of papers by Rodabaugh, Eklund and other authors, see e.g. [6,7,37,38].

The objects of **FUZZ** are triples (X, L, τ) where X is a set, L is a completely distributive lattice endowed with an order reversing involution $^c : \mathrm{L} \to \mathrm{L}$ and $\tau \subseteq \mathrm{L}^X$ satisfies the following axioms:

(1) $0_\mathrm{L}, 1_\mathrm{L} \in \tau$;
(2) $U, V \in \tau \Longrightarrow U \wedge V \in \tau$;
(3) $U_i \in \tau,\ \forall i \in \mathcal{I} \Longrightarrow \bigvee_{i\in\mathcal{I}} U_i \in \tau$.

The morphisms in **FUZZ** are pairs

$$(f, \varphi) : (X_1, \mathrm{L}_1, \tau_1) \to (X_2, \mathrm{L}_2, \tau_2),$$

where $f : X_1 \to X_2$ is a mapping of sets, $\varphi : \mathrm{L}_1 \to \mathrm{L}_2$ is a morphism in the category $\mathbf{AIIDL}^{op}$ and $\varphi(f^{\leftarrow}(\tau_2)) \subseteq \tau_1$. Now the category **FUZZ** of variable-basis fuzzy topological spaces can be characterized as the full subcategory D-$\mathbf{AS^2}(\mathbf{H^{SET}})$ of the category $\mathbf{AS^2}(\mathbf{AIIDL^{SET}})$ whose objects are self-dual **2**-approximate systems.

7.7. Category of variable-basis (L, 𝕄)-fuzzy topological spaces

Generalizing in a natural way category D-$\mathbf{AS^2}(\mathbf{AIIDL^{SET}})$ introduced in the previous subsection by replacing **2** with an arbitrary completely distributive lattice $\mathbb{M}$ we obtain the category D-$\mathbf{AS}^{\mathbb{M}}(\mathbf{AIIDL^{SET}})$ which can be identified with the category of variable-basis $(\mathrm{L}, \mathbb{M})$-fuzzy topological spaces first mentioned in [29].

8. Categories related to rough sets

8.1. Rough sets

Let $\rho \subseteq X \times X$ be a binary relation on a set X and let $R(x) = \{x' | x\rho x'\}$ be the right ρ-class of $x \in X$. Given $A \in 2^X$ let operators $u : 2^X \to 2^X$ and $l : 2^X \to 2^X$ be defined, respectively, by

$$u(A) =: A^{\blacktriangle} = \{x | R(x) \cap A \neq \emptyset\},$$

$$l(A) =: A^{\blacktriangledown} = \{x | R(x) \subseteq A\}.$$

In case ρ is reflexive, that is

$$x\rho x, \quad \forall x \in X,$$

and transitive, that is

$$x\rho x', x'\rho x'' \Longrightarrow x\rho x'',$$

$u : 2^X \to 2^X$ and $l : 2^X \to 2^X$ satisfy, respectively, the axioms of Definitions 3.1 and 3.2. Thus they are, respectively, an upper and a lower $\bullet$-approximate operators on $2^X = \mathcal{P}(X)$ (see e.g. [26–28].) As the result we interpret the triple $(2, ^{\blacktriangledown}, ^{\blacktriangle})$ as an $\bullet$-approximate system while $(X, 2, ^{\blacktriangledown}, ^{\blacktriangle})$ is viewed as an $\bullet$-approximate space. Besides, one can easily see that in case ρ is also symmetric, that is

$$x\rho x' \iff x'\rho x$$

the system $(\mathbf{2}^X, ^{\blacktriangledown}, ^{\blacktriangle})$ is self-dual:

$$A^{c\blacktriangle} = A^{\blacktriangledown c},$$

where $A^c = X \setminus A$ for any $A \subseteq X$, see e.g. [26,28]. Such operators and corresponding approximate spaces in case when $\rho : X \times X$ is an equivalence relation (that is a reflexive, transitive, and symmetric relation) were first introduced by Pawlak [34] under the name "a rough set". Further approximate operators induced by binary relations, either general or satisfying additional properties, were studied by different authors, see e.g. [26–28,20,49,50]. Note, however, that in case ρ is not reflexive or transitive, such operators may fail to be approximate operators in our sense.

In case when ρ is only reflexive, Järvinen and Kortelainen [28] along with operators $l(A) = A^{\blacktriangledown}$ and $u(A) = A^{\blacktriangle}$ consider also operators $u' : 2^X \to 2^X$ and $l' : 2^X \to 2^X$ defined, respectively, by

$$u'(A) = A^{\triangle} = \{x | R^{-1}(x) \cap A \neq \emptyset\}$$

and

$$l'(A) = A^{\triangledown} = \{x | R^{-1}(x) \subseteq A\},$$

where $R^{-1}(x) = \{x' | x' \rho x\}$, and show that the pair (u, l') and (u', l) forms Galois connection:

$$u(a) \leq b \iff a \leq l'(b) \quad \text{and} \quad l(a) \leq b \iff a \leq u'(b).$$

Thus in case ρ is also transitive, we obtain self-dual $\bullet$-approximate systems

$$(\mathbf{2}^X, ^{\blacktriangledown}, ^{\triangle}) \quad \text{and} \quad (\mathbf{2}^X, ^{\triangledown}, ^{\blacktriangle}).$$

To consider rough sets as a category of approximate systems we have to specify its morphisms. Let **REL** be the category whose objects are sets endowed with a reflexive and transitive relation, that is pairs (X, ρ) and whose morphisms $f : (X, \rho_X) \to (Y, \rho_Y)$ are mappings $f : X \to Y$ such that

$$x \rho_X x' \Longrightarrow f(x) \rho_Y f(x'), \quad \forall\, x, x' \in X.$$

Further let **SREL** be the full subcategory of **REL** whose objects are sets with symmetric relations. We define the category $\mathbf{AS}^\bullet(2^{\mathbf{REL}})$ as follows. The objects of $\mathbf{AS}^\bullet(2^{\mathbf{REL}})$ are triples

$$(2^{(X,\rho)}, u = \blacktriangle, l = \blacktriangledown)$$

defined as above; the morphisms in $\mathbf{AS}^\bullet(2^{\mathbf{REL}})$ are backward operators [38]:

$$F = f^{\leftarrow} : (2^{(X_1 \rho_1)}, u_1 = \blacktriangle, l_1 = \blacktriangledown)) \to (2^{(X_2, \rho_2)}, u_2 = \blacktriangle, l_2 = \blacktriangledown)$$

induced by morphisms $f : (X_1, \rho_1) \to (X_2, \rho_2)$ of the category **REL**. We show that $\mathbf{AS}^\bullet(2^{\mathbf{REL}})$ thus obtained is indeed a category of $\bullet$-approximate system. To verify that

$$u_1(f^{-1}(B)) \subseteq f^{-1}(u_2(B)), \quad \forall B \in 2^{X_2}$$

let $x \in u_1(f^{-1}(B))$, then there exists $x' \in f^{-1}(B)$ such that $x \rho_1 x'$. However, this means that $f(x') \in B$ and $f(x) \rho_2 f(x')$. Therefore $f(x) \in u_2(B)$ and hence $x \in f^{-1}(u_2(B))$.

Now, to verify that

$$f^{-1}(l_2(B)) \subseteq l_1(f^{-1}(B)), \quad \forall B \in \mathbf{2}^{X_2},$$

let $x \in f^{-1}(l_2(B))$. Then $f(x) \in l_2(B)$ and hence $R(f(x)) \subseteq B$. However, this means that if $x' \rho_1 x$, then, since $f(x') \rho_2 f(x)$, it follows that $f(x') \in B$. Hence $x' \in f^{-1}(B)$. We conclude from here that $R(x) \subseteq f^{-1}(B)$ and hence $x \in l_1(f^{-1}(B))$.

Thus we have a category of approximate systems $\mathbf{AS}^\bullet(\mathbf{2}^{\mathbf{REL}})$ which can be identified with the category of rough sets **ROUGH**. In case we restrict to its full subcategory $\mathbf{AS}^\bullet\mathbf{2}^{\mathbf{SREL}}$ determined by symmetric relations the corresponding approximate systems are self-dual.

Along with the category $\mathbf{AS}^\bullet(\mathbf{2}^{\mathbf{REL}})$ one can consider the category

$$\mathbf{AS}^\bullet(\mathbf{2}^{\mathbf{REL}^{\triangle}}),$$

whose objects are self-dual approximate systems

$$(\mathbf{2}^{(X,\rho)}, u = \triangle, l = \blacktriangledown)$$

and the category $\mathbf{AS}^\bullet(\mathbf{2}^{\mathbf{REL}^\triangledown})$ whose objects are self-dual approximate systems

$$(\mathbf{2}^{(X,\rho)}, u = \blacktriangle, l = \triangledown).$$

The morphisms of these categories are defined in the same way as in the category $\mathbf{AS}^\bullet(\mathbf{2}^{\mathbf{REL}})$.

8.2. L-*rough sets*

In this subsection we extend the construction of approximate systems induced by rough sets for the case when the underlying set is equipped with an L-relation (see e.g. [51,3,4]). As an appropriate context for such construction we consider the structure of a cl-monoid on L. Thus, let $\mathrm{L} = (\mathrm{L}, \leq, \wedge, \vee, *)$ be a fixed cl-monoid and let X be a set. An L-relation on a set X is a mapping $R : X \times X \to \mathrm{L}$. An L-relation is called *reflexive* if

$$R(x, x) = 1 \quad \text{for all } x \in X,$$

transitive, if

$$R(x, x') * R(x'x'') \leq R(x, x'') \quad \text{for all } x, x', x'' \in X,$$

and *symmetric*, if

$$R(x, x') = R(x', x) \quad \text{for all } x, x' \in X.$$

Let $\mathbf{REL}(\mathrm{L})$ be the category whose objects are sets with L-relations, that is pairs (X, R) where X is a set and $R : X \times X \to L$ is an L-relation on it, and whose morphisms are mappings $f : (X, R_X) \to (Y, R_Y)$ such that

$$R_X(x, x') \leq R_Y(f(x), f(x')), \ \forall x, x' \in X.$$

Given a set X and a reflexive transitive L-relation $R : X \times X \to \mathrm{L}$, for every $x \in X$ we define a mapping $\mathcal{R}_x : X \to \mathrm{L}$ by

$$\mathcal{R}_x(x') = R(x, x') \quad \text{for all } x' \in X.$$

We define operators $u : \mathrm{L}^X \to \mathrm{L}^X$ and $l : \mathrm{L}^X \to \mathrm{L}^X$ as follows: Given $A \in \mathrm{L}^X$, let

$$l(A)(x) = \inf_{x' \in X} (\mathcal{R}(x)(x') \mapsto A(x')),$$

$$u(A)(x) = \sup_{x' \in X} (\mathcal{R}(x)(x') * A(x')).$$

One can show that (L^X, u, l) is an $\mathbb{L}$-approximate system where $\mathbb{L} = \mathrm{L}^X$. We refer to $\mathbb{L}$-approximate system of such type as an $\mathbb{L}$-rough system induced by the L-relation R. In case $(\mathrm{L}, \wedge, \vee, *)$ is a Girard monoid, the system (L^X, u, l) is self-dual. Further, if $\mathrm{L} = \mathbf{2}$ is a two-point lattice we come to the situation described in the previous subsection. In a natural way we define morphisms for the category $\mathbf{ROUGH}(\mathrm{L})$ of L-rough systems and characterize it as a category of $\mathbb{M}$-approximate systems. The detailed study of L-approximate systems of such type will be the subject of a subsequent paper.

9. Defuzzification approximate operators

Finally we sketch how the concept of an approximate systems can be applied for fuzzy sets themselves.

Let $\mathrm{L} = (\mathrm{L}, \wedge, \vee, \leq)$ be a complete lattice, X be a set and $\mathbb{L} = \mathrm{L}^X$. Define $u : \mathbb{L} \times \mathrm{L} \to \mathbb{L}$ and $l : \mathbb{L} \times \mathrm{L} \to \mathbb{L}$ as follows: Given $A \in \mathrm{L}^X$ let

$$u(A, \alpha) = A \vee 1_{A_\alpha},$$

$$l(A, \alpha) = \alpha_X \wedge A,$$

where $A_\alpha = \{x \in X | A(x) \geq \alpha\}$ is the α-cut of the L-set A and α_X is the constant function of X taking value α. In this way we obtain L-approximate operators on the L-powerset $\mathbb{L} = \mathrm{L}^X$ of a set X which can be interpreted as, respectively, upper and lower L-defuzzification operators and the corresponding L-approximate system $(\mathbb{L}, u, l)$ as an L-defuzzification system.

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