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PhD Thesis

Aggregation of fuzzy structures based on equivalence relations

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Abstract

The thesis deals with the special constructions of general aggregation operators, which are based on an equivalence relation. A need for such operators acting on fuzzy sets could arise dealing with particular decision making problems, multi-objective optimization and other problems, where it is important to take into account an equivalence relation between the objects of aggregation. But previously existed constructions of a general aggregation operator don't provide this possibility.

Initially, the general aggregation operator based on an equivalence relation has appeared in our research while we had been studying a choice of optimal solution for bilevel linear programming problems. We have suggested the construction of an aggregation of the lower level objectives taking into account a crisp equivalence relation generated by the upper level objective. This allowed us to obtain the tool, which helps to analyse bilevel linear programming problem's solving parameters in order to choose the optimal solution.

We generalize this concept by involving a fuzzy equivalence relation instead of a crisp one. As a result, we consider upper and lower general aggregation operators based on a fuzzy equivalence relation. These constructions provide upper and lower approximations of the pointwise and t-norm extension of an ordinary aggregation operator. We consider different properties of these operators, including its connection with extensional fuzzy sets. We demonstrate with a numerical example possible applications of these constructions in decision making. Another part is devoted to the case, when inputs are in the form of fuzzy real numbers.

Finally, we describe approximate systems induced by upper and lower general aggregation operators, considering the lattice of all general aggregation operators and the lattice of all fuzzy equivalence relations. It provides a generalized view on the suggested constructions and allows to use approximate system tools for further research.

Key words and phrases: Aggregation operator, general aggregation operator, fuzzy equivalence relation, extensional fuzzy set, upper and lower approximate operators, approximate system

Anotācija

Promocijas darbs veltīts vispārinātā agregācijas operatora speciālai konstrukcijai, kas balstīta uz ekvivalences attiecības. Nepieciešamība pēc šāda operatora var rasties, meklējot risinājumu konkrētajam lēmumu pieņemšanas uzdevumam, vairāku mērķa funkciju optimizācijas uzdevumam un citām problēmām, kad ir svarīgi ņemt vērā ekvivalences attiecību starp agregācijas objektiem.

Sākotnēji uz ekvivalences attiecības balstīts vispārinātais agregācijas operators darbā tiek lietots pētot optimālā atrisinājuma izvēli divu līmeņu lineārās programmēšanas uzdevumos. Tiek piedāvāta apakšējā līmeņa mērķa funkciju agregācijas konstrukcija, kas ņem vērā striktu ekvivalences attiecību, ģenerētu ar augšējā līmeņa mērķa funkcijas palīdzību. Tas ļauj iegūt rīku, ar kuru var analizēt divu līmeņu lineārās programmēšanas uzdevuma parametrus, lai izvēlētos optimālo atrisinājumu.

Tālāk šī koncepcija tiek vispārināta, ieviešot nestriktu ekvivalences attiecību striktās attiecības vietā. Rezultātā tiek iegūts apakšējais un augšējais vispārinātais agregācijas operators, kas balstās uz nestriktas ekvivalences attiecības. Šī konstrukcija nodrošina augšējo un apakšējo aproksimāciju parastā agregācijas operatora punktveida un t-turpinājumam. Tiek pētītas šo operatoru īpašības. Ar skaitlisku piemēru tiek parādīts šīs konstrukcijas iespējamais pielietojums lēmumu pieņemšanā. Atsevišķa nodaļa veltīta gadījumam, kad ieejas lielumi ir nestrikto reālo skaitļu formā.

Visbeidzot, tiek aprakstīta augšējā un apakšējā vispārinātā agregācijas operatora inducēta aproksimatīvā sistēma, aplūkojot visu vispārināto agregācijas operatoru un visu nestrikto ekvivalences attiecību režģus. Tas dod vispārīgo skatījumu uz izstrādātajām konstrukcijām un ļauj izmantot aproksimatīvo sistēmu rīkus tālākajos pētījumos.

Atslēgas vārdi un frāzes: agregācijas operators, vispārinātais agregācijas operators, nestrikta ekvivalences attiecība, ekstensionālās nestriktas kopas, augšējais un apakšējais aproksimatīvie operatori, aproksimatīvā sistēma

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Contents

Introduction

Aggregation is the process of combining several numerical values into a single representative value. For example, aggregations are widely used in pattern recognition and image processing, researches on neural networks, decision making theory as well as many other applied fields of physics and natural sciences. As the widely used examples of aggregation operators we could mention arithmetic and geometric mean, minimum and maximum operators, t-norms and others (see, e.g., [4, 7, 14]). Aggregation operators have been widely used during all the history of physics, probability theory and statistics, economics and finance. Since 1980's aggregation operators have become a substantive research field due to rapid development of computer technology. Numerous papers and books have been published on both theoretical investigations and applications of aggregation operators. Mathematically aggregation operator is a function that maps multiple inputs from a set into a single output from the same set, which somehow characterizes these inputs. A problem of choosing the right class of aggregation operators for a particular problem is a difficult task itself. The choice could depend on the properties of inputs as well as the properties of aggregation operators. In many cases one should construct an appropriate aggregation operator with the required properties.

Our thesis deals with a concept of aggregation of fuzzy sets. One of the facts, which significantly restricts the use of theoretical mathematics in modelling, researches and forecasting in real-world problems, is that theoretical mathematics traditionally deals with two-valued logics, and therefore is based on alternative nature of a set: each element either belongs to the set or does not belong. However, in real-world processes typically are situations, when some set of objects has vague boundaries: some object could be a member of the set with some degree. In order to deal with this shortcoming and to allow to develop theoretical mathematics in the direction, which is more suitable for modelling real-world processes, professor Lotfi A. Zadeh of the University of California, Berkley introduced the concept of a fuzzy set.

The development of the fuzzy set theory started in 1965, when the paper entitled

"Fuzzy Sets" w as published in journal *Information and control*. This publication serves as a foundation of the development of new mathematical theory. In his paper Zadeh extended the classical notion on Cantor set, allowing the membership function to take values not only 0 and 1, but any value from interval [0,1]. Such sets are called "fuzzy". In 1968 J.A. Goguen [13] developed and improved the ideas of Zadeh by introducing the notion of L-fuzzy set. He allowed the membership function to take value not only on internal [0,1], but on general lattice L. The geographical spread of investigations on many-valued mathematical structures is very wide. A lot of countries has specialized institutes and laboratories, which are involved in this subject (USA, Spain, Czech Republic, Slovakia, Poland, Germany, Austria, South Korea, South Africa and others). It is worth to particularly mention Japan, where in the 1980-s were established two large specialized institutes, where fundamental researches in theoretical mathematics of many-valued structures have been performed in collaboration with industrial companies (Honda, Kawasaki Steel, Toshiba, Nissan Motors, Canon and others). A lot of laboratories are financed by big enterprises form the outside of Japan, among them are Bosch, Zeiss, Siemens, Audi and Volkswagen. In Europe many investigations in this area deals with the problems of artificial intelligence, information theory, pattern recognition and image analysis, big data and others [38].

The notion of a general aggregation operator acting on fuzzy structures was introduced by A. Takači in [36]. He defined an aggregation operator acting on fuzzy sets and investigated pointwise and min-extensions of aggregation operator with respect to different orderings between fuzzy sets.Later the research group at the University of Latvia started investigations in this area. J. Lebedinska in her works (see, e.g., [23, 24]) developed the concept of general aggregation operator. She considered different constructions of a general aggregation operator, investigated behaviour of this operator acting on different types of fuzzy sets and also explored various properties. Another view on aggregation of fuzzy relations was provided by O. Grigorenko [15, 16] by using the idea of t-norm extension of aggregation operator. Recently, Z. Takač proposed an aggregation operator acting on fuzzy truth values [37] and investigated it's properties. Non of the authors has considered aggregation operators where an equivalence between the objects is taken into account. For example, the need for such constructions could arise dealing with particular decision making problems, multi-objective optimization and other.

In the thesis we define and develop the concept of general aggregation operators with respect to an equivalence relation both in crisp and fuzzy cases. First, we study general aggregation operators based on a crisp equivalence relation. The idea is to aggregate fuzzy sets in accordance with classes of equivalence generated by this crisp equivalence relation. Initially, the need for such operator raised while we were investigating how to assess an optimal solution of a bilevel linear programming problem. The fuzzy solution approach to this problem led to the need of aggregating membership functions of the lower level objectives taking into account the satisfactory level of the upper level objective. In the thesis we generalize this concept by involving fuzzy equivalence relation instead of a crisp one. We consider two operators as upper and lower approximations of a general aggregation operator and study its properties. Taking into account that fuzzy equivalence relations represent the fuzzification of equivalence relations and extensional fuzzy subsets play the role of fuzzy equivalence classes, we consider the upper and lower general aggregation operators in the context of extensional fuzzy sets. It is important that the results of upper and lower general aggregation operators corresponding to a fuzzy equivalence are extensional with respect to this relation. In some cases while aggregating extensional fuzzy sets it could be necessary to obtain as a result an extensional fuzzy set as well, but an ordinary general aggregation does not ensure this property. We describe also upper and lower approximations of the t-norm extension of an ordinary aggregation operator. Finally, the constructions of upper and lower general aggregation operators allow us to describe an approximate system induced by these operators (see, e.g., [11, 17, 34, 35]). Among the most important examples of approximate systems are approximate systems induced by fuzzy equivalence relations. These approximate systems are related to fuzzy rough sets (see, e.g., [10, 28, 39]). The constructed approximate systems allow to perform research on connection between these areas.

Goal and objectives

The main goal of the thesis is to develop the theory of aggregation operators acting on fuzzy sets by constructing an aggregation operator involving an equivalence relation between objectives.

The tasks of the thesis are:

1. to describe a construction scheme of an aggregation operator acting on fuzzy sets, which takes into account an equivalence relation between these fuzzy sets,
2. to apply the proposed construction for analysis of solving parameters of a bilevel linear programming problem, in order to choose an optimal solution of the problem and to improve the existing fuzzy algorithm for solving such kind of problems,

3. to generalize the proposed construction by using a fuzzy equivalence relation and to investigate properties of the obtained operators,
4. to investigate the case when inputs of general aggregation operators are in the form of fuzzy real numbers,
5. to construct an approximate system based on upper and lower general aggregation operators.

Thesis structure

This thesis is structured in the following way. Chapter 1 presents a general overview of necessary preliminaries. The notion of an aggregation operator is described, several examples of widely used aggregation operators are given and the most important properties are defined. The definition of a fuzzy set and some important operators in the context of fuzzy mathematics are provided. Finally, some basic concepts from the lattice theory are given.

Chapter 2 is devoted to operators in the case of a crisp equivalence relation. Here we recall the construction of a general aggregation operator, which aggregates fuzzy sets in accordance with classes of equivalence generated by a crisp equivalence relation ρ. We show that all properties of the definition of a general aggregation operator such as the boundary conditions and the monotonicity hold for the defined operator. We show the possible application of this construction for the analysis of solving parameters for bilevel linear programming problems, which was studied in details in [50, 51]. This section is based on the interactive method of solution of bilevel linear programming problems introduced by M. Sakawa and I. Nishizaki [32, 33] and involving some parameters for the upper and lower level objectives. Several important properties of the output of this aggregation operator are proved. These properties help us in the process of choosing the solving parameters. In the final section the particular numerical example of such analysis is provided on the base of a mixed production planning problem. We use numerical values for the parameters of this problem to describe how the analysis of the solving parameters could be performed.

In Chapter 3 we give the definition of upper and lower general aggregation operators, which are based on a fuzzy equivalence relation, and show that these operators are general aggregation operators themselves. We illustrate by numerical examples how these constructions operate. We consider properties of upper and lower general aggregation operators derived from the properties of the corresponding ordinary aggregation operator. We study different types of monotonicity for these operators. We investigate the constructions of upper and lower general aggregation operators in the context of extensional fuzzy sets. These operators are considered as aggregations which take values in the class of all extensional fuzzy sets, which could be important in some particular problems. Finally, we consider the constructions of upper and lower general aggregation operators based on a t-norm extension of an ordinary aggregation operator.

Chapter 4 is devoted to aggregation of fuzzy real numbers. First, we describe a general aggregation operator which allows us to aggregate such inputs and to preserve its properties. We also show that upper and lower general aggregation operators preserve properties of fuzzy real numbers. Finally, we consider the preservation of the form of initial inputs in the case of fuzzy real numbers by upper and lower general aggregation operators based on a t-norm extension of an ordinary aggregation operator.

In Chapter 5 we recall the definition of $\mathbb{M}$-approximate system, which provides an alternative view on the relations between fuzzy sets and rough sets. In the context of $\mathbb{M}$-approximate systems two lattices $\mathbb{L}$ and $\mathbb{M}$ play the fundamental role. We provide the constructions of $\mathbb{M}$-approximate system induced by a fuzzy equivalence relation and based on upper and lower general aggregation operators. These constructions use the lattice $\mathbb{L}$ of all general aggregation operators and the lattice $\mathbb{M}$ of all fuzzy equivalence relations.

Approbation

The results obtained in the process of thesis writing have been presented at 17 international conferences:

- IFSA-EUSFLAT 2015: 16th World Congress of the International Fuzzy Systems Association (IFSA), 9th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT), Gijon, Spain, June 30 - July 3, 2015 [49],
- MMA 2015: 20th International Conference on Mathematical Modelling and Analysis, Sigulda, Latvia, May 26 - 29, 2015 [A1],
- IPMU 2014: 15th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, Montpellier, France, July 15 - 19, 2014 [46],
- FUZZ-IEEE 2014: IEEE International Conference on Fuzzy Systems, Beijing, China, July 6 - 11, 2014 [47],

- FSTA 2014: 12th International Conference on Fuzzy Set Theory and Applications, Liptovsky Jan, Slovakia, January 26 - 31, 2014 [A3],
- EUSFLAT 2013: 8th Conference of the European Society for Fuzzy Logic and Technology, Milan, Italy, September 11 - 13, 2013 [51],
- AGOP 2013: 7th International Summer School on Aggregation Operators, Pamplona, Spain, July 16 - 20, 2013 [50],
- MMA 2013: 18th International Conference on Mathematical Modelling and Analysis, Tartu, Estonia, May 27 - 30, 2013 [A4],
- EURO 2012: 25th European Conference on Operational Research, Vilnius, Lithuania, July 8 - 11, 2012 [A5],
- FSTA 2012: 11th International Conference on Fuzzy Set Theory and Applications, Liptovsky Jan, Slovakia, January 30 - February 3, 2012 [A7],
- EUSFLAT 2011: 7th Conference of the European Society for Fuzzy Logic and Technology, Aix-les-Bains, France, July 18 - 22, 2011 [45],
- MMA 2011: 16th International Conference on Mathematical Modelling and Analysis, Sigulda, Latvia, May 26 - 28, 2011. [A8]
- APLIMAT 2011: 10th International Conference International Conference on Applied Mathematics, Bratislava, Slovakia, February 1 - 4, 2011 [44],
- MMA 2010: 15th International Conference on Mathematical Modelling and Analysis, Druskininkai, Lithuania, May 26 - 29, 2010 [A9],
- 8th Latvian Mathematical Conference, Valmiera, Latvia, April 9 - 10, 2010 [A10],
- FSTA 2010: 10th International Conference on Fuzzy Set Theory and Applications, Liptovsky Jan, Slovakia, February 1 - 5, 2010 [A11],
- MMA 2009: 14th International Conference on Mathematical Modelling and Analysis, Daugavpils, Latvia, May 27 - 30, 2009 [A12],

as well as at 10 domestic conferences:

- 73th Conference of the University of Latvia, Riga, February 26, 2015,

- 10th Latvian Mathematical Conference, Liepaja, April 10 - 12, 2014 [A2],
- 72th Conference of the University of Latvia, Riga, March 6, 2014,
- 71th Conference of the University of Latvia, Riga, February 28, 2013,
- 9th Latvian Mathematical Conference, Jelgava, March 30 - 31, 2012 [A6],
- 70th Conference of the University of Latvia, Riga, February 23, 2012,
- 69th Conference of the University of Latvia, Riga, March 10, 2011,
- 68th Conference of the University of Latvia, Riga, March 4, 2010,
- 67th Conference of the University of Latvia, Riga, February 11, 2009,
- 7th Latvian Mathematical Conference, Rezekne, April 18 - 19, 2008, [A13].

The results of the research have been presented on the regular seminars *Many-valued structures in algebra, topology and analysis* at the University of Latvia and on the 1st Czech-Latvian Seminar on Advanced Methods in Soft Computing (Trojanovice, Czech Republic, 2008) [A13].

The main results of the research have been reflected in 8 scientific publications:

- P. Orlovs, S. Asmuss, On some properties of general aggregation operators based on a fuzzy equivalence relation, Advances in Intelligent Systems Research, Vol. 89, Atlantis Press, 2015, p. 1354 - 1361. (Web of Science, to appear in Scopus)
- P. Orlovs, S. Asmuss, General aggregation operators based on a fuzzy equivalence relation in the context of approximate systems, Fuzzy Sets and Systems, 2015. (Scopus, article in press)
- P. Orlovs, S. Asmuss, On extensional fuzzy sets generated by factoraggregation, Information Processing and Management of Uncertainty in Knowledge-Based Systems, Series: Communications in Computer and Information Science, Vol. 444(3), Springer, 2014, p. 317 - 326. (Scopus, Web of Science)
- P. Orlovs, S. Asmuss, Upper and lower generalized factoraggregations based on fuzzy equivalence relation, Proceedings of the 2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), Beijing, China, 2014, p. 1772 - 1777. (Scopus, Web of Science)

- P. Orlovs, O. Montvida and S. Asmuss, A choice of bilevel linear programming solving parameters: factoraggregation approach, Advances in Intelligent Systems Research, Vol. 32, Atlantis Press, 2013, p. 489 - 496. (Scopus, Web of Science)
- P. Orlovs, O. Montvida and S. Asmuss, An analysis of bilevel linear programming solving parameters based on factoraggregation approach, Aggregation Functions in Theory and in Practise, Series: Advances in Intelligent Systems and Computing, Vol. 228, Springer Berlin Heidelberg, 2013, p. 345 - 354. (Scopus, Web of Science)
- P. Orlovs, General aggregation operators acting on the L-fuzzy real line, Proceedings of the EUSFLAT-LFA 2011 conference, Aix-les-Bains, France, 2011, p. 47 - 52. (Scopus, Web of Science)
- P. Orlovs, On aggregation of L-fuzzy real numbers, APLIMAT - Journal of Applied Mathematics, Vol. 4 (3), 2011, p. 127 - 137.

Chapter 1

Preliminaries

In this chapter we present general definitions and results which are important for our further considerations. First, we give the definition, main properties and examples of aggregation operator, which is the basic notion throughout the thesis. These topics were widely studied in [4, 7, 14]. Second, we provide a brief overview of fuzzy set theory, while fuzzy sets are the objects, which are involved in aggregation process in our work. For more fundamental results on fuzzy set theory we refer the reader to [8, 9, 38, 43]. Finally, we recall some basic concepts of the lattice theory (see, e.g., [3, 6, 38]) and recall the notion of a t-norm operator, which is important both in the context of fuzzy sets and general aggregation operators (see, e.g., [22, 38]).

1.1 Aggregation operators

Aggregation is the process of combining several numerical values into a single representative value. Mathematically aggregation operator is a function that maps multiple inputs from a set into a single output from this set. In classical the case [4, 7] aggregation operators are defined on interval $[0,1]$.

Definition 1.1.1. *A mapping* $A : \bigcup_n [0,1]^n \to [0,1]$ *is called an aggregation operator if the following conditions hold:*

(A1) $A(0,\dots,0) = 0$*;*

(A2) $A(1,\dots,1) = 1$*;*

(A3) for all $n \in \mathbb{N}$ *and for all* $x_1,\dots,x_n,y_1,\dots,y_n \in [0,1]$*:*

$$x_i \leq y_i, i = 1,\dots,n \Longrightarrow A(x_1,\dots,x_n) \leq A(y_1,\dots,y_n).$$

Conditions (A1) and (A2) are called boundary conditions of A, but (A3) means the monotonicity of A. One can consider a case, when instead of $[0,1]$ an arbitrary closed interval $[a,b] \subset [-\infty,+\infty]$ is used. Let us denote by $A_{(n)}$ an aggregation operator of n arguments: $A_{(n)}\colon [0,1]^n \to [0,1]$.

Now we will refer to some properties of aggregation operators. There are particular examples of aggregation operators [4, 7] which satisfy some of the properties, but do not satisfy another.

(1) Symmetry:

$$\forall x_1,x_2,\ldots,x_n \in [0,1] \quad \forall \pi \quad A(x_1,\ldots,x_n) = A(x_{\pi(1)},\ldots,x_{\pi(n)}),$$

where $\pi\colon N \to N$ is a permutation and $N = \{1,\ldots,n\}$.

(2) Associativity:

$$\forall x_1,x_2,x_3 \in [0,1] \quad A(x_1,x_2,x_3) = A(A(x_1,x_2),x_3) = A(x_1,A(x_2,x_3)).$$

(3) Idempotence:

$$\forall x \in [0,1] \quad A(x,x,\ldots,x) = x.$$

(4) Existence of an absorbent element:

$$\exists a \in [0,1] \quad \forall i \in 1,\ldots,n \quad \forall x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n \in [0,1]$$

$$A(x_1,\ldots,x_{i-1},a,x_{i+1},\ldots,x_n) = a.$$

Such element a is called an absorbent element (or annihilator) of operator A.

(5) Existence of a neutral element:

$$\exists e \in [0,1] \quad \forall i \in 1,\ldots,n \quad \forall x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n \in [0,1]$$

$$A(x_1,\ldots,x_{i-1},e,x_{i+1},\ldots,x_n) = A(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n).$$

Such element e is called a neutral element of operator A.

(6) Homogeneity with respect to a multiplication with a non-negative number:

$$\forall r \geq 0, \quad \forall x_1,x_2,\ldots,x_n \in [0,1] \quad r \cdot x_i \in [0,1],\ i = 1,\ldots,n \Longrightarrow$$

$$\Longrightarrow A(r \cdot x_1,\ldots,r \cdot x_n) = r \cdot A(x_1,\ldots,x_n).$$

(7) Stability for a non-negative linear transformation:

$$\forall r \geq 0, \quad \forall t, x_1, x_2, \ldots, x_n \in [0,1] \quad r \cdot x_i + t \in [0,1],\ i = 1, \ldots, n \Longrightarrow$$

$$\Longrightarrow A(r \cdot x_1 + t, \ldots, r \cdot x_n + t) = r \cdot A(x_1, \ldots, x_n) + t.$$

(8) Continuity of an aggregation operator A is ordinary continuity of all n-argument operators $A_{(n)}$ in the sense of continuity defined for n-argument functions.

Finally, we consider some of the most well-known and important examples of aggregation operators [7]. Arguments of these operators could be taken from any closed interval of the real line, one should be aware of defining boundary conditions.

1. Arithmetic mean:
$$A_M(x_1, x_2, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

2. Weighted average:
$$W_{w_1,\ldots,w_n}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} w_i \cdot x_i,$$
where $w_1, \ldots, w_n \geq 0$ and $\sum\limits_{j=1}^{n} w_j = 1$.

3. Minimum and maximum:
$$MIN(x_1, x_2, \ldots, x_n) = \min\{x_1, x_2, \ldots, x_n\},$$
$$MAX(x_1, x_2, \ldots, x_n) = \max\{x_1, x_2, \ldots, x_n\}.$$

4. Geometric mean:
$$A_G(x_1, x_2, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}}, \ x_1, \ldots, x_n \geq 0.$$

5. Harmonic mean:
$$A_H(x_1, x_2, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}, \ x_1, \ldots, x_n > 0.$$

6. Operator of addition $A_+ : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$:

$$A_+(x_1,\dots,x_n) = \sum_{i=1}^{n} x_i, \quad x_1,x_2,\dots,x_n \in \overline{\mathbb{R}},$$

where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,+\infty\}$.

Boundary conditions (A1) and (A2) in this case should be described as follows:

(A1) $A_+(-\infty,\dots,-\infty) = -\infty$;

(A2) $A_+(+\infty,\dots,+\infty) = +\infty$.

In order to make this definition correct we agree that $-\infty$ is an absorbent element of A_+:

$$\forall i \in 1,\dots,n \quad \forall x_1,\dots,x_{i-1},x_{i+1},\dots,x_n \in \mathbb{R}$$

$$A_+(x_1,\dots,x_{i-1},-\infty,x_{i+1},\dots,x_n) = -\infty.$$

In this case $A_+(-\infty,+\infty) = -\infty$.

In this Chapter we will consider t-norm and t-conorm operators defined on arbitrary lattice L. The notion of a lattice will also appear later.

1.2 Fuzzy sets

The fuzzy set theory started in 1965, when the paper entitled *Fuzzy Sets* by L.A. Zadeh was published in journal *Information and Control*. This publication serves as a foundation of the development of the new mathematical theory. In his paper Zadeh extended the classical notion on Cantor set, allowing the membership function to take values not only 0 and 1, but any value from interval [0,1]. Such sets are called "fuzzy". In 1968 J.A. Goguen [13] developed and improved the ideas of Zadeh by introducing the notion of L-set. He allowed the membership function to take values not only on internal [0,1], but on a general lattice L. The notion of fuzzy set acquired extraordinary interest between mathematicians, as well as specialists, who applied mathematical ideas, concepts and results to model various real-world processes. We could mention several areas of application of fuzzy sets such as decision making, soft computing, optimization problems, control theory, pattern recognition, image processing and many others.

Mathematically fuzzy sets could be defined in the following way:

Definition 1.2.1. *A fuzzy set μ in the universe X (a fuzzy subset of X) is defined as a mapping $\mu\colon X \to [0,1]$. The set of all fuzzy subsets of X is denoted by $[0,1]^X$.*

Crisp sets are a special case of fuzzy sets. The membership function of a crisp set takes values on lattice $L = \{0,1\}$. It means that each point $x \in X$ could be assigned with numbers 1 or 0, when this point belongs or does not belong to μ respectively.

Operations on fuzzy sets are defined by using a t-norm T and a t-conorm S [38]. These notions will be described in the next section.

Definition 1.2.2. *An intersection of fuzzy sets μ and ν is defined as a fuzzy set $\mu \cap \nu$ such that*

$$(\mu \cap \nu)(x) = T(\mu(x), \nu(x)).$$

Definition 1.2.3. *A union $\mu \cup \nu$ of fuzzy sets μ and ν is defined as a fuzzy set $\mu \cup \nu$ such that*

$$(\mu \cup \nu)(x) = S(\mu(x), \nu(x)).$$

A complementary set of a fuzzy set is defined by an involution [38]:

Definition 1.2.4. *A function $N : [0,1] \to [0,1]$ is called an order reversing involution if it satisfies the following conditions for all $x, y \in [0,1]$:*

- $N(N(x)) = x$,
- $N(x) \geq N(y)$ *whenever* $x \leq y$.

Definition 1.2.5. *A complement of a fuzzy set μ is defined as a fuzzy set μ^c such that*

$$\mu^c(x) = N(\mu(x)).$$

In case of lattice $L = [0,1]$ a complement of a fuzzy set μ is usually defined as

$$\mu^c(x) = 1 - x.$$

In fuzzy mathematics the notions of a fuzzy number and a fuzzy interval play an important role. These are special types of fuzzy sets, which domain is a subset of the real line. Fuzzy numbers and intervals are widely used both in theoretical researches and applications. There exist several approaches how to define fuzzy numbers. As an example of widely used fuzzy numbers we could mention trapezoidal fuzzy numbers:

$$\mu(x) = \begin{cases} 0, \ x \leq a, \\ \frac{x-a}{b-a}, \ a \leq x \leq b, \\ 1, \ b \leq x \leq c, \\ \frac{x-d}{c-d}, \ c \leq x \leq d, \\ 0, \ d \leq x, \end{cases}$$

where $a < b \leq c < d$. In the case when $b = c$ we obtain a triangular fuzzy number.

In the thesis we consider fuzzy real numbers defined by B. Hutton [18] and then studied by other authors (see, e.g., [25, 26, 29, 30]).

Definition 1.2.6. *A fuzzy real number is defined as a function* $z\colon \mathbb{R} \to [0,1]$ *such that*

(N1) *z is non-increasing:* $x_1 \geq x_2 \Longrightarrow z(x_1) \leq z(x_2)$ *for all* $x_1, x_2 \in \mathbb{R}$*;*

(N2) *z is bounded:* $\inf\limits_{x\in\mathbb{R}} z(x) = 0, \ \sup\limits_{x\in\mathbb{R}} z(x) = 1;$

(N3) *z is left semi-continuous:* $\inf\limits_{x<x_0} z(x) = z(x_0)$ *for all* $x_0 \in \mathbb{R}$.

The set of all fuzzy real numbers is called the fuzzy real line and it is denoted by $\mathbb{R}([0,1])$.

1.3 Lattices and triangular norms

In this section we give an overview of some basic notions of the lattice theory [6, 38].

Definition 1.3.1. *A non-empty set L in which a binary relation* $\leq$ *is defined, which satisfies for all* $a,b,c \in L$ *the following properties:*

- $a \leq a$ *(reflexivity);*
- $a \leq b$ *and* $b \leq a \Longrightarrow a = b$ *(antisymmetry);*
- $a \leq b$ *and* $b \leq c \Longrightarrow a \leq c$ *(transitivity),*

is called a partially ordered set (a poset). A poset could be denoted by $(L, \leq)$.

Definition 1.3.2. *A poset L (or* $(L,\leq)$*) is said to be bounded, if*

- *there exists an element* 1_L *such that for all* $a \in L$ *it holds* $a \leq 1_L$*;*
- *there exists an element* 0_L *such that for all* $a \in L$ *it holds* $0_L \leq a$*.*

Elements 1_L *and* 0_L *are the greatest element of* L *(or maximum) and the least element of* L *(or minimum) respectively.*

Definition 1.3.3. *An element* $a \vee b \in L$ *is called a join of two elements* a *and* b $(a, b \in L)$ *if*

- $a \leq a \vee b$ *and* $b \leq a \vee b$*;*
- *for all* $c \in L$ *such that* $a \leq c$ *and* $b \leq c$ *it holds* $a \vee b \leq c$*.*

Definition 1.3.4. *An element* $a \wedge b \in L$ *is called a meet of two elements* a *and* b $(a, b \in L)$ *if*

- $a \wedge b \leq a$ *and* $a \wedge b \leq b$*;*
- *for all* $c \in L$ *such that* $c \leq a$ *and* $c \leq b$ *it holds* $c \leq a \wedge b$*.*

It is clear that for each $a \in L$ it holds $a \wedge 1_L = a$ and $a \vee 0_L = a$, if such 1_L and 0_L exist.

Definition 1.3.5. *A poset* L *is called a lattice if for each two elements* $a, b \in L$ *there exists a meet and a join:*

$$\forall a, b \in L \qquad \exists a \vee b \in L \ \& \ \exists a \wedge b \in L.$$

Definition 1.3.6. *A lattice* L *is called a complete lattice if for each set* $A = \{a_i \mid i \in I\}$*, where* I *is an arbitrary index set, there exists a meet and a join:*

$$\forall I \ \ \forall A = \{a_i \mid i \in I\} \subset L \ \ \exists \bigvee \{a_i \mid i \in I\} \in L \ \& \ \exists \bigwedge \{a_i \mid i \in I\} \in L.$$

A join and a meet of a set $A = \{a_i \mid i \in I\}$ are denoted by $\bigvee A$ and $\bigwedge A$ respectively, and are defined by analogy with Definition 1.3.3. and 1.3.4. A complete lattice L is always bounded, i.e. there exists the greatest and the least elements:

$$\bigvee L = 1_L, \ \ \bigwedge L = 0_L.$$

Proposition 1.3.1. *In each lattice* L *for all* $a, b, c, \in L$ *it holds*

$$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c).$$

Definition 1.3.7. *A lattice L is called a distributive lattice if for any $a,b,c,\in L$ it holds*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Definition 1.3.8. *A lattice L is said to be infinitely distributive, if L is complete and for all $a \in L$ and $\{b_i \mid i \in I\} \subseteq L$ it holds*

- $a \wedge (\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \wedge b_i)$;
- $a \vee (\bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a \vee b_i)$.

Definition 1.3.9. *A lattice L is called a completely distributive if for each set of elements $\{a_{ji} \mid j \in J, i \in I_j\} \subseteq L$ it holds*

$$\bigwedge_{j\in J} \bigvee_{i\in I_j} a_{ji} = \bigvee_{f\in F} \bigwedge_{j\in J} a_{jf(j)},$$

where F is the set of choice functions f choosing for each index $j \in J$ some $f(j) \in I_j$.

Now we consider the notion of a triangular norm (or a t-norm for short). This notion (see, e.g., [22, 38]) is fundamental in fuzzy mathematics. By using a t-norm it is possible to define operations with fuzzy sets [38]. A concept of a t-norm was introduced by K. Menger in 1942 in order to generalize the triangle inequality from classical metric spaces to probabilistic metric spaces [22].

Definition 1.3.10. *A function $T : L \times L \to L$ is called a t-norm if it satisfies the following conditions for all $a,b,c \in L$:*

- $T(a,b) = T(b,a)$ *(symmetry);*
- $T(a,T(b,c)) = T(T(a,b),c)$ *(associativity);*
- $T(a,b) \leq T(a,c)$ *whenever* $b \leq c$ *(monotonicity);*
- $T(a,1_L) = a$ *(neutral element).*

Now let us consider some important examples of t-norms [22], which will be used later throughout the thesis.

(1) Minimum t-norm T_M, which is defined on an arbitrary lattice L:

$$T_M(x,y) = x \wedge y.$$

(2) Weak t-norm T_W, which is defined on an arbitrary lattice L:

$$T_W(x,y) = \begin{cases} x \wedge y, \text{ if } x \vee y = 1_L \\ 0, \text{ if } x \vee y < 1_L. \end{cases}$$

The next t-norms could be defined only in the case, when lattice L is interval $[0,1]$ or it's subset.

(3) Lukasiewicz t-norm T_L on lattice $L = [0,1]$:

$$T_L(x,y) = \max\{x+y-1,0\}.$$

(4) Product t-norm T_P on lattice $L = [0,1]$:

$$T_P(x,y) = xy.$$

Taking into account the associativity, we are able to use a t-norm in the case of n arguments:

$$T(x_1,x_2,\ldots,x_n) = T(T(x_1,x_2,\ldots,x_{n-1}),x_n) \text{ for all } x_1,x_2,\ldots,x_n \in L, \ n \geq 3.$$

Definition 1.3.11. *A t-norm $T\colon L \times L \to L$ is said to be lower semi-continuous if for each set $\{x_i \mid i \in I\} \subset L$ and for each $y \in L$ it is satisfied*

$$\bigvee_{i \in I} T(x_i,y) = T(\bigvee_{i \in I} x_i, y).$$

Definition 1.3.12. *A t-norm $T\colon L \times L \to L$ is said to be upper semi-continuous if for each set $\{x_i \mid i \in I\} \subset L$ and for each $y \in L$ it is satisfied*

$$\bigwedge_{i \in I} T(x_i,y) = T(\bigwedge_{i \in I} x_i, y).$$

Definition 1.3.13. *A t-norm $T\colon L \times L \to L$ is said to be continuous if it is both lower and upper semi-continuous.*

Let us also define a dual operator for a t-norm — t-conorm [22, 38]. A t-conorm always concords with the respective t-norm.

Definition 1.3.14. *A function $S : L \times L \to L$ is called a triangular conorm (t-conorm for short) if it satisfies the following conditions for all $a,b,c \in L$:*

- $S(a,b) = S(b,a)$ *(symmetry);*
- $S(a,S(b,c)) = S(S(a,b),c)$ *(associativity);*
- $S(a,b) \leq S(a,c)$ *whenever* $b \leq c$ *(monotonicity);*
- $S(a,0_L) = a$ *(neutral element).*

Again, by the associativity, we are able to use a t-conorm in the case of n arguments:

$$S(x_1,x_2,\ldots,x_n) = S(S(x_1,x_2,\ldots,x_{n-1}),x_n) \text{ for all } x_1,x_2,\ldots,x_n \in L,\ n \geq 2.$$

The most important examples of t-conorms are

(1) maximum t-conorm S_M, which is defined on an arbitrary lattice L:

$$S_M(x,y) = x \vee y,$$

(2) weak t-conorm S_W, which is defined on an arbitrary lattice L:

$$S_W(x,y) = \begin{cases} x \vee y, \text{ if } x \wedge y = 0_L \\ 0, \text{ if } x \vee y < 1_L, \end{cases}$$

(3) Lukasiewicz t-conorm S_L on lattice $L = [0,1]$:

$$S_L(x,y) = \min\{x+y,1\},$$

(4) product t-conorm S_P on lattice $L = [0,1]$:

$$S_P(x,y) = x + y - xy.$$

Definition 1.3.15. *Let T be a left continuous t-norm. The residuum $\overrightarrow{T}$ of T is defined for all $x,y \in [0,1]$ by*

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0,1] \mid T(\alpha,x) \leq y\}.$$

We recall the following basic properties of the residuum:

($\overrightarrow{T}$1) $\overrightarrow{T}(x|y) = 1$ if and only if $x \leq y$;
($\overrightarrow{T}$2) $\overrightarrow{T}(1|y) = y$;
($\overrightarrow{T}$3) $\overrightarrow{T}(0|y) = 1$;
($\overrightarrow{T}$4) if $y = 0$ and $x \neq 0$, then $\overrightarrow{T}(x|y) = 0$;

$(\overrightarrow{T}5)$ the residuum is a non-increasing function with respect to the first argument and a non-decreasing function with respect to the second argument:

$$x_1 \le x_2 \Longrightarrow \overrightarrow{T}(x_1|y) \ge \overrightarrow{T}(x_2|y);$$

$$y_1 \le y_2 \Longrightarrow \overrightarrow{T}(x|y_1) \le \overrightarrow{T}(x|y_2);$$

$(\overrightarrow{T}6)$ $T(z,x) \le y \Longleftrightarrow z \le \overrightarrow{T}(x|y)$
(here $x, x_1, x_2, y, y_1, y_2, z \in [0,1]$).
Some important examples are the residua of t-norms T_L, T_M and T_P, which are given respectively by

$$\overrightarrow{T}_L(x|y) = \begin{cases} 1, & x \le y, \\ 1 - x + y, & x > y; \end{cases}$$

$$\overrightarrow{T}_M(x|y) = \begin{cases} 1, & x \le y, \\ y, & x > y; \end{cases}$$

$$\overrightarrow{T}_P(x|y) = \begin{cases} 1, & x \le y, \\ y/x, & x > y. \end{cases}$$

Chapter 2

General aggregation operator based on a crisp equivalence relation

This chapter is devoted to a specially constructed general aggregation operator based on a crisp equivalence relation. Initially, the need for such operator has appeared while we have been investigating how to assess an optimal solution of bilevel linear programming problem. The fuzzy solution approach to this problem led to the need of aggregating the membership functions of the lower level objectives taking into account the satisfactory level of the upper level objective. The existing construction methods of a general aggregation operator did not provide such opportunity. We suggested another construction of a general aggregation operator, initially named as factoraggregation, which was based on a crisp equivalence relation generated by the upper level objective. This allowed to obtain the tool, which helped us to analyse bilevel linear programming problem's solving parameters in order to choose the optimal solution. Later, we came up with a more general formulation of such construction and investigated it's aggregational properties. The results presented in this chapter were published in [50, 51]. Further in our work we are dealing with the construction of a general aggregation operator based on a fuzzy equivalence relation, instead of a crisp one.

2.1 General aggregation operator

The notion of general aggregation operator $\tilde{A}$ acting on $[0,1]^X$, where $[0,1]^X$ is the set of all fuzzy subsets of a set X, was introduced in 2003 by A. Takači [36]. He defined an aggregation operator acting on fuzzy sets and investigated the pointwise and min-extensions of an aggregation operator with respect to different orderings between fuzzy sets. J. Lebedinska in her works (see, e.g., [23, 24]) developed the concept of general aggregation operator. She considered different constructions of a general aggregation operator, investigated the behaviour of this operator acting on different types of fuzzy sets and also explored various properties. Another view on aggregation of fuzzy relations was provided by O. Grigorenko [15, 16] by using the idea of t-norm based extension of an aggregation operator. Recently, Z. Takač proposed an aggregation operator acting on fuzzy truth values [37] and investigated it's properties.

We will start with the definition of a general aggregation operator [36]. We denote a partial order on $[0,1]^X$ by $\preceq$. The least and the greatest elements of this order are denoted by $\tilde{0}$ and $\tilde{1}$, which are indicators of $\varnothing$ and X respectively, i.e.

$$\tilde{0}(x) = 0 \text{ and } \tilde{1}(x) = 1 \text{ for all } x \in X.$$

Definition 2.1.1. *A mapping $\tilde{A}\colon \bigcup_n([0,1]^X)^n \to [0,1]^X$ is called a general aggregation operator if and only if the following conditions hold:*
$(\tilde{A}1)$ $\tilde{A}(\tilde{0},\ldots,\tilde{0}) = \tilde{0}$*;*
$(\tilde{A}2)$ $\tilde{A}(\tilde{1},\ldots,\tilde{1}) = \tilde{1}$*;*
$(\tilde{A}3)$ *for all $n \in \mathbb{N}$ and for all $\mu_1,...,\mu_n,\eta_1,...,\eta_n \in [0,1]^X$:*

$$\mu_1 \preceq \eta_1,\ldots,\mu_n \preceq \eta_n \Longrightarrow \tilde{A}(\mu_1,\ldots,\mu_n) \preceq \tilde{A}(\eta_1,\ldots,\eta_n).$$

We consider the case:

$$\mu \preceq \eta \text{ if and only if } \mu(x) \leq \eta(x) \text{ for all } x \in X,$$

for $\mu,\eta \in [0,1]^X$.

A mapping $\tilde{A}\colon ([0,1]^X)^n \to [0,1]^X$ is called n-ary general aggregation operator. In the thesis we do not stress out when such aggregation operator is used. There exist several approaches to construct a general aggregation operator $\tilde{A}$ based on an ordinary aggregation operator A. The simplest one is the pointwise extension of an aggregation operator A:

$$\tilde{A}(\mu_1,...,\mu_n)(x) = A(\mu_1(x),...,\mu_n(x)), \tag{2.1}$$

where $\mu_1,...,\mu_n \in [0,1]^X$ are fuzzy sets and $x \in X$.

A widely used approach to constructing a general aggregation operator $\tilde{A}$ is the T -extension [36], whose idea comes from the classical extension principle and uses a t-

norm T (see, e.g., [22]):

$$\tilde{A}^T(\mu_1,\ldots,\mu_n)(x) = \sup_{x=A(x_1,\ldots,x_n)} T(\mu_1(x_1),\ldots,\mu_n(x_n)). \tag{2.2}$$

Here $\mu_1,\ldots,\mu_n \in [0,1]^X$ and $x, x_1 \ldots, x_n \in X$, where $X = [0,1]$. As a set X one could take any closed interval.

2.2 Construction of general aggregation operator based on a crisp equivalence relation

General aggregation operator $\tilde{A}_\rho$ based on a crisp equivalence relation ρ is a specially designed construction of a general aggregation operator based on an ordinary aggregation operator A. Initially, the idea of such construction appeared in the research while we were exploring an analysis of optimal solution of bilevel linear programming problems. It seemed important to aggregate the lower level objective functions taking into account the equivalence relation between them generated by the upper level objective. In general, such aggregation operator could be used, when it is important to take into account an equivalence relation between elements of universe X.

The construction of general aggregation operator based on a crisp equivalence relation is formulated in the following definition:

Definition 2.2.1. *Let $A\colon [0,1]^n \to [0,1]$ be an aggregation operator and ρ be a crisp equivalence relation defined on a set X. The general aggregation operators $\tilde{A}_\rho$ is defined by*

$$\tilde{A}_\rho(\mu_1,\ldots,\mu_n)(x) = \sup_{x'\in X:(x',x)\in\rho} A(\mu_1(x'),\ldots,\mu_n(x')), \tag{2.3}$$

where $x \in X$ and $\mu_1,\ldots,\mu_n \in [0,1]^X$.

In the case when crisp equivalence relation is defined in the following way:

$$\rho_0(x,y) = \begin{cases} 1, & x = y, \\ 0, & \text{otherwise}, \end{cases}$$

we have, that the general aggregation operators based on ρ_0 turn into the pointwise extensions of ordinary aggregation operator.

Let us demonstrate that construction (2.3) gives us a general aggregation operator. We must show that conditions $(\tilde{A}1)$, $(\tilde{A}2)$ and $(\tilde{A}3)$ are satisfied.

Proposition 2.2.1. *Operator $\tilde{A}_\rho$ (see 2.3) is a general aggregation operator.*

Proof. First we prove the boundary conditions:

1)
$$\tilde{A}_\rho(\tilde{0},\ldots,\tilde{0})(x) = \sup_{(x',x)\in\rho} A(\tilde{0}(x'),\ldots,\tilde{0}(x')) = A(0,\ldots,0) = \tilde{0}(x),$$

2)
$$\tilde{A}_\rho(\tilde{1},\ldots,\tilde{1})(x) = \sup_{(x',x)\in\rho} A(\tilde{1}(x'),\ldots,\tilde{1}(x')) = A(1,\ldots,1) = \tilde{1}(x).$$

To prove the monotonicity of $\tilde{A}_\rho$ we use the monotonicity of A:

$$\mu_i \preceq \eta_i,\ i = 1,2,\ldots,n \Longrightarrow A(\mu_1(x'),\ldots,\mu_n(x')) \le A(\eta_1(x'),\ldots,\eta_n(x')) \text{ for all } x' \in X \Longrightarrow$$

$$\Longrightarrow \sup_{(x',x)\in\rho} A(\mu_1(x'),\ldots,\mu_n(x')) \le \sup_{(x',x)\in\rho} A(\eta_1(x'),\ldots,\eta_n(x')) \text{ for all } x \in X \Longrightarrow$$

$$\Longrightarrow \tilde{A}_\rho(\mu_1,\ldots,\mu_n) \preceq \tilde{A}_\rho(\eta_1,\ldots,\eta_n).$$

□

In our papers [50, 51] the operator $\tilde{A}_\rho$ was named as *factoraggregation*. The motivation of using the name *factoraggregation* for $\tilde{A}_\rho$ is that ρ factorizes X into the classes of equivalence. Operator $\tilde{A}_\rho$ aggregates fuzzy sets $\mu_1,\ldots,\mu_n \in [0,1]^X$ in accordance with these classes of equivalence. In this construction for evaluation of $\tilde{A}_\rho(\mu_1,\ldots,\mu_n)(x)$ we take the supremum of aggregation A of values $\mu_1(x'),\ldots,\mu_n(x')$ on the set of all points x', which are equivalent to x with respect to ρ, i.e. we consider all elements $x' \in X$ such that $(x',x) \in \rho$.

2.3 Bilevel linear programming problem fuzzy solution approach

The construction of a general aggregation operator based on a crisp equivalence appeared while we were approaching bilevel linear programming problems (BLPP), which are a special type of multi-objective linear programming problems (MOLP). We observe BLPP with one objective on the upper level P^U with the higher priority in optimization than multiple objectives on the lower level $P^L = (P_1^L, P_2^L, ..., P_n^L)$:

$$\begin{aligned} P^U: \;\; & y_0(x) = c_{01}x_1 + c_{02}x_2 + ... + c_{0k}x_k \longrightarrow \min \\ P_1^L: \;\; & y_1(x) = c_{11}x_1 + c_{12}x_2 + ... + c_{1k}x_k \longrightarrow \min \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdots \\ P_n^L: \;\; & y_n(x) = c_{n1}x_1 + c_{n2}x_2 + ... + c_{nk}x_k \longrightarrow \min \end{aligned}$$

$$D: \begin{cases} a_{j1}x_1 + a_{j2}x_2 + ... + a_{jk}x_k \le b_j, \; j = \overline{1,m}, \\ x_l \ge 0, \; l = \overline{1,k}, \end{cases}$$

where $k,m,n \in \mathbb{N}$, $a_{jl}, b_j, c_{il} \in \mathbb{R}$, $j = \overline{1,m}$, $l = \overline{1,k}$, $i = \overline{0,n}$, and $x = (x_1, ..., x_k) \in \mathbb{R}^k$. We assume that $D \subset \mathbb{R}^k$ is non-empty and bounded, and we consider the case when the constraint $x \in D$ is related to all levels.

As all objectives rarely reach their optimal values in a single point, a compromise solution should be found. In multi-objective optimization Pareto optimality (see, e.g., [21]) is a concept that allows us to characterize an acceptable solution.

Definition 2.3.1. *An element $x^* \in D$ is said to be a Pareto optimal solution if and only if there does not exist another $x \in D$ such that $y_i(x) \le y_i(x^*)$ for all $i = \overline{0,n}$ and $y_j(x) \neq y_j(x^*)$ for at least one j.*

In 1978 H.J. Zimmermann [42] proposed a fuzzy solution approach for MOLP by introducing membership functions of the objectives. The membership function characterises the degree of satisfaction for each objective, i.e. it shows how the objective function is close to it's optimal value (i.e. to it's individual minimum). The construction of the membership function of objective y_i is based on the following function:

$$z_i(t) = \begin{cases} 1, & t < y_i^{min}, \\ \dfrac{t - y_i^{max}}{y_i^{min} - y_i^{max}}, & y_i^{min} \le t \le y_i^{max}, \\ 0, & t > y_i^{max}, \end{cases}$$

where y_i^{min} and y_i^{max} are the individual minimum and the individual maximum of the objective y_i respectively:

$$y_i^{min} = \min_{x \in D} y_i(x), \; y_i^{max} = \max_{x \in D} y_i(x), \;\; i = \overline{0,n}.$$

We obtain the membership functions of the objectives by denoting

$$\mu_i(x) = z_i(y_i(x)), i = \overline{0,n}.$$

Here $\mu_0, \mu_1, ..., \mu_n : D \to [0,1]$ are fuzzy subsets of D:

$$\mu_i \in [0,1]^D, i = \overline{0,n}.$$

A solution x^* for the MOLP

$$y_0(x), y_1(x), \ldots, y_n(x) \longrightarrow \min_{x \in D}$$

without any hierarchy could be found by solving the following linear programming problem:

$$\min(\mu_0(x), \mu_1(x), \ldots, \mu_n(x)) \longrightarrow \max_{x \in D}$$

or, in general:

$$A(\mu_0(x), \mu_1(x), \ldots, \mu_n(x)) \longrightarrow \max_{x \in D},$$

where A is an aggregation operator. However, in case when the objectives are divided between two levels of hierarchy, the present method does not reflect any priority of the upper level objective over the lower level.

Considering the case when there is one objective function on the upper level and multiple objectives on the lower level, we suggest a special aggregation. This aggregation observes objective functions on the lower level considering the classes of equivalence generated by the function on the upper level:

$$\tilde{A}_{\mu_0}(\mu_1, \mu_2, ..., \mu_n)(x) = \max_{\mu_0(x) = \mu_0(u)} A(\mu_1(u), \mu_2(u), ..., \mu_n(u)),$$

where

$$x, u \in D, \ \ \mu_0, \mu_1, ..., \mu_n \in [0,1]^D.$$

As one can see, general aggregation operator $\tilde{A}_{\mu_0}$ is based on equivalence relation ρ_{μ_0}:

$$u \, \rho_{\mu_0} v \Longleftrightarrow \mu_0(u) = \mu_0(v),$$

which factorizes D into the classes D^α of equivalence:

$$D^\alpha = \{x \in D \mid \mu_0(x) = \alpha\}, \ \ \alpha \in [0,1].$$

Operator $\tilde{A}_{\mu_0}$ aggregates fuzzy sets in accordance with these classes of equivalence.

2.4 General aggregation operator applied for analysis of bilevel linear programming problem solving parameters

By using membership functions $\mu_0, \mu_1, ..., \mu_n$ the multi-objective linear programming problem can be reduced to the classical linear programming (LPP):

$$\sigma \longrightarrow \max_{x,\sigma}$$

$$\begin{cases} \mu_i(x) \geq \sigma, \quad i = \overline{0,n}, \\ x \in D, \end{cases} \qquad (2.3)$$

which is equivalent to the following problem:

$$\min(\mu_0(x), \mu_1(x), \ldots, \mu_n(x)) \longrightarrow \max_{x \in D}$$

(here we use the additional real variable σ). Let us denote the solution of this LPP by (x^*, σ^*). In [21] there is described how to verify whether x^* is Pareto optimal.

For bilevel linear programming problems M. Sakawa and I. Nishizaki [32],[33] proposed the interactive method of solution by involving some parameters for the upper and lower level objectives. The algorithm specifies an optimal solution x^{**} for BLPP according to the chosen values of positive real parameters $\delta, \Delta_L, \Delta_U$, where

$$\mu_0(x^{**}) \geq \delta,$$

$$\Delta_L \leq \Delta = \frac{\min\{\mu_1(x^{**}), ..., \mu_n(x^{**})\}}{\mu_0(x^{**})} \leq \Delta_U.$$

By this method (see [32],[33]) we solve the linear programming problem

$$\sigma \longrightarrow \max_{x,\sigma}$$

$$\begin{cases} \mu_0 \geq \delta, \\ \mu_i(x) \geq \sigma, \quad i = \overline{1,n}, \\ x \in D, \end{cases}$$

for a given δ, afterwards we check whether $\Delta \in [\Delta_L, \Delta_U]$ and specify the parameters again if it is necessary.

Parameter δ describes the minimal satisfactory level for membership function μ_0, but Δ characterizes the overall balance between the upper and lower levels. Taking into account that all three parameters are dependent one on another, the problem of the choice of parameters becomes important. Let us consider the following BLPP.

Example 2.4.1.

$$P^U: \ \ y_0(x) = -x_2 \longrightarrow \min$$

$$P_1^L: \ \ y_1(x) = -3\sqrt{3}x_1 + 3x_2 \longrightarrow \min$$

$$P_2^L: \ \ y_2(x) = 3\sqrt{3}x_1 + 3x_2 \longrightarrow \min$$

$$D: \begin{cases} 3\sqrt{3}x_1 + 3x_2 \leq 18\sqrt{3}, \\ -3\sqrt{3}x_1 + 3x_2 \leq 0, \\ x_l \geq 0, \ x_2 \geq 0. \end{cases}$$

Fig. 2.1 shows how parameter δ depends on parameter Δ. On Fig.2.1 we can see, that if we choose parameter $\delta = 0.6$ and interval $[\Delta_L, \Delta_U] = [0.6, 0.7]$, then a solution of the problem doesn't exist. If we first choose interval $[\Delta_L, \Delta_U] = [0.6, 0.7]$, then the maximal possible value for δ is 0.46. But in case, when the value of δ is 0.6, then the maximal possible value of Δ is 0.33.

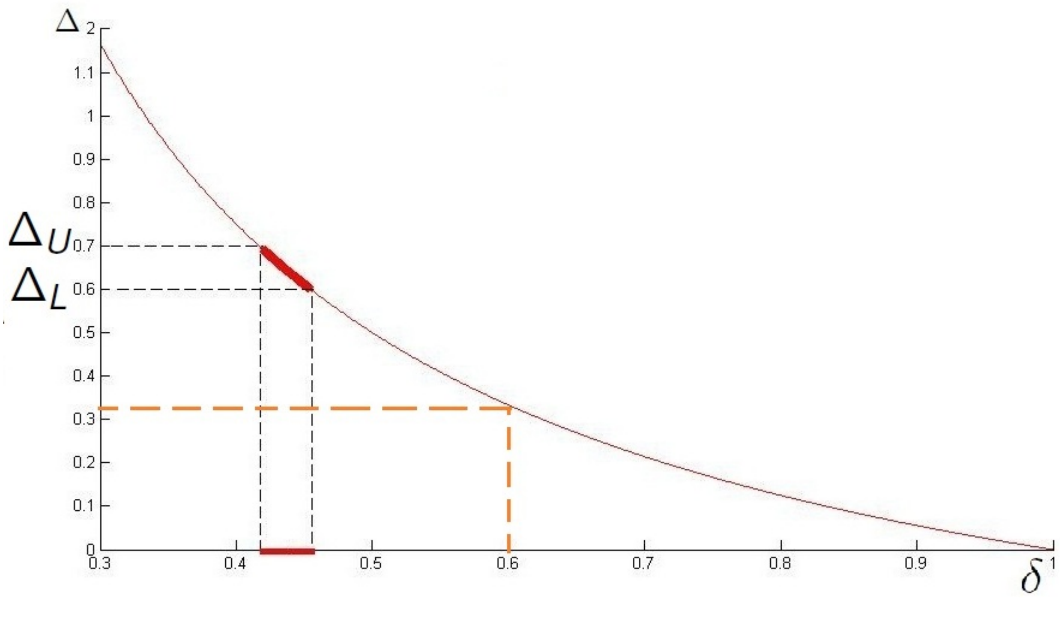

Figure 2.1: Dependence between δ and Δ for Example 2.4.1.

We specify the general construction of aggregation operator $\tilde{A}_{\mu_0}$ by taking $A = \min$ in order to apply it for the analysis of the parameters of the BLPP solving algorithm:

$$\tilde{A}_{\mu_0}(\mu_1,\mu_2,\dots,\mu_n)(x) = \max_{\mu_0(x)=\mu_0(u)} \min(\mu_1(u),\mu_2(u),\dots,\mu_n(u)), \qquad (2.4)$$

where

$$\mu_1,\dots,\mu_n \in [0,1]^D,\ x,u \in D.$$

We rewrite μ as

$$\mu(x) = \tilde{A}_{\mu_0}(\mu_1,\mu_2,\dots,\mu_n)(x) = z(y_0(x)), \qquad (2.5)$$

where z is the function defined on the interval $[y_0^{min}, y_0^{max}]$ as follows: for $t \in [y_0^{min}, y_0^{max}]$ we take $x \in D$ such, that $t = y_0(x)$, and set $z(t) = \tilde{A}_{\mu_0}(\mu_1,\mu_2,\dots,\mu_n)(x)$ (it is easy to see, that this value doesn't depend on x).

Now we consider two functions: z_0 (introduced in previous section) and z (defined by the result of aggregation). The graphical analysis of these functions (i.e. the graphical analysis of two lines $\alpha = z_0(t)$ and $\alpha = z(t)$) helps us to choose parameters Δ_L, Δ_U and δ correctly.

Let us consider the following example.

Example 2.4.2.

$$P^U:\ \ y_0(x) = x_1 - x_2 \longrightarrow \min$$

$$P_1^L:\ \ y_1(x) = -0.2x_1 - x_2 \longrightarrow \min$$

$$P_2^L:\ \ y_2(x) = x_2 \longrightarrow \min$$

$$D: \begin{cases} x_2 \le 6, \\ 5x_1 + x_2 \le 15, \\ x_l \ge 0,\ x_2 \ge 0. \end{cases}$$

The graphical analysis of the parameters could be performed by Fig. 2.2. The intersection of lines $\alpha = z_0(t)$ and $\alpha = z(t)$ on Fig.2.2 points out the optimal solution x^* of the corresponding MOLP problem without any hierarchy between objectives: $t^* = y_0(x^*)$. In our case we are dealing with BLPP, when objective function y_0 is minimized with the higher priority than objectives y_1 and y_2. The compromised solution x^* gives us the degree of satisfaction of the upper level objective $\delta = 0.51$. But the analysis of Fig. 2.2 allows us to see, that a minor decrease by 0.0224 in the degree of minimization on the

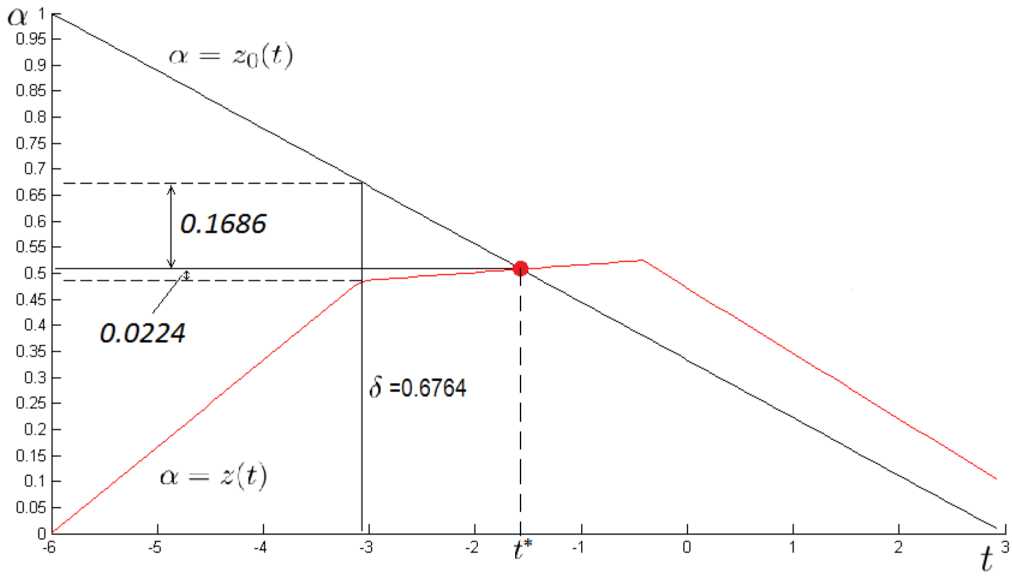

Figure 2.2: Analysis of solving parameters for Example 2.4.2.

lower level, which is characterized by the result of general aggregation $z(t)$, will give us a significant increase by 0.1686 in the degree of minimization δ on the upper level. It means, that we would rather choose point $t^{**} = -3.05$ to obtain the optimal solution x^{**} for this BLPP, than point t^*, which gives us the solution without priority for the upper level objective. The similar graphical analysis could be performed, when we first choose the values of parameters Δ_L and Δ_U, which characterize the degree of minimization on the lower level, and then we can find out the possible values of the degree of minimization δ for the upper level objective.

We consider some properties of function z, which is defined above and describes the output value of general aggregation of the lower level objectives. Function z_0 used for representation of the upper level objective membership function is decreasing on interval $[y_0^{min}, y_0^{max}]$. Now we consider properties of function z on interval $[y_0^{min}, y_0^{max}]$.

Theorem 2.4.1. *Let $t^* = y_0(x^*)$, where (x^*, σ^*) is the solution of the MOLP*

$$y_0(x), y_1(x), \ldots, y_n(x) \longrightarrow \min_{x \in D}$$

without any hierarchy between objectives reduced to the form (2.3). Then the following holds true:

1. $\sigma^* = \min\{z(t^*), z_0(t^*)\}$;

2. $\max\limits_{t \in [y_0^{min}, t^*]} z(t) = z(t^*)$;

3. *function z is concave (convex upwards) on interval $[y_0^{min}, y_0^{max}]$,*

where z is the function defined by (2.4) – (2.5), $y_0^{min} = \min_{x \in D} y_0(x)$, $y_0^{max} = \max_{x \in D} y_0(x)$.

Proof. 1. We recall that

$$\sigma^* = \min\{\mu_0(x^*), ..., \mu_n(x^*)\}$$

and consider two cases.
1) If $\sigma^* = \mu_0(x^*)$, then on the one hand

$$\sigma^* = z_0(t^*),$$

but on the other hand

$$z_0(t^*) = \mu_0(x^*) \leq \min\{\mu_1(x^*), ..., \mu_n(x^*)\} \leq \max_{\mu_0(x^*) = \mu_0(u)} \min\{\mu_1(u), ..., \mu_n(u)\} =$$

$$= \mu(x^*) = z(t^*),$$

which means

$$z_0(t^*) = \min\{z_0(t^*),\ z(t^*)\}.$$

2) Now let us suppose

$$\sigma^* = \min\{\mu_1(x^*), ..., \mu_n(x^*)\} < \mu_0(x^*)$$

and show that

$$\sigma^* = \mu(x^*) = z(t^*) < z_0(t^*).$$

Considering

$$\sigma^* < \mu(x^*) = \max_{\mu_0(x^*) = \mu_0(u)} \min\{\mu_1(u), ..., \mu_n(u)\}$$

and taking into account that $\sigma^* < \mu_0(x^*)$, we got the contradiction since σ^* is the solution of MOLP.

2. Let us suppose that there exists

$$\tau \in [y_0^{min},\ t^*[\text{ such that } z(\tau) > z(t^*) \geq \sigma^*.$$

Then we can find $u^* \in D$ such that

$$z(\tau) = \max_{z_0(\tau) = \mu_0(u)} \min\{\mu_1(u), ..., \mu_n(u)\} = \min\{\mu_1(u^*), ..., \mu_n(u^*)\} > \sigma^*$$

and $\mu_0(u^*) = z_0(\tau) > \sigma^*$. As a result we got the contradiction since σ^* is the solution of MOLP.

3. We have to prove that

$$z(\lambda t^1 + (1-\lambda)t^2) \geq \lambda z(t^1) + (1-\lambda)z(t^2)$$

for all $t^1, t^2 \in [y_0^{min}, y_0^{max}]$ and for all $\lambda \in [0,1]$. Let us take such x^1 and x^2 that

$$t^1 = y_0(x^1),\ t^2 = y_0(x^2),$$

$$z(t^1) = \min\{\mu_1(x^1), ..., \mu_n(x^1)\}$$

and

$$z(t^2) = \min\{\mu_1(x^2), ..., \mu_n(x^2)\}.$$

Then

$$\mu_i(\lambda x^1 + (1-\lambda)x^2) = \lambda\mu_i(x^1) + (1-\lambda)\mu_i(x^2) \geq$$

$$\geq \lambda \min\{\mu_1(x^1), ..., \mu_n(x^1)\} + (1-\lambda)\min\{\mu_1(x^2), ..., \mu_n(x^2)\} = \lambda z(t^1) + (1-\lambda)z(t^2)$$

for all $i = 1, 2, ..., n$. Therefore

$$z(\lambda t^1 + (1-\lambda)t^2) = \max_{\mu_0(u) = z_0(\lambda t^1 + (1-\lambda)t^2)} \min\{\mu_1(u), ..., \mu_n(u)\} \geq$$

$$\geq \min\{\mu_1(\lambda x^1 + (1-\lambda)x^2), ..., \mu_n(\lambda x^1 + (1-\lambda)x^2)\} \geq \lambda z(t^1) + (1-\lambda)z(t^2).$$

□

Let us illustrate these properties with two examples.

Example 2.4.3.

$$P^U: \ y_0(x) = -3x_1 - x_2 \longrightarrow \min$$

$$P_1^L: \ y_1(x) = 3x_1 - 2x_2 \longrightarrow \min$$

$$P_2^L: \ y_2(x) = x_1 - x_2 \longrightarrow \min$$

$$D: \begin{cases} -x_1 + 3x_2 \leq 21, \\ x_1 + 3x_2 \leq 27, \\ 4x_1 + 3x_2 \leq 45, \\ 3x_1 + x_2 \leq 30, \\ x_l \geq 0,\ x_2 \geq 0. \end{cases}$$

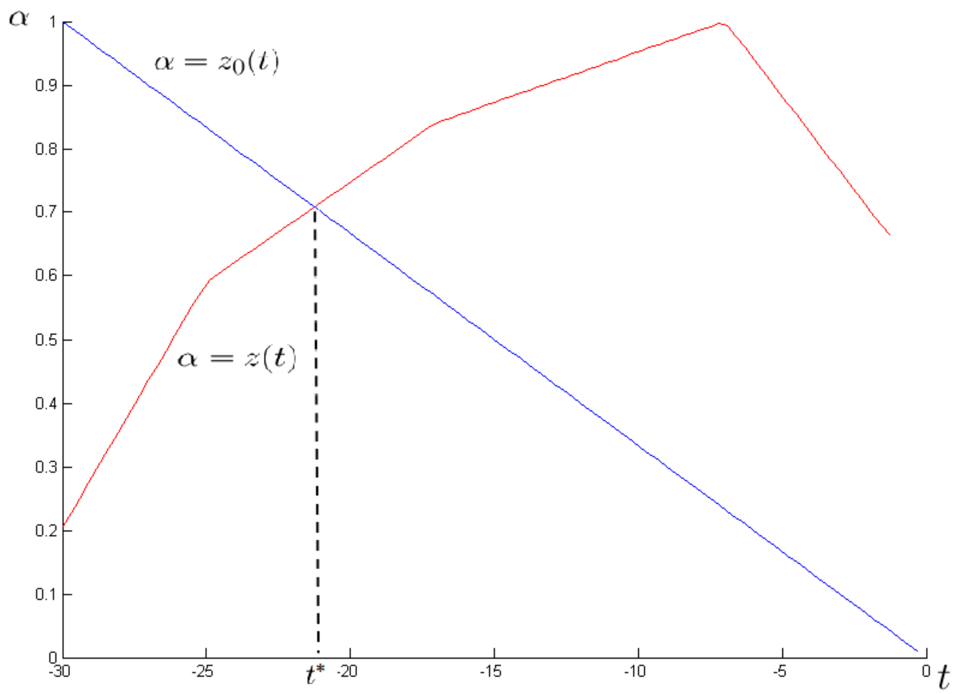

Figure 2.3: Lines $\alpha = z_0(t)$ and $\alpha = z(t)$ for Example 2.4.3.

Line $\alpha = z(t)$ on Fig.2.3 is concave and monotony increasing till the intersection point t^*, where the compromise solution could be found.

Example 2.4.4.

$$P^U: \ \ y_0(x) = x_1 - x_2 \longrightarrow \min$$

$$P_1^L: \ \ y_1(x) = 3x_1 - 2x_2 \longrightarrow \min$$

$$P_2^L: \ \ y_2(x) = -3x_1 - x_2 \longrightarrow \min$$

$$D: \begin{cases} -x_1 + 3x_2 \leq 21, \\ x_1 + 3x_2 \leq 27, \\ 4x_1 + 3x_2 \leq 45, \\ 3x_1 + x_2 \leq 30, \\ x_l \geq 0, \ \ x_2 \geq 0. \end{cases}$$

Line $\alpha = z(t)$ on Fig.2.4 is concave too, but the compromise solution now could be found in point t^*, which is not the intersection of lines $\alpha = z(t)$ and $\alpha = z_0(t)$. And line $\alpha = z(t)$ is monotony increasing till this point.

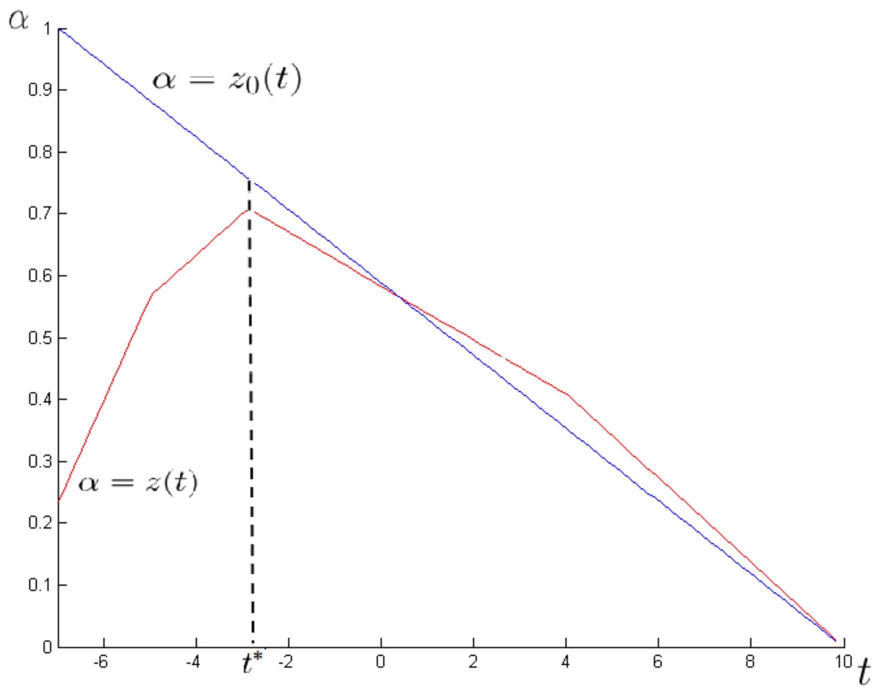

Figure 2.4: Lines $\alpha = z_0(t)$ and $\alpha = z(t)$ for Example 2.4.4.

2.5 Mixed production planning problem

We consider the following modification of the mixed production planning problem described by J.C. Figueroa-Garcia et al. in [12]. The goal of the mixed production planning problem is to determine the most profitable manufacturing plan at the same time minimizing environmentally dangerous products and the dependence on the outsource companies:

$$P^U : \sum_{j\in\mathbb{N}_J} \sum_{i\in\mathbb{N}_I} sp_{ji}(x^r_{ji} + x^o_{ji} + x^s_{ji}) - (cp^r_{ji}x^r_{ji} + cp^o_{ji}x^o_{ji} + cp^s_{ji}x^s_{ji}) \longrightarrow \max$$

$$P^L_1 : \sum_{j\in\mathbb{N}_J} \sum_{i\in\mathbb{N}_I} ec_{ji}(x^r_{ji} + x^o_{ji} + x^s_{ji}) \longrightarrow \min$$

$$P^L_2 : \sum_{j\in\mathbb{N}_J} \sum_{i\in\mathbb{N}_I} w_{ji}x^s_{ji} \longrightarrow \min$$

$$\sum_{j\in\mathbb{N}_J} rm_{jir}(x^r_{ji} + x^o_{ji}) \le am_{ir}, \quad i \in \mathbb{N}_I, \;\; r \in \mathbb{N}_R,$$

$$x^r_{ji}, x^o_{ji}, x^s_{ji} \ge 0, \;\; x^s_{ji} \le as_{ji}, \quad j \in \mathbb{N}_J, \;\; i \in \mathbb{N}_I,$$

$$d_{ji}^{(-)} \leq x_{ji}^{r} + x_{ji}^{o} + x_{ji}^{s} \leq d_{ji}^{(+)}, \quad j \in \mathbb{N}_J, \; i \in \mathbb{N}_I.$$

Index sets:
set $\mathbb{N}_R = \{1, 2, ..., R\}$ of all resources $r \in \mathbb{N}_R$,
set $\mathbb{N}_J = \{1, 2, ..., J\}$ of all products $j \in \mathbb{N}_J$,
set $\mathbb{N}_I = \{1, 2, ..., I\}$ of all periods $i \in \mathbb{N}_I$.
Decision variables:
x_{ji}^{r} – quantity of product j to be manufactured in regular time in period i,
x_{ji}^{o} – quantity of product j to be manufactured in overtime in period i,
x_{ji}^{s} – quantity of product j to be manufactured by outsourcing in period i.
Parameters:
$SP = (sp_{ji} \mid j \in \mathbb{N}_J, i \in \mathbb{N}_I)$, where sp_{ji} is a sell price of product j in period i,
$CP^r = (cp_{ji}^{r} \mid j \in \mathbb{N}_J, i \in \mathbb{N}_I)$, where cp_{ji}^{r} is a product j production cost in the period i for regular time,
$CP^o = (cp_{ji}^{o} \mid j \in \mathbb{N}_J, i \in \mathbb{N}_I)$, where cp_{ji}^{o} is a product j production cost in the period i for overtime,
$CP^s = (cp_{ji}^{s} \mid j \in \mathbb{N}_J, i \in \mathbb{N}_I)$, where cp_{ji}^{s} is a product j production cost in the period i for outsourcing,
$EC = (ec_{ji} \mid j \in \mathbb{N}_J, i \in \mathbb{N}_I)$, where ec_{ji} is an evaluation of damage to environment caused by product j in period i,
$W = (w_{ji} \mid j \in \mathbb{N}_J, i \in \mathbb{N}_I)$, where w_{ji} is a weight of outsource product j in period i,
$RM = (rm_{jir} \mid j \in \mathbb{N}_J, i \in \mathbb{N}_I, r \in \mathbb{N}_R)$, where rm_{jir} is an amount of the r raw material units used to manufacture product j in period i,
$AM = (am_{ir} \mid r \in \mathbb{N}_R, i \in \mathbb{N}_I)$, where am_{ir} is an availability of the raw material type r in period i,
$AS = (as_{ji} \mid j \in \mathbb{N}_J, i \in \mathbb{N}_I)$, where as_{ji} is a number of available outsourced units of product j in period i,
$D^- = (d_{ji}^{(-)} \mid j \in \mathbb{N}_J, i \in \mathbb{N}_I)$, where $d_{ji}^{(-)}$ is a minimum demand of product j in period i,
$D^+ = (d_{ji}^{(+)} \mid j \in \mathbb{N}_J, i \in \mathbb{N}_I)$, where $d_{ji}^{(+)}$ is a maximum (potential) demand of product j in period i.

Maximization problem P^U is reduced to the minimization problem by taking the profit function with the minus sign. This numerical example uses the following values of the parameters:
$I = 1, J = 10, R = 5$,
$AM = (9 \cdot 10^6, 4 \cdot 10^6, 4.5 \cdot 10^6, 3 \cdot 10^6, 5.5 \cdot 10^6)$,

$EC = (8, 9, 5, 4, 3, 7, 2, 1, 6, 10)$.
Table 2.1 contains the values of RM. The values of other parameters are given by Table 2.2.

$j \backslash r$	1	2	3	4	5
1	50.47	83.37	90.29	133.27	71.75
2	53.46	79.93	84.88	133.87	55.69
3	106.49	75.30	101.81	113.06	96.03
4	125.26	103.13	94.35	59.82	134.97
5	93.96	120.50	100.36	134.71	78.87
6	137.24	87.68	40.55	110.17	93.26
7	136.14	112.83	67.93	96.40	77.01
8	72.47	53.75	124.05	110.74	99.43
9	56.53	42.53	44.42	66.05	97.82
10	98.72	109.48	56.77	103.07	95.72

Table 2.1: Values of parameters rm_{jr} for the mixed production planning problem

j	cp_j^r	cp_j^s	sp_j	as_j	$d_j^{(+)}$	$d_j^{(-)}$	w_j
1	255	260	350	1237	8775	3900	0.07
2	165	200	300	1107	7650	3400	0.15
3	160	185	280	1519	6075	2700	0.07
4	105	130	210	2636	7875	1500	0.05
5	205	240	300	1979	6300	3400	0.05
6	175	190	305	1617	7650	4000	0.08
7	160	210	270	1442	5000	1500	0.05
8	225	245	315	1527	6300	4000	0.05
9	53	105	190	2266	4725	1000	0.18
10	74	120	220	2500	8775	2600	0.25

Table 2.2: Values of parameters for the mixed production planning problem

The graphical analysis of lines $\alpha = z_0(t)$ and $\alpha = z(t)$ is given by Fig. 2.5. The intersection of lines $\alpha = z_0(t)$ and $\alpha = z(t)$ points out the optimal solution x^* of the corresponding MOLP problem without any hierarchy between objectives: $t^* = y_0(x^*)$. By

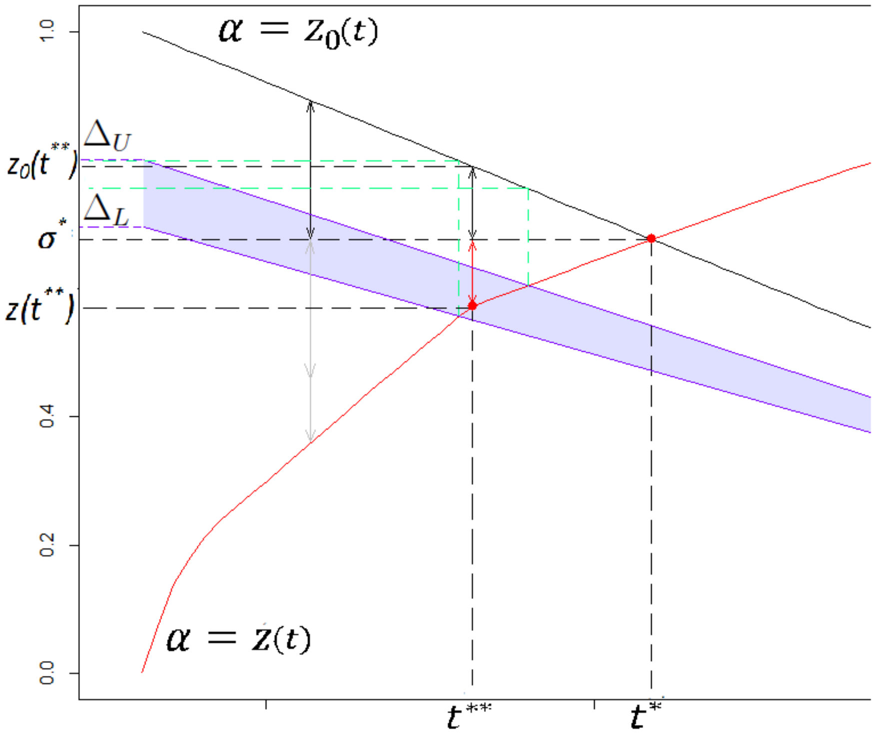

Figure 2.5: Analysis of solving parameters for the mixed production planning problem

setting $\Delta_L = 0.7$ and $\Delta_U = 0.8$ we can observe that δ should lie in interval $[0.765, 0.799]$. Otherwise if $\delta > 0.799$ then a solution does not exist. The graphical analysis shows that as the optimal solution of the mixed production planning problem it is natural to take x^{**} such that $t^{**} = y_0(x^{**})$.

Chapter 3

Upper and lower general aggregation operators based on a fuzzy equivalence relation

In the previous chapter we defined and described the construction of a general aggregation operator which was based on a crisp equivalence relation. In the current chapter we generalize this construction by using a fuzzy equivalence relation instead of a crisp one. The need for such operators may appear dealing with different problems. For example, in decision making if fuzzy sets represent evaluation of some objects provided by several experts, and at the same time we have a fuzzy equivalence relation between these objects. In order to obtain the evaluation of some object it is important to take into account how the experts evaluated equivalent objects. So if we want to aggregate several experts evaluations for this object taking into account the equivalence relation, it could be performed by the proposed operators. Using the idea of upper and lower approximation operators we describe two types of upper and lower general aggregation operators based on fuzzy equivalence relation. Operators of the first type are considered as upper and lower approximations of the pointwise extension of an ordinary aggregation operator. In other case, these operators serve as upper and lower approximations of the t-norm extension of an ordinary aggregation operator. We consider aggregational properties of upper and lower general aggregation operators, as well as investigate the connection between them and extensional fuzzy sets. The results presented in this chapter could be found in [46, 47, 48].

3.1 Fuzzy equivalence relation

First, let us describe a particular real-world example where one could use a general aggregation operator based on a fuzzy equivalence relation. For example, we could consider an investment firm or a bank, which investigates investment opportunities in different countries. One of the key components of risk management in such institutions is a country risk evaluation. Management of the institution approves maximal limits for risk exposures to be taken in different countries. The evaluation of these risk limits is usually performed by risk analysts (experts) taking into account economical, financial, political and social background of the particular countries. It is usually important to know, which limit was assigned previously to some similar in many aspects country (or equivalent with some degree). For example, considering Asian region, Japan and North Korea are equivalent with very low degree (close to 0), while Japan and South Korea could be considered equivalent with high degree by many factors. Therefore, to obtain the evaluation of some country for management approval, a risk manager could obtain the aggregated result, taking into account the evaluation for equivalent with high degree countries, by using upper and lower general aggregation operators.

Let us recall the definition of a fuzzy equivalence relation. Fuzzy equivalence relations were introduced in 1971 by L.A. Zadeh [41] for the strongest t-norm T_M and later were developed and applied by several authors in more general cases.

Definition 3.1.1. *Let T be a t-norm and E be a fuzzy relation on a set X, i.e. E is a fuzzy subset of $X \times X$. A fuzzy relation E is called a T-fuzzy equivalence relation if and only if for all $x,y,z \in X$ it holds*
$(E1)$ $E(x,x) = 1$ *(reflexivity);*
$(E2)$ $E(x,y) = E(y,x)$ *(symmetry);*
$(E3)$ $T(E(x,y),E(y,z)) \leq E(x,z)$ *(T-transitivity).*

Dealing with fuzzy equivalence relations usually extensional fuzzy sets attract an additional attention. These sets correspond to the fuzzification of classical classes of equivalence, they play the role of fuzzy equivalence classes altogether with their intersections and unions.

Definition 3.1.2. *Let T be a t-norm and E be a T-fuzzy equivalence relation on a set X. A fuzzy subset $\mu \in [0,1]^X$ is called extensional with respect to E if*

$$T(E(x,y),\mu(y)) \leq \mu(x) \text{ for all } x,y \in X.$$

Extensional fuzzy subsets have been widely studied in the literature [5, 19, 20, 27].

3.2 Constructions of upper and lower general aggregation operators

The constructions of upper and lower general aggregation operators are provided in the following definition:

Definition 3.2.1. *Let $A\colon [0,1]^n \to [0,1]$ be an aggregation operator, T be a left continuous t-norm, $\overrightarrow{T}$ be the residuum of T and E be a T-fuzzy equivalence relation defined on a set X. The upper and lower general aggregation operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ are defined respectively by*

$$\tilde{A}_{E,T}(\mu_1,\ldots,\mu_n)(x) = \sup_{x'\in X} T(E(x,x'),A(\mu_1(x'),\ldots,\mu_n(x'))), \tag{3.1}$$

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n)(x) = \inf_{x'\in X} \overrightarrow{T}(E(x,x')|A(\mu_1(x'),\ldots,\mu_n(x'))), \tag{3.2}$$

where $x \in X$ and $\mu_1,\ldots,\mu_n \in [0,1]^X$.

Further, in Proposition 3.2.1 we will prove that operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ actually are general aggregation operators.

Now we will demonstrate how such constructions could be used in applications. We consider the following problem. Let X be the set of all countries in the world or in some particular region. Let $\mu_i(x)$, where $\mu_i\colon X \to [0,1]$, be normalized evaluation of country's $x \in X$ risk level by the i-th expert (country is considered as more risky if the evaluation is closer to 1). As ordinary aggregation operator A one could take the arithmetic mean or the weighted arithmetic mean aggregation operators. It is important to define appropriate fuzzy equivalence relation $E\colon X \times X \to [0,1]$ between the objects of X, which could be a complex problem itself.

We want to obtain an assessment of the risk level of some country by taking arithmetic mean of the experts evaluations of other countries taking into account fuzzy equivalence relation between these countries and to compare it with ordinary arithmetic mean operator.

Let us consider the discrete universe which consists of 8 countries from some region:

$$X = \{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\}$$

and the following T_M-fuzzy (T_M is the minimum t-norm) equivalence relation E between these countries, given in a matrix form:

$$E = \begin{pmatrix} 1 & 0.9 & 0.8 & 0.8 & 0.8 & 0.7 & 0.1 & 0.1 \\ 0.9 & 1 & 0.8 & 0.8 & 0.8 & 0.7 & 0.1 & 0.1 \\ 0.8 & 0.8 & 1 & 0.8 & 0.8 & 0.7 & 0.1 & 0.1 \\ 0.8 & 0.8 & 0.8 & 1 & 0.8 & 0.7 & 0.1 & 0.1 \\ 0.8 & 0.8 & 0.8 & 0.8 & 1 & 0.7 & 0.1 & 0.1 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 1 & 0.9 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.9 & 1 \end{pmatrix}.$$

Suppose, we have evaluations of the risk level for each country given by 5 experts, expressed in the form of fuzzy sets $\mu_i : X \to [0,1]$, $i = 1, \ldots, 5$:

$$\mu_1 = \begin{pmatrix} 1 \\ 0.9 \\ 0.8 \\ 0.6 \\ 1 \\ 0.6 \\ 0 \\ 0.9 \end{pmatrix}, \ \mu_2 = \begin{pmatrix} 0.8 \\ 0.8 \\ 0.7 \\ 0.8 \\ 0.4 \\ 0.1 \\ 0.1 \\ 0.8 \end{pmatrix}, \ \mu_3 = \begin{pmatrix} 1 \\ 0.6 \\ 0.7 \\ 0.6 \\ 0.6 \\ 0.5 \\ 0.1 \\ 0.9 \end{pmatrix}, \ \mu_4 = \begin{pmatrix} 0.9 \\ 1 \\ 0.9 \\ 0.8 \\ 0.6 \\ 0.7 \\ 0 \\ 0.8 \end{pmatrix}, \ \mu_5 = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.7 \\ 0.6 \\ 0.6 \\ 0.6 \\ 0.1 \\ 0.7 \end{pmatrix}.$$

First, we calculate the arithmetic mean, thus obtaining the aggregated evaluation:

$$AVG(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)(x) = AVG(\mu_1(x), \mu_2(x), \mu_3(x), \mu_4(x), \mu_5(x)).$$

Then we apply the upper general aggregation operator in order to obtain the upper approximation of AVG taking into account equivalence relation E. This result will give us the most conservative assessment of risk levels for each country, which is the goal of a risk manager. Let us take the strongest t-norm $T = T_M$:

$$\tilde{A}_{E,T_M}(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)(x) = \max_{x' \in X} T_M(E(x,x'), AVG(\mu_1(x'), \mu_2(x'), \mu_3(x'), \mu_4(x'), \mu_5(x'))).$$

As a result we obtain the following fuzzy sets in a vector form:

$$AVG = \begin{pmatrix} 0.88 \\ 0.76 \\ 0.76 \\ 0.68 \\ 0.64 \\ 0.50 \\ 0.06 \\ 0.82 \end{pmatrix}, \ \tilde{A}_{E,T_M} = \begin{pmatrix} 0.88 \\ 0.88 \\ 0.80 \\ 0.80 \\ 0.80 \\ 0.70 \\ 0.82 \\ 0.82 \end{pmatrix}.$$

As one can see, the risk level assessments for most of the countries is more conservative. For example, the average risk level for country x_7 is 0.06, while the upper approximation taking into account the equivalence relation give us more conservative result 0.82, because this country is equivalent with high degree to the much more risky country x_8. Depending on the problem specifics and the construction of T-fuzzy equivalence relation E one could choose different t-norms T and aggregation operators A, which will influence the result.

Now let us show that constructions (3.1) and (3.2) give us general aggregation operators. We must show that conditions $(\tilde{A}1)$, $(\tilde{A}2)$ and $(\tilde{A}3)$ are satisfied.

Proposition 3.2.1. *Operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ (see (3.1) and (3.2)) are general aggregation operators.*

Proof. First we consider the boundary conditions:
$(\tilde{A}1)$:

$$\tilde{A}_{E,T}(\tilde{0},\dots,\tilde{0})(x) = \sup_{x'\in X} T(E(x,x'),A(\tilde{0}(x'),\dots,\tilde{0}(x'))) =$$
$$= \sup_{x\in X} T(E(x,x'),A(0,\dots,0)) = \sup_{x'\in X} T(E(x,x'),0) = \tilde{0}(x);$$
$$\tilde{A}_{E,\overrightarrow{T}}(\tilde{0},\dots,\tilde{0})(x) = \inf_{x'\in X} \overrightarrow{T}(E(x,x')|A(\tilde{0}(x'),\dots,\tilde{0}(x'))) =$$
$$= \inf_{x'\in X} \overrightarrow{T}(E(x,x')|A(0,\dots,0)) = \inf_{x'\in X} \overrightarrow{T}(E(x,x')|0) = \tilde{0}(x);$$

$(\tilde{A}2)$:

$$\tilde{A}_{E,T}(\tilde{1},\dots,\tilde{1})(x) = \sup_{x'\in X} T(E(x,x'),A(\tilde{1}(x'),\dots,\tilde{1}(x'))) =$$
$$= \sup_{x'\in X} T(E(x,x'),A(1,\dots,1)) = \sup_{x'\in X} T(E(x,x'),1) = \sup_{x'\in X} E(x,x') = \tilde{1}(x);$$

$$\tilde{A}_{E,\overrightarrow{T}}(\tilde{1},\ldots,\tilde{1})(x)=\inf_{x'\in X}\overrightarrow{T}(E(x,x')|A(\tilde{1}(x'),\ldots,\tilde{1}(x')))=$$
$$=\inf_{x'\in X}\overrightarrow{T}(E(x,x')|A(1,\ldots,1))=\inf_{x'\in X}\overrightarrow{T}(E(x,x')|1)=\tilde{1}(x).$$

$(\tilde{A}3)$: To prove the monotonicity of $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ we use the monotonicity of A and T, and the basic properties of $\overrightarrow{T}$:

$$\mu_i\preceq\eta_i,\ i=1,...,n\Longrightarrow A(\mu_1(x'),...,\mu_n(x'))\le A(\eta_1(x'),...,\eta_n(x'))\Longrightarrow$$
$$\Longrightarrow\ T(E(x,x'),A(\mu_1(x'),...,\mu_n(x')))\le T(E(x,x'),A(\eta_1(x'),...,\eta_n(x')))\Longrightarrow$$
$$\Longrightarrow\ \sup_{x'\in X}T(E(x,x'),A(\mu_1(x'),\ldots,\mu_n(x')))\le$$
$$\le\sup_{x'\in X}T(E(x,x'),A(\eta_1(x'),\ldots,\eta_n(x')))\Longrightarrow$$
$$\Longrightarrow\ \tilde{A}_{E,T}(\mu_1,\ldots,\mu_n)\preceq\tilde{A}_{E,T}(\eta_1,\ldots,\eta_n);$$
$$\mu_i\preceq\eta_i,\ \ i=1,...,n\Longrightarrow A(\mu_1(x'),...,\mu_n(x'))\le A(\eta_1(x'),...,\eta_n(x'))\Longrightarrow$$
$$\Longrightarrow\overrightarrow{T}(E(x,x')|A(\mu_1(x'),...,\mu_n(x')))\le\overrightarrow{T}(E(x,x')|A(\eta_1(x'),...,\eta_n(x')))\Longrightarrow$$
$$\Longrightarrow\inf_{x'\in X}\overrightarrow{T}(E(x,x')|A(\mu_1(x'),\ldots,\mu_n(x')))\le$$
$$\le\inf_{x'\in X}\overrightarrow{T}(E(x,x')|A(\eta_1(x'),\ldots,\eta_n(x')))\Longrightarrow$$
$$\Longrightarrow\ \tilde{A}_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n)\preceq\tilde{A}_{E,\overrightarrow{T}}(\eta_1,\ldots,\eta_n).$$

□

We could consider (3.1) and (3.2) as upper and lower approximations of a general aggregation operator $\tilde{A}$, which is the pointwise extension of an ordinary aggregation operator A. It is clear, that for all $\mu_1,\ldots,\mu_n\in[0,1]^X$ it holds

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n)\le\tilde{A}(\mu_1,\ldots,\mu_n)\le\tilde{A}_{E,T}(\mu_1,\ldots,\mu_n).$$

Indeed, for all $x\in X$

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n)(x)=\inf_{x'\in X}\overrightarrow{T}(E(x,x')|A(\mu_1(x'),\ldots,\mu_n(x')))\le$$
$$\le\overrightarrow{T}(E(x,x)|A(\mu_1(x'),\ldots,\mu_n(x')))=A(\mu_1(x),\ldots,\mu_n(x))=$$
$$=T(E(x,x),A(\mu_1(x),\ldots,\mu_n(x)))\le\sup_{x'\in X}T(E(x,x'),A(\mu_1(x'),\ldots,\mu_n(x')))=$$
$$=\tilde{A}_{E,T}(\mu_1,\ldots,\mu_n)(x).$$

Let ρ be a crisp equivalence relation defined on a set X. We take $E = E_\rho$, where

$$E_\rho(x,y) = \begin{cases} 1, & (x,y) \in \rho, \\ 0, & (x,y) \notin \rho, \end{cases}$$

and for an ordinary aggregation operator A and for any t-norm T obtain $\tilde{A}_{E_\rho,T}$ and $\tilde{A}_{E_\rho,\overrightarrow{T}}$, which does not depend on T:

$$\tilde{A}_{E_\rho,T}(\mu_1,\dots,\mu_n)(x) = \sup_{x' \in X:(x',x)\in\rho} A(\mu_1(x'),\dots,\mu_n(x')),$$

$$\tilde{A}_{E_\rho,\overrightarrow{T}}(\mu_1,\dots,\mu_n)(x) = \inf_{x' \in X:(x',x)\in\rho} A(\mu_1(x'),\dots,\mu_n(x')).$$

It is easy to see that $\tilde{A}_{E_\rho,T} = \tilde{A}_\rho$. In the case when crisp equivalence relation is defined in the following way:

$$\rho_0(x,y) = \begin{cases} 1, & x = y, \\ 0, & \text{otherwise}, \end{cases}$$

we have, that both upper and lower general aggregation operators based on E_{ρ_0} turn into the pointwise extension of ordinary aggregation operator.

Now let us consider the following construction in compare with (3.1):

$$\tilde{A}_{E,B}(\mu_1,\dots,\mu_n)(x) = \sup_{x' \in X} B(E(x,x'),A(\mu_1(x'),\dots,\mu_n(x'))),\ x \in X,\ \mu_1,\dots,\mu_n \in [0,1]^X,$$

where A and B are aggregation operators, E is a T-fuzzy equivalence relation.

Dealing with such construction, it is natural to state the following problem: what properties aggregation operator B should fulfil in order to operator $\tilde{A}_{E,B}$ represent an upper general aggregation operator (3.1). It is clear that if $B = T$, then $\tilde{A}_{E,B}$ is the upper general aggregation operator. But if the symmetry of a t-norm is omitted, we obtain the necessary result if the following properties of B hold:

$$B(\alpha,0) = 0 \quad \text{for all } \alpha \in [0,1].$$

Similar considerations could be obtained by studying operator $\tilde{A}_{E,C}$ in compare with (3.2):

$$\tilde{A}_{E,C}(\mu_1,\dots,\mu_n)(x) = \inf_{x' \in X} C(E(x,x'),A(\mu_1(x'),\dots,\mu_n(x'))),\ x \in X,\ \mu_1,\dots,\mu_n \in [0,1]^X,$$

where C is some aggregation operator. In this case it is enough to demand that the following properties are fulfilled:

1) $C(1,0) = 0$;

2) $C(\alpha,1) = 1$ for all $\alpha \in [0,1]$.

While $\overrightarrow{T}$ is not an aggregation operator, it is also possible to weaken the monotonicity condition for C and to demand only the monotonicity of C with respect to the second argument. With the conditions described above operators $\tilde{A}_{E,B}$ and $\tilde{A}_{E,C}$ provide upper and lower approximations of a general aggregation operator $\tilde{A}$.

Now we illustrate the upper and lower general aggregation operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ with some particular examples.

Example 3.2.1. *Numerical inputs here are taken from [27]. Let us consider the discrete universe*

$$X = \{x_1, x_2, x_3, x_4, x_5\}$$

and the following T_M-fuzzy (T_M is the minimum t-norm) equivalence relation E, given in a matrix form:

$$E = \begin{pmatrix} 1 & 0.9 & 0.7 & 0.4 & 0.2 \\ 0.9 & 1 & 0.7 & 0.4 & 0.2 \\ 0.7 & 0.7 & 1 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.4 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 1 \end{pmatrix}.$$

Of course, this equivalence relation is also T_L-transitive and T_P-transitive, i.e. transitive with respect to the Lukasiewicz t-norm T_L and the product t-norm T_P respectively. Relation E has a noteworthy feature: elements x_4 and x_5 are equivalent to other elements with relatively lower degree, than elements x_1, x_2 and x_3. We illustrate that this fact has significant impact on the result of general aggregations.

Let us take the following fuzzy subsets of X:

$$\mu_1 = \begin{pmatrix} 0.9 \\ 0.5 \\ 0.6 \\ 0.8 \\ 0.3 \end{pmatrix}, \mu_2 = \begin{pmatrix} 0.2 \\ 0.0 \\ 0.2 \\ 0.6 \\ 0.9 \end{pmatrix}, \mu_3 = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.1 \\ 0.8 \\ 0.6 \end{pmatrix}, \mu_4 = \begin{pmatrix} 0.1 \\ 0.9 \\ 0.2 \\ 0.8 \\ 0.5 \end{pmatrix}.$$

Now we consider the minimum aggregation operator $A = MIN$ and obtain the following upper general aggregation operator:

$$\tilde{A}_{E,T}(\mu_1,\mu_2,\mu_3,\mu_4)(x) = \max_{x' \in X} T(E(x,x'), \min(\mu_1(x'),\mu_2(x'),\mu_3(x'),\mu_4(x'))).$$

Taking $T = T_L$, $T = T_M$ and $T = T_P$ we obtain as results fuzzy subsets μ_{T_L}, μ_{T_M} and μ_{T_P} respectively:

$$\mu_{T_L} = \begin{pmatrix} 0.1 \\ 0.0 \\ 0.1 \\ 0.6 \\ 0.3 \end{pmatrix}, \mu_{T_M} = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \\ 0.6 \\ 0.3 \end{pmatrix}, \mu_{T_P} = \begin{pmatrix} 0.24 \\ 0.24 \\ 0.24 \\ 0.6 \\ 0.3 \end{pmatrix}.$$

First, let us note, that result μ_T at points x_4 and x_5 does not depend on the choice of the t-norm. It could be explained with low degrees of equivalence for these points with respect to other elements. Therefore the result of $\tilde{A}_{E,T}$ is not effected by these other elements. Second, the values of μ_T at points x_1, x_2 and x_3 depend on each other, since the degree of equivalence between any two of these points is relatively high, and at the same time the results of ordinary aggregation of $\mu_1(x)$, $\mu_2(x)$, $\mu_3(x)$ and $\mu_4(x)$ at these points are relatively small.

Taking the arithmetic mean aggregation operator $A = AVG$ as an ordinary aggregation operator, we obtain the following upper general aggregation operator:

$$\tilde{A}_{E,T}(\mu_1,\mu_2,\mu_3,\mu_4)(x) = \max_{x' \in X} T(E(x,x'), AVG(\mu_1(x'),\mu_2(x'),\mu_3(x'),\mu_4(x'))).$$

Taking $T = T_L$, $T = T_M$ and $T = T_P$ we obtain as results the following fuzzy subsets:

$$\mu_{T_L} = \begin{pmatrix} 0.475 \\ 0.475 \\ 0.275 \\ 0.750 \\ 0.575 \end{pmatrix}, \mu_{T_M} = \begin{pmatrix} 0.475 \\ 0.475 \\ 0.475 \\ 0.750 \\ 0.575 \end{pmatrix}, \mu_{T_P} = \begin{pmatrix} 0.475 \\ 0.475 \\ 0.333 \\ 0.750 \\ 0.575 \end{pmatrix}.$$

Here again, one can see, that result μ_T at points x_4 and x_5 does not depend on the choice of the t-norm and is not effected by other points x_1, x_2 and x_3. The values of the upper general aggregation at points x_1 and x_2 depend on each other because of the high equivalence degree between these two elements. The dependence of the value of μ_T at point x_3 on the values at points x_1 and x_2 is effected by the choice of the t-norm.

Similarly, we will calculate several results for the following lower general aggregation operator:

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1,\mu_2,\mu_3,\mu_4)(x) = \min_{x' \in X} \overrightarrow{T}(E(x,x')|AVG(\mu_1(x'),\mu_2(x'),\mu_3(x'),\mu_4(x'))).$$

As results of aggregation $\tilde{A}_{E,\overrightarrow{T}}$ for $T = T_L$, $T = T_M$ and $T = T_P$ we obtain the fuzzy subsets $\mu_{\overrightarrow{T}_L}$, $\mu_{\overrightarrow{T}_M}$ and $\mu_{\overrightarrow{T}_P}$ respectively:

$$\mu_{\overrightarrow{T}_L} = \begin{pmatrix} 0.475 \\ 0.475 \\ 0.275 \\ 0.750 \\ 0.575 \end{pmatrix}, \; \mu_{\overrightarrow{T}_M} = \begin{pmatrix} 0.275 \\ 0.275 \\ 0.275 \\ 0.275 \\ 0.575 \end{pmatrix}, \; \mu_{\overrightarrow{T}_P} = \begin{pmatrix} 0.393 \\ 0.393 \\ 0.275 \\ 0.688 \\ 0.575 \end{pmatrix}.$$

In this case the low degree of equivalence has major impact only at point x_5. The result of the lower general aggregation $\tilde{A}_{E,\overrightarrow{T}}$ at point x_4 now is also effected by other elements x_1, x_2 and x_3, while changing the t-norm.

3.3 Properties derived from an ordinary aggregation operator

In this section we describe some of the most important properties of upper and lower general aggregation operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ derived from the properties of ordinary aggregation operator A (see [4, 7, 14]). These properties were studied in [49].

Symmetry. Let $\pi\colon N \to N$ be a permutation, where $N = \{1,...,n\}$. If for all elements $t_1,...,t_n \in [0,1]$ the following property holds:

$$A(t_1,\ldots,t_n) = A(t_{\pi(1)},\ldots,t_{\pi(n)}),$$

then the following equalities

$$\tilde{A}_{E,T}(\mu_1,\ldots,\mu_n) = \tilde{A}_{E,T}(\mu_{\pi(1)},\ldots,\mu_{\pi(n)}),$$

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n) = \tilde{A}_{E,\overrightarrow{T}}(\mu_{\pi(1)},\ldots,\mu_{\pi(n)})$$

hold for all $\mu_1,\ldots,\mu_n \in [0,1]^X$.

Associativity. The associativity of A in general is not preserved by upper and lower general aggregation operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$. Let us provide an example to illustrate this fact.

For simplicity we consider the discrete two-point universe X, and we use a vector form for fuzzy sets and a matrix form for T-fuzzy equivalence relation. Let us take

T_L-fuzzy equivalence relation (T_L is Lukasiewicz t-norm)

$$E = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$$

and fuzzy sets μ_1, μ_2 and μ_3:

$$\mu_1 = \begin{pmatrix} 0.4 \\ 0.2 \end{pmatrix},\ \mu_2 = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix},\ \mu_3 = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix}.$$

We obtain the following results for the upper general aggregation operator:

$$A\widetilde{V}G_{E,T_L}(A\widetilde{V}G_{E,T_L}(\mu_1,\mu_2),\mu_3) = \begin{pmatrix} 0.375 \\ 0.675 \end{pmatrix},$$

$$A\widetilde{V}G_{E,T_L}(\mu_1, A\widetilde{V}G_{E,T_L}(\mu_2,\mu_3)) = \begin{pmatrix} 0.45 \\ 0.50 \end{pmatrix},$$

and the following results for the lower general aggregation operator:

$$A\widetilde{V}G_{E,\overrightarrow{T}_L}(A\widetilde{V}G_{E,\overrightarrow{T}_L}(\mu_1,\mu_2),\mu_3) = \begin{pmatrix} 0.225 \\ 0.525 \end{pmatrix},$$

$$A\widetilde{V}G_{E,\overrightarrow{T}_L}(\mu_1, A\widetilde{V}G_{E,\overrightarrow{T}_L}(\mu_2,\mu_3)) = \begin{pmatrix} 0.30 \\ 0.35 \end{pmatrix},$$

where $A = AVG$ is the ordinary arithmetic mean aggregation operator. As one can see, the associativity does not hold in both cases.

Now we will look at the existence of absorbent, neutral and idempotent elements.

Absorbent element. Let M_{ab} be the set of all absorbent elements (or annihilators) of ordinary aggregation operator A, i.e. for all $d \in M_{ab} \subset [0,1]$, for all $i \in \{1,\ldots,n\}$ and for all $t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n \in [0,1]$

$$A(t_1,\ldots,t_{i-1},d,t_{i+1},\ldots,t_n) = d.$$

Then any fuzzy set $\tilde{d} \in [0,1]^X$, such that $\tilde{d}$ is extensional with respect to E and $\tilde{d}(x) \in M_{ab}$ for all $x \in X$, is an absorbent element of $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$. Indeed, for all $i \in \{1,\ldots,n\}$ and for all $\mu_1,\ldots,\mu_{i-1},\mu_{i+1},\ldots,\mu_n \in [0,1]^X$ we have

$$\tilde{A}_{E,T}(\mu_1,\ldots,\mu_{i-1},\tilde{d},\mu_{i+1},\ldots,\mu_n)(x) =$$

$$= \sup_{x' \in X} T(E(x,x'), A(\mu_1(x'),\ldots,\mu_{i-1}(x'),\tilde{d}(x'),\mu_{i+1}(x'),\ldots,\mu_n(x'))) =$$

$$= \sup_{x' \in X} T(E(x,x'),\tilde{d}(x')) = \tilde{d}(x),$$

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_{i-1},\tilde{d},\mu_{i+1},\ldots,\mu_n)(x) =$$

$$= \inf_{x' \in X} \overrightarrow{T}(E(x,x')|A(\mu_1(x'),...,\mu_{i-1}(x'),\tilde{d}(x'),\mu_{i+1}(x'),...,\mu_n(x'))) =$$

$$= \inf_{x' \in X} \overrightarrow{T}(E(x,x')|\tilde{d}(x')) = \tilde{d}(x).$$

Neutral element. Let M_{ne} be a set of all neutral elements of ordinary aggregation operator A, i.e. for all $e \in M_{ne} \subset [0,1]$, for all $i \in \{1,...,n\}$ and for all elements $t_1,...,t_{i-1},t_{i+1},...,t_n \in [0,1]$

$$A(t_1,\ldots,t_{i-1},e,t_{i+1},\ldots,t_n) = A(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n).$$

Then any fuzzy set $\tilde{e} \in [0,1]^X$, such that $\tilde{e}$ is extensional with respect to E and $\tilde{e}(x) \in M_{ne}$ for all $x \in X$, is a neutral element of $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$. Indeed, for all $i \in \{1,\ldots,n\}$ and for all $\mu_1,\ldots,\mu_{i-1},\mu_{i+1},\ldots,\mu_n \in [0,1]^X$ we have

$$\tilde{A}_{E,T}(\mu_1,\ldots,\mu_{i-1},\tilde{e},\mu_{i+1},\ldots,\mu_n) =$$

$$= \sup_{x' \in X} T(E(x,x'),A(\mu_1(x'),\ldots,\mu_{i-1}(x'),\tilde{e}(x'),\mu_{i+1}(x'),\ldots,\mu_n(x'))) =$$

$$= \sup_{x' \in X} T(E(x,x'),A(\mu_1(x'),\ldots,\mu_{i-1}(x'),\mu_{i+1}(x'),\ldots,\mu_n(x'))) =$$

$$= \tilde{A}_{E,T}(\mu_1,\ldots,\mu_{i-1},\mu_{i+1},\ldots,\mu_n),$$

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_{i-1},\tilde{e},\mu_{i+1},\ldots,\mu_n) =$$

$$= \inf_{x' \in X} \overrightarrow{T}(E(x,x')|A(\mu_1(x'),\ldots,\mu_{i-1}(x'),\tilde{e}(x'),\mu_{i+1}(x'),\ldots,\mu_n(x'))) =$$

$$= \inf_{x' \in X} \overrightarrow{T}(E(x,x')|A(\mu_1(x'),\ldots,\mu_{i-1}(x'),\mu_{i+1}(x'),\ldots,\mu_n(x'))) =$$

$$= \tilde{A}_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_{i-1},\mu_{i+1},\ldots,\mu_n).$$

Idempotent element. Let M_{id} be a set of all idempotent elements of ordinary aggregation operator A, i.e. for all $t \in M_{id} \subset [0,1]$

$$A(t,\ldots,t) = t.$$

Then any fuzzy set $\mu \in [0,1]^X$, such that μ is extensional with respect to E and $\mu(x) \in M_{id}$ for all $x \in X$, is an idempotent element of $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$. Indeed, we have

$$\tilde{A}_{E,T}(\mu,\ldots,\mu)(x) = \sup_{x' \in X} T(E(x,x'),A(\mu(x'),\ldots,\mu(x'))) = \sup_{x' \in X} T(E(x,x'),\mu(x')) = \mu(x),$$

and analogously

$$\tilde{A}_{E,\overrightarrow{T}}(\mu,\ldots,\mu)(x) = \inf_{x' \in X} \overrightarrow{T}(E(x,x')|A(\mu(x'),\ldots,\mu(x'))) = \inf_{x' \in X} \overrightarrow{T}(E(x,x')|\mu(x')) = \mu(x).$$

Now we will illustrate the properties mentioned above for some particular ordinary aggregation operators A.

$\mathbf{A = min}$. Taking $A = \min$ we will obtain symmetric aggregation operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ with neutral element $\tilde{e} = \tilde{1}$ and absorbent element $\tilde{d} = \tilde{0}$. In this case all extensional fuzzy sets with respect to E are idempotent elements for these operators.

$\mathbf{A = max}$. Taking $A = \max$ we obtain operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ with similar properties to the case, when $A = \min$ with a remark, that neutral and absorbent elements are $\tilde{e} = \tilde{0}$ and $\tilde{d} = \tilde{1}$ respectively.

$\mathbf{E = E_\rho}$ In the case of crisp equivalence relation ρ general aggregation operators $\tilde{A}_{E_\rho,T}$ and $\tilde{A}_{E_\rho,\overrightarrow{T}}$ inherit such properties of ordinary aggregation operator A as symmetry, associativity, existence of absorbent, neutral and idempotent elements. Let us note that in this case fuzzy set μ is extensional with respect to E_ρ if for all $x, y \in X$ it holds

$$\mu(x) = \mu(y) \Longleftrightarrow (x,y) \in \rho.$$

3.4 Different types of monotonicity for upper and lower general aggregation operators

In this section we look at the monotonicity of upper and lower general aggregation operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ with respect to ordinary aggregation operator A, t-norm T and T-fuzzy equivalence relation E. The monotonicity with respect to inputs is ensured by property $(\tilde{A}3)$.

In order to consider the monotonicity with respect to ordinary aggregation operator A, we define that

$$A^1 \leq A^2 \Longleftrightarrow A^1(t_1,\ldots,t_n) \leq A^2(t_1,\ldots,t_n) \text{ for all } t_1,\ldots,t_n \in [0,1].$$

We also define that

$$\tilde{A}^1_{E,T} \leq \tilde{A}^2_{E,T} \Longleftrightarrow \tilde{A}^1_{E,T}(\mu_1,\ldots,\mu_n) \leq \tilde{A}^2_{E,T}(\mu_1,\ldots,\mu_n) \text{ for all } \mu_1,\ldots,\mu_n \in [0,1]^X,$$

$\tilde{A}^1_{E,\overrightarrow{T}} \leq \tilde{A}^2_{E,\overrightarrow{T}} \Longleftrightarrow \tilde{A}^1_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n) \leq \tilde{A}^2_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n)$ for all $\mu_1,\ldots,\mu_n \in [0,1]^X$.

We obtain that, if it holds $A^1 \leq A^2$, then $\tilde{A}^1_{E,T} \leq \tilde{A}^2_{E,T}$ and $\tilde{A}^1_{E,\overrightarrow{T}} \leq \tilde{A}^2_{E,\overrightarrow{T}}$.

Now we consider the monotonicity of upper and lower general aggregation operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ with respect to a left-continuous t-norm T. It is easy to show, that if $T_1 \leq T_2$, then $\tilde{A}_{E,T_1} \leq \tilde{A}_{E,T_2}$ and $A_{E,\overrightarrow{T}_1} \geq \tilde{A}_{E,\overrightarrow{T}_2}$.

To study the monotonicity of $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ with respect to a T-fuzzy equivalence relation E, we consider the inequality between fuzzy equivalence relations defined by:

$$E_1 \leq E_2 \Longleftrightarrow E_1(x,y) \leq E_2(x,y) \text{ for all } x,y \in X.$$

Taking into account the monotonicity of T, we obtain that if $E_1 \leq E_2$, then for all $x \in X$ it holds

$$\tilde{A}_{E_1,T}(\mu_1,\ldots,\mu_n)(x) = \sup_{x' \in X} T(E_1(x,x'),A(\mu_1(x'),\ldots,\mu_n(x'))) \leq$$

$$\leq \sup_{x' \in X} T(E_2(x,x'),A(\mu_1(x'),\ldots,\mu_n(x'))) = \tilde{A}_{E_2,T}(\mu_1,\ldots,\mu_n)(x).$$

It is true that if $E_1 \leq E_2$, then for all $x \in X$ it holds

$$\tilde{A}_{E_1,\overrightarrow{T}}(\mu_1,\ldots,\mu_n)(x) = \inf_{x' \in X} \overrightarrow{T}(E_1(x,x')|A(\mu_1(x'),\ldots,\mu_n(x'))) \geq$$

$$\geq \inf_{x' \in X} \overrightarrow{T}(E_2(x,x')|A(\mu_1(x'),\ldots,\mu_n(x'))) = \tilde{A}_{E_2,\overrightarrow{T}}(\mu_1,\ldots,\mu_n)(x).$$

In some cases it is important to observe T-fuzzy equivalence relation E on some particular α-levels, $\alpha \in [0,1]$. For example, dealing with lower general aggregation operator, elements with low degrees of equivalence have major impact on the output of aggregation and we could obtain a distorted result. In such case it is reasonable to use T-fuzzy equivalence relation E only above some α-level, in that way ignoring elements with lower degrees of equivalence. We can consider T-fuzzy equivalence relation E^α:

$$E^\alpha(x,y) = \begin{cases} E(x,y), & E(x,y) \geq \alpha, \\ 0, & E(x,y) < \alpha, \end{cases}$$

or crisp equivalence relation E^α_ρ:

$$E^\alpha_\rho(x,y) = \begin{cases} 1, & E(x,y) \geq \alpha, \\ 0, & E(x,y) < \alpha. \end{cases}$$

For both cases the following implications hold:

$$\alpha_1 \leq \alpha_2 \Longrightarrow E^{\alpha_1} \geq E^{\alpha_2},$$

$$\alpha_1 \leq \alpha_2 \Longrightarrow E_\rho^{\alpha_1} \geq E_\rho^{\alpha_2}$$

for all $\alpha_1, \alpha_2 \in [0,1]$. Thereby if $E^{\alpha_1} \geq E^{\alpha_2}$, then $\tilde{A}_{E^{\alpha_1},T} \geq \tilde{A}_{E^{\alpha_2},T}$ and $\tilde{A}_{E^{\alpha_1},\overrightarrow{T}} \leq \tilde{A}_{E^{\alpha_2},\overrightarrow{T}}$, as well as if $E_\rho^{\alpha_1} \geq E_\rho^{\alpha_2}$, then $\tilde{A}_{E_\rho^{\alpha_1},T} \geq \tilde{A}_{E_\rho^{\alpha_2},T}$ and $\tilde{A}_{E_\rho^{\alpha_1},\overrightarrow{T}} \leq \tilde{A}_{E_\rho^{\alpha_2},\overrightarrow{T}}$.

As we will see later, the properties described in this section are important, when we are dealing with approximate systems, based on upper and lower general aggregation operators.

3.5 Upper and lower general aggregation operators and extensional fuzzy sets

In this section we describe some important results regarding the connection between upper and lower general aggregation operators and extensional fuzzy sets. This connection was studied in [46]. We recall two approximation operators ϕ_E and ψ_E, which appear in a natural way in the theory of fuzzy rough sets (see, e.g., [10], [28], [39]). Fuzzy sets $\phi_E(\mu)$ and $\psi_E(\mu)$ were introduced to provide upper and lower approximation of a fuzzy set μ with respect to fuzzy equivalence relation E.

Definition 3.5.1. *Let T be a left-continuous t-norm and E be a T-fuzzy equivalence relation on a set X. The maps $\phi_E\colon [0,1]^X \to [0,1]^X$ and $\psi_E\colon [0,1]^X \to [0,1]^X$ are defined by*

$$\phi_E(\mu)(x) = \sup_{y\in X} T(E(x,y),\mu(y)),$$

$$\psi_E(\mu)(x) = \inf_{y\in X} \overrightarrow{T}(E(x,y)|\mu(y)),$$

for all $x \in X$ and for all $\mu \in [0,1]^X$, where $\overrightarrow{T}$ is the residuum of T.

It is known that for all $\mu \in [0,1]^X$ fuzzy sets $\phi_E(\mu)$ and $\psi_E(\mu)$ are extensional with respect to E. Let us note that we always obtain extensional fuzzy sets as results of the upper and lower general aggregation operators.

Proposition 3.5.1. *Let T be a left-continuous t-norm, $\overrightarrow{T}$ be a corresponding residuum, E be a T-fuzzy equivalence relation on a set X. Let $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\overrightarrow{T}}$ be general aggregation operators defined by (3.1) and (3.2) respectively. Then fuzzy sets $\tilde{A}_{E,T}(\mu_1,\ldots,\mu_n)$ and $\tilde{A}_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n)$ are extensional with respect to E for all $\mu_1,...,\mu_n \in [0,1]^X$.*

Aggregating extensional fuzzy sets it could be necessary to obtain as a result an extensional fuzzy set as well. It could be effectively done by the upper and lower general aggregation operators, while an arbitrary general aggregation does not provide us this advantage. It is easy to show that by taking an arbitrary aggregation of extensional fuzzy sets we will not necessary obtain an extensional fuzzy set. If we take the following T_M-fuzzy equivalence relation E and fuzzy sets μ_a and μ_b, which are extensional with respect to E:

$$E = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix},\ \mu_a = \begin{pmatrix} 0.7 \\ 0.8 \end{pmatrix},\ \mu_b = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix},$$

and apply the arithmetic mean aggregation operator AVG to μ_a and μ_b, then the result

$$\mu = AVG(\mu_a, \mu_b) = \begin{pmatrix} 0.45 \\ 0.5 \end{pmatrix}$$

is not extensional fuzzy set with respect to E.

It is important that results of the upper and lower general aggregation operators are extensional with respect to E even in the case, when aggregated fuzzy sets are not extensional.

We consider the class of all general aggregations operators, which take values in the set of all extensional fuzzy sets. We will show that operators $\tilde{A}_{E,T}$ and $\tilde{A}_{E,\vec{T}}$ in this class of operators provide us with the best upper and lower approximations of a general aggregation operator $\tilde{A}$.

We will denote by X_E the set of all extensional fuzzy subsets of X with respect to a T-fuzzy equivalence relation E. We will show that

1) for any general aggregation operator $\tilde{A}_l\colon ([0,1]^X)^n \to X_E$

$$\tilde{A}_l \leq \tilde{A} \Longrightarrow \tilde{A}_l \leq \tilde{A}_{E,\vec{T}},$$

2) for any general aggregation operator $\tilde{A}_u\colon ([0,1]^X)^n \to X_E$

$$\tilde{A}_u \geq \tilde{A} \Longrightarrow \tilde{A}_u \geq \tilde{A}_{E,T}.$$

1) To prove, that

$$\tilde{A}_l(\mu_1, \ldots, \mu_n) \leq \tilde{A}_{E,\vec{T}}(\mu_1, \ldots, \mu_n) \text{ for all } \mu_1, \ldots, \mu_n \in [0,1]^X,$$

we take into account that for all $x, x' \in X$ it holds

$$T(E(x,x'), \tilde{A}_l(\mu_1, \ldots, \mu_n)(x)) \leq \tilde{A}_l(\mu_1, \ldots, \mu_n)(x'),$$

which is equivalent to

$$\tilde{A}_l(\mu_1,\ldots,\mu_n)(x) \leq \overrightarrow{T}(E(x,x')|\tilde{A}_l(\mu_1,\ldots,\mu_n)(x')).$$

Therefore

$$\tilde{A}_l(\mu_1,\ldots,\mu_n)(x) \leq \inf_{x'\in X} \overrightarrow{T}(E(x,x')|\tilde{A}_l(\mu_1,\ldots,\mu_n)(x')) \leq$$

$$\leq \inf_{x'\in X} \overrightarrow{T}(E(x,x')|\tilde{A}(\mu_1,\ldots,\mu_n)(x')) = \tilde{A}_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n)(x).$$

2) We will show, that

$$\tilde{A}_u(\mu_1,\ldots,\mu_n) \geq \tilde{A}_{E,T}(\mu_1,\ldots,\mu_n) \text{ for all } \mu_1,\ldots,\mu_n \in [0,1]^X.$$

For all $x,x' \in X$ it holds

$$T(E(x,x'),\tilde{A}_u(\mu_1,\ldots,\mu_n)(x')) \leq \tilde{A}_u(\mu_1,\ldots,\mu_n)(x),$$

and thus

$$\tilde{A}_u(\mu_1,\ldots,\mu_n)(x) \geq \sup_{x'\in X} T(E(x,x'),\tilde{A}_u(\mu_1,\ldots,\mu_n)(x')) \geq$$

$$\geq \sup_{x'\in X} T(E(x,x'),\tilde{A}(\mu_1,\ldots,\mu_n)(x')) = \tilde{A}_{E,T}(\mu_1,\ldots,\mu_n)(x).$$

And now we will propose a method to approximate an arbitrary fuzzy set by an extensional one. The approximations $\phi_E(\mu)$ and $\psi_E(\mu)$ in general are not the best approximations of μ by extensional fuzzy subsets. It is important to understand how $\phi_E(\mu)$ and $\psi_E(\mu)$ should be aggregated to obtain an aggregation with good approximative properties. In [27] the authors provided and compared two methods for the case of Archimedean t-norms: one is by taking the weighted quasi-arithmetic mean of $\phi_E(\mu)$ and $\psi_E(\mu)$ and another by taking powers with respect to a t-norm of lower approximation $\psi(\mu)$. Unfortunately, the proposed methods could not be applied in the case when t-norm T does not satisfy the required restriction.

Our approach is to apply to $\phi_E(\mu)$ and $\psi_E(\mu)$ the upper general aggregation operator based on fuzzy equivalence relation E, thus obtaining an approximation of μ by extensional fuzzy subset

$$\tilde{A}_{E,T}(\phi_E(\mu),\psi_E(\mu)).$$

As ordinary aggregation operator A one should take an aggregation operator, which satisfies the property of internality (or compensation) (see e.g. [14]). In the case when $A = MAX$ we obtain the upper approximation:

$$\tilde{A}_{E,T_M}(\phi_E(\mu),\psi_E(\mu))(x) = \phi_E(\mu).$$

By analogy, in the case when $A = MIN$ we obtain the lower approximation:

$$\tilde{A}_{E,T_M}(\phi_E(\mu), \psi_E(\mu))(x) = \psi_E(\mu).$$

Now we will illustrate our approach with several examples. Let us consider E and fuzzy sets $\mu_1, \mu_2, \mu_3, \mu_4$ from the previous Section 3.2. (see Example 3.2.1.) In the case of finite X for evaluation of the error of approximation we use the Euclidean distance between original fuzzy set $\mu \in [0,1]^X$ and the result of upper general aggregation operator:

$$d_T(A, \mu) = \|\mu - \tilde{A}_{E,T}(\phi_E(\mu), \psi_E(\mu))\|.$$

For example, by taking $T = T_M$ and $A = AVG$ and applying the corresponding upper general aggregation operator $\tilde{A}_{E,T_M}(\phi_E(\mu), \psi_E(\mu))$, we obtain the approximations of fuzzy sets $\mu_1, \mu_2, \mu_3, \mu_4$, and then evaluate $d_{T_M}(AVG, \mu_i)$ and compare with $d_{T_M}(MAX, \mu_i)$, $d_{T_M}(MIN, \mu_i)$ for different i:

$$d_{T_M}(MAX, \mu_1) = 0.4123, \;\; d_{T_M}(MIN, \mu_1) = 0.4123, \;\; d_{T_M}(AVG, \mu_1) = 0.3000,$$

$$d_{T_M}(MAX, \mu_2) = 0.4899, \;\; d_{T_M}(MIN, \mu_2) = 1.1180, \;\; d_{T_M}(AVG, \mu_2) = 0.6344,$$

$$d_{T_M}(MAX, \mu_3) = 0.6325, \;\; d_{T_M}(MIN, \mu_3) = 1.1225, \;\; d_{T_M}(AVG, \mu_3) = 0.6124,$$

$$d_{T_M}(MAX, \mu_4) = 0.9434, \;\; d_{T_M}(MIN, \mu_4) = 1.1402, \;\; d_{T_M}(AVG, \mu_4) = 0.7566.$$

The approximation by upper general aggregation operator of $\phi_E(\mu)$ and $\psi_E(\mu)$ for some initial fuzzy sets provides better results than upper and lower approximations and could be improved by involving weights into averaging operator. The problem of choosing optimal weights remains beyond the frames of this work.

Next we take $T = T_L$ and $A = MAX$, $A = MIN$, $A = AVG$, and evaluate the approximation errors:

$$d_{T_L}(MAX, \mu_1) = 0.3000, \;\; d_{T_L}(MIN, \mu_1) = 0.3000, \;\; d_{T_L}(AVG, \mu_1) = 0.2121,$$

$$d_{T_L}(MAX, \mu_2) = 0.1000, \;\; d_{T_L}(MIN, \mu_2) = 0.1414, \;\; d_{T_L}(AVG, \mu_2) = 0.0866,$$

$$d_{T_L}(MAX, \mu_3) = 0.3162, \;\; d_{T_L}(MIN, \mu_3) = 0.3317, \;\; d_{T_L}(AVG, \mu_3) = 0.2179,$$

$$d_{T_L}(MAX, \mu_4) = 0.8062, \;\; d_{T_L}(MIN, \mu_4) = 0.7071, \;\; d_{T_L}(AVG, \mu_4) = 0.5362.$$

As one can see, the smallest errors for all the given sets are obtained for the aggregation $A = AVG$, and these results are comparable with the approximations obtained in [27].

Our approach has been tested for one particular fuzzy equivalence relation E and for four fuzzy subsets of the discrete universe. A deeper analysis should be performed to make a conclusion on approximative properties of upper general aggregation operator depending on the choice of ordinary aggregation operator A.

3.6 Upper and lower general aggregation operators based on a t-norm extension of ordinary aggregation operator

Operators (3.1) and (3.2) give upper and lower approximations of a general aggregation operator $\tilde{A}$, which is the pointwise extension of an ordinary aggregation operator A. In this section we will define upper and lower approximations of the T-extension $\tilde{A}^T$ of ordinary aggregation operator A:

$$\tilde{A}^T(\mu_1,\dots,\mu_n)(x) = \sup_{x=A(x_1,\dots,x_n)} T(\mu_1(x_1),\dots,\mu_n(x_n)),$$

here $x,x_1,\dots,x_n \in [0,1]$ and $\mu_1,\dots,\mu_n \in [0,1]^{[0,1]}$. By the associativity of a t-norm here we consider T as n-ary operator.

Definition 3.6.1. *Let $A\colon [0,1]^n \to [0,1]$ be an aggregation operator, which takes all values from $[0,1]$, T be a left continuous t-norm, $\overrightarrow{T}$ be the residuum of T, and E be a T-fuzzy equivalence relation defined on $[0,1]$. The upper and lower general aggregation operators $\tilde{A}^T_{E,T}$ and $\tilde{A}^T_{E,\overrightarrow{T}}$ are defined respectively by*

$$\tilde{A}^T_{E,T}(\mu_1,\dots,\mu_n)(x) = \sup_{x'=A(x_1,\dots,x_n)} T(E(x,x'),T(\mu_1(x_1),\dots,\mu_n(x_n))), \tag{3.3}$$

$$\tilde{A}^T_{E,\overrightarrow{T}}(\mu_1,\dots,\mu_n)(x) = \inf_{x'=A(x_1,\dots,x_n)} \overrightarrow{T}(E(x,x')|T(\mu_1(x_1),\dots,\mu_n(x_n))), \tag{3.4}$$

where $x,x',x_1,\dots,x_n \in [0,1]$ and $\mu_1,\dots,\mu_n \in [0,1]^{[0,1]}$.

Further, in Proposition 3.6.1 we will prove that operators $\tilde{A}^T_{E,T}$ and $\tilde{A}^T_{E,\overrightarrow{T}}$ actually are general aggregation operators.

We consider such constructions taking into account that operator (3.3) based on crisp equivalence relation E_{ρ_0} turns into the T-extension of ordinary aggregation operator A.

While usually aggregation operators could be defined on the whole real line $\mathbb{R}$, it is also possible to modify the previous definition for the extended version of aggregation operator $A\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty,+\infty\}$.

Let us demonstrate that constructions (3.3) and (3.4) give us general aggregation operators. We must show that conditions $(\tilde{A}1)$, $(\tilde{A}2)$ and $(\tilde{A}3)$ are satisfied.

Proposition 3.6.1. *Operators $\tilde{A}^T_{E,T}$ and $\tilde{A}^T_{E,\overrightarrow{T}}$ (see (3.3),(3.4)) are general aggregation operators.*

Proof. First we consider the boundary conditions:
$(\tilde{A}1)$:

$$\tilde{A}^T_{E,T}(\tilde{0},\ldots,\tilde{0})(x) = \sup_{x'=A(x_1,\ldots,x_n)} T(E(x,x'),T(\tilde{0}(x_1),\ldots,\tilde{0}(x_n))) =$$

$$= \sup_{x'=A(x_1,\ldots,x_n)} T(E(x,x'),0) = 0 = \tilde{0}(x);$$

$$\tilde{A}^T_{E,\overrightarrow{T}}(\tilde{0},\ldots,\tilde{0})(x) = \inf_{x'=A(x_1,\ldots,x_n)} \overrightarrow{T}(E(x,x')|T(\tilde{0}(x_1),\ldots,\tilde{0}(x_n))) =$$

$$= \inf_{x'=A(x_1,\ldots,x_n)} \overrightarrow{T}(E(x,x')|0) = 0 = \tilde{0}(x);$$

$(\tilde{A}2)$:

$$\tilde{A}^T_{E,T}(\tilde{1},\ldots,\tilde{1})(x) = \sup_{x'=A(x_1,\ldots,x_n)} T(E(x,x'),T(\tilde{1}(x_1),\ldots,\tilde{1}(x_n))) =$$

$$= \sup_{x'=A(x_1,\ldots,x_n)} T(E(x,x'),1) = \sup_{x'=A(x_1,\ldots,x_n)} E(x,x') = 1 = \tilde{1}(x);$$

$$\tilde{A}^T_{E,\overrightarrow{T}}(\tilde{1},\ldots,\tilde{1})(x) = \inf_{x'=A(x_1,\ldots,x_n)} \overrightarrow{T}(E(x,x')|T(\tilde{1}(x_1),\ldots,\tilde{1}(x_n))) =$$

$$= \inf_{x'=A(x_1,\ldots,x_n)} \overrightarrow{T}(E(x,x')|1) = 1 = \tilde{1}(x).$$

$(\tilde{A}3)$: To prove the monotonicity of $\tilde{A}^T_{E,T}$ and $\tilde{A}^T_{E,\overrightarrow{T}}$ we use the monotonicity of T and the basic properties of $\overrightarrow{T}$:

$$\mu_i \preceq \eta_i,\ i=1,...,n \Longrightarrow T(\mu_1(x_1),...,\mu_n(x_n)) \le T(\eta_1(x_1),...,\eta_n(x_n)) \Longrightarrow$$

$$\Longrightarrow\ T(E(x,x'),T(\mu_1(x_1),...,\mu_n(x_n))) \le T(E(x,x'),T(\eta_1(x_1),...,\eta_n(x_n))) \Longrightarrow$$

$$\Longrightarrow \sup_{x'=A(x_1,\ldots,x_n)} T(E(x,x'),T(\mu_1(x_1),\ldots,\mu_n(x_n))) \le$$

$$\le \sup_{x'=A(x_1,\ldots,x_n)} T(E(x,x'),T(\eta_1(x_1),\ldots,\eta_n(x_n))) \Longrightarrow$$

$$\Longrightarrow \tilde{A}^T_{E,T}(\mu_1,\ldots,\mu_n) \preceq \tilde{A}^T_{E,T}(\eta_1,\ldots,\eta_n);$$

$$\mu_i \preceq \eta_i,\ \ i=1,...,n \Longrightarrow T(\mu_1(x_1),...,\mu_n(x_n)) \le T(\eta_1(x_1),...,\eta_n(x_n)) \Longrightarrow$$

$$\Longrightarrow \overrightarrow{T}(E(x,x')|T(\mu_1(x_1),...,\mu_n(x_n))) \le \overrightarrow{T}(E(x,x')|T(\eta_1(x_n),...,\eta_n(x_n))) \Longrightarrow$$

$$\Longrightarrow \inf_{x'=A(x_1,\ldots,x_n)} \overrightarrow{T}(E(x,x')|T(\mu_1(x_1),\ldots,\mu_n(x_n))) \leq$$

$$\leq \inf_{x'=A(x_1,\ldots,x_n)} \overrightarrow{T}(E(x,x')|T(\eta_1(x_1),\ldots,\eta_n(x_n))) \Longrightarrow$$

$$\Longrightarrow \tilde{A}^T_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n) \preceq \tilde{A}^T_{E,\overrightarrow{T}}(\eta_1,\ldots,\eta_n).$$

□

Upper and lower operators $\tilde{A}^T_{E,T}$ and $\tilde{A}^T_{E,\overrightarrow{T}}$ are respectively upper and lower approximations for the T-extension $\tilde{A}^T$ of an ordinary aggregation operator A. Indeed, it is easy to show that for all $\mu_1,\ldots,\mu_n \in [0,1]^{[0,1]}$

$$\tilde{A}^T_{E,\overrightarrow{T}}(\mu_1,\ldots,\mu_n) \leq \tilde{A}^T(\mu_1,\ldots,\mu_n) \leq \tilde{A}^T_{E,T}(\mu_1,\ldots,\mu_n).$$

Chapter 4

Aggregation of fuzzy real numbers

In the previous chapters we considered general aggregation operators acting on arbitrary fuzzy sets, but in real-world applications usually specific types of fuzzy sets are used. For example, dealing with various problems, fuzzy numbers have a wide range of applications. There exist different ways how define a fuzzy number – e.g. triangular, trapezoidal fuzzy numbers and other. Previously constructed general aggregation operators do not preserve the shape of such numbers in general. In this chapter we consider how these operators act on fuzzy real numbers in the sense of Hutton. Such numbers also have wide possibilities of applications (see, e.g., [1, 2, 31]). Here we investigate the problem of preservation of the shape of fuzzy numbers in the sense of Hutton in an aggregation process. For the main results presented in this chapter we refer the reader to [44, 48].

4.1 General aggregation of fuzzy real numbers

The fuzzy real numbers were first defined by B. Hutton [18] and then studied thoroughly in a series of papers (see, e.g., [25, 26, 29, 30]). Further we will use the notion *fuzzy real numbers* for fuzzy real numbers in the sense of Hutton (see Definition 1.2.8.).

We consider general aggregation operator $\tilde{A}^T$ defined by using the T-extension of an ordinary aggregation operator A (see formula (2.2)). We assume that $A\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ is a continuous aggregation operator and T is a continuous t-norm. Let us also assume that A takes real values if all the arguments are real numbers, i.e. $A(\mathbb{R}^n) = \mathbb{R}$.

We analyse the result of aggregation

$$\tilde{A}^T(z_1,\dots,z_n)(x) = \sup_{x=A(x_1,\dots,x_n)} T(z_1(x_1),\dots,z_n(x_n)), \tag{4.1}$$

where $z_1, z_2, \dots, z_n \in \mathbb{R}([0,1])$, $x, x_1, x_2, \dots, x_n \in \overline{\mathbb{R}}$. Here we assume that all $z \in \mathbb{R}([0,1])$ are defined as $z\colon \overline{\mathbb{R}} \to [0,1]$, additionally assuming that $z(-\infty) = 1$ and $z(+\infty) = 0$. The inequality for fuzzy real numbers is used in the following sense:

$$z_1 \le z_2 \Longleftrightarrow z_1(x) \le z_2(x) \text{ for all } x \in \mathbb{R}.$$

Now we will show that by using $\tilde{A}^T$ we get a fuzzy real number as the result .

Theorem 4.1.1. *Let $A\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ be a continuous aggregation operator such as $A(\mathbb{R}^n) = \mathbb{R}$, T be a continuous t-norm. Then $\tilde{A}^T(z_1, \dots, z_n) \in \mathbb{R}([0,1])$ (see formula (4.1)) for all $z_1, \dots, z_n \in \mathbb{R}([0,1])$.*

Proof. We denote $z = \tilde{A}^T(z_1, \dots, z_n)$. We must check properties (N1) – (N3) for z.
(N1): We should prove that for all $x_1, x_2 \in \mathbb{R}$

$$x_1 \le x_2 \Longrightarrow z(x_1) \ge z(x_2).$$

Let us fix arbitrary $t_1, \dots, t_n$ such that $x_2 = A(t_1, ..., t_n)$. Taking into account the boundary conditions and continuity of A, we obtain that

$$\exists(\tau_1, \dots, \tau_n) \in \prod_{i=1}^{n} [-\infty, t_i] : \; A(\tau_1, ..., \tau_n) = x_1.$$

Taking into account that $z_i(\tau_i) \ge z_i(t_i)$ for $\tau_i \le t_i$, $i = 1, \dots, n$, we obtain

$$T(z_1(\tau_1), \dots, z_n(\tau_n)) \ge T(z_1(t_1), \dots, z_n(t_n)),$$

$$z(x_1) = \sup_{x_1 = A(u_1, \dots, u_n)} T(z_1(u_1), \dots, z_n(u_n)) \ge T(z_1(t_1), \dots, z_n(t_n)),$$

$$z(x_1) \ge \sup_{x_2 = A(t_1, \dots, t_n)} T(z_1(t_1), \dots, z_n(t_n)) = z(x_2).$$

(N2): First, we show that $\sup\limits_{x \in \mathbb{R}} z(x) = 1$:

$$\sup_{x \in \mathbb{R}} \sup_{x = A(x_1, \dots, x_n)} T(z_1(x_1), \dots, z_n(x_n)) \ge \sup_{m \in \mathbb{N}} T(z_1(-m), \dots, z_n(-m)) =$$

$$= \lim_{m \to +\infty} T(z_1(-m), \dots, z_n(-m)) = T(z_1(-\infty), \dots, z_1(-\infty)) = T(1, \dots, 1) = 1.$$

Second, we show that $\inf\limits_{x \in \mathbb{R}} z(x) = 0$. Taking into account, that $\inf\limits_{x \in \mathbb{R}} z(x) = \lim\limits_{x \to +\infty} z(x)$, we need to prove that

$$\lim_{x \to +\infty} \sup_{x = A(x_1, \dots, x_n)} T(z_1(x_1), \dots, z_n(x_n)) = 0.$$

It is easy to see that

$$A(x_1,\dots,x_n) \to +\infty \Longrightarrow \max\{x_1,\dots,x_n\} \to +\infty.$$

Indeed, if this implication does not hold, then

$$\exists p \in \mathbb{R}: \ \forall q \in \mathbb{R} \ \exists x_1,\dots,x_n \in \mathbb{R}: \ A(x_1,\dots,x_n) > q \ \& \ \max\{x_1,\dots,x_n\} \le p,$$

which leads to a contradiction, because

$$\forall x_1,\dots,x_n \in (-\infty, p]: \ A(x_1,\dots,x_n) \le A(p,\dots,p) \in \mathbb{R}.$$

Now we need to show that

$$\max\{x_1,\dots,x_n\} \to +\infty \Longrightarrow T(z_1(x_1),\dots,z_n(x_n)) \to 0.$$

Let us take arbitrary $\varepsilon > 0$. Then there exists $t \in \mathbb{R}$ such that for all $i = 1,\dots,n$ it holds:

$$x_i > t \Longrightarrow z(x_i) < \varepsilon.$$

Therefore

$$\max\{x_1,\dots,x_n\} > t \Longrightarrow \min\{z_1(x_1),\dots,z_n(x_n)\} < \varepsilon \Longrightarrow T(z_1(x_1),\dots,z_n(x_n)) < \varepsilon.$$

Let us also show, that $z(-\infty) = 1$ and $z(+\infty) = 0$. The equality $A(x_1,...,x_n) = +\infty$ implies that there exists such $i \in \{1,\dots,n\}$ that $x_i = +\infty$. Then

$$\sup_{+\infty = A(x_1,\dots,x_{i-1},+\infty,x_{i-1},\dots,x_n)} T\left(z_1(x_1),\dots,z_{i-1}(x_{i-1}),z_i(+\infty),z_{i+1}(x_{i+1}),\dots,z_n(x_n)\right) = 0,$$

which implies $z(+\infty) = 0$. For $z(-\infty)$ we have:

$$z(-\infty) = \sup_{-\infty = A(x_1,\dots,x_n)} T(z_1(x_1,\dots,z_n(x_n)) \ge T(z_1(-\infty),\dots,z_n(-\infty)) = T(1,\dots,1) = 1.$$

(N3): Let us consider $z(x_0 - 0) = \inf\limits_{x < x_0} z(x)$ for $x_0 \in \mathbb{R}$. If $z(x_0 - 0) = 0$, then the equality $z(x_0 - 0) = z(x_0)$ is obvious. Let us consider the case, when $z(x_0 - 0) > 0$. We will show, that for all $\delta > 0$ it is true that $z(x_0) \ge z(x_0 - 0) - \delta$. Thus we will show, that $z(x_0 - 0) = z(x_0)$. It is enough to consider $\delta < z(x_0 - 0)$.

Let us take $x_m = x_0 - \frac{1}{m}$, $m \in \mathbb{N}$. Since

$$z(x_m) = \sup_{x_m = A(t_1,\dots,t_n)} T(z_1(t_1),\dots,z_n(t_n)),$$

there exists such $t^m = (t_1^m, \ldots, t_n^m) \in \overline{\mathbb{R}}^n$ that $x_m = A(t_1^m, \ldots, t_n^m)$ and

$$T(z_1(t_1^m), \ldots, z_n(t_n^m)) > z(x_m) - \delta \geq z(x_0 - 0) - \delta.$$

One could find such subsequence $(t^{m_k})_{k\in\mathbb{N}}$ that there exists $\lim\limits_{k\to\infty} t_i^{m_k} \in \overline{\mathbb{R}}$ for all $i = 1, ..., n$. Let us denote $t_i^0 = \lim\limits_{k\to\infty} t_i^{m_k}$, $i = 1, ...n$. Let us define new sequences $(\tau_i^k)_{k\in\mathbb{N}}$:

$$\tau_i^k = \min\{t_i^{m_k},\, t_i^0\},\ k \in \mathbb{N},\ \ i = 1, ..., n.$$

Then $\tau_i^k \leq t_i^{m_k}$ for all $k \in \mathbb{N}$, and $\lim\limits_{k\to\infty} \tau_i^k = t_i^0$, $i = 1, ..., n$. Therefore

$$T(z_1(\tau_1^k), \ldots, z_n(\tau_n^k)) \geq T(z_1(t_1^{m_k}), \ldots, z_n(t_n^{m_k})) \geq z(x_0 - 0) - \delta.$$

Taking into account that $x_0 = A(t_1^0, \ldots, t_n^0)$, we have

$$z(x_0) \geq T(z_1(t_1^0), \ldots, z_n(t_n^0)) = \lim_{k\to\infty} T(z_1(\tau_1^k), \ldots, z_n(\tau_n^k)) \geq z(x_0 - 0) - \delta.$$

□

Later we will use Theorem 4.1.1 for the proofs of Theorem 4.3.1 and Theorem 4.3.2.

4.2 Upper and lower general aggregation operators acting on fuzzy real numbers

In order to show that the upper and lower general aggregation operators acting on fuzzy real numbers give us a fuzzy real number as the result, we specify the additional restrictions on equivalence relation E. We assume that fuzzy equivalence relation E could be represented by using a non-increasing function

$$\phi\colon [0, +\infty) \to [0, 1] \qquad (4.2)$$

with the following properties:

1) $\phi(0) = 1$;

2) there exists such $h > 0$ that $\phi(h) = 0$;

3) function ϕ is consistent with t-norm T in the following sense: for all $\alpha, \beta > 0$ it holds
$$T(\phi(\alpha), \phi(\beta)) \leq \phi(\alpha + \beta).$$

We assume that

$$E(x,x') = \begin{cases} \phi(|x-x'|), & \text{if} \quad x,x' \in \mathbb{R}, \\ 1, & \text{if} \quad x = x' \ \text{ and } \ x,x' \in \overline{\mathbb{R}}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Let us take an ordinary aggregation operator $A\colon [0,1]^n \to [0,1]$. We analyse the results of the following aggregations:

$$\tilde{A}_{E,T}(z_1,\ldots,z_n)(x) = \sup_{x' \in \overline{\mathbb{R}}} T(E(x,x'),A(z_1(x'),\ldots,z_n(x'))), \tag{4.4}$$

$$\tilde{A}_{E,\overrightarrow{T}}(z_1,\ldots,z_n)(x) = \inf_{x' \in \overline{\mathbb{R}}} \overrightarrow{T}(E(x,x')|A(z_1(x'),\ldots,z_n(x'))), \tag{4.5}$$

where $x \in \overline{\mathbb{R}}$ and $z_1,\ldots,z_n \in \mathbb{R}([0,1])$.

Theorem 4.2.1. *Let $A\colon [0,1]^n \to [0,1]$ be a continuous aggregation operator, T be a continuous t-norm, and E be the T-fuzzy equivalence relation defined on $\overline{\mathbb{R}}$ by a continuous function ϕ (see (4.2)) according to (4.3). Then $\tilde{A}_{E,T}(z_1,\ldots,z_n) \in \mathbb{R}([0,1])$ (see formula (4.4)) for all $z_1,\ldots,z_n \in \mathbb{R}([0,1])$.*

Proof. Let us denote $z = \tilde{A}_{E,T}(z_1,\ldots,z_n)$. We have

$$z(x) = \sup_{x' \in \mathbb{R}} T(E(x,x'),A(z_1(x'),\ldots,z_n(x'))) =$$

$$= \sup_{t \in \mathbb{R}} T(\phi(|t|),A(z_1(x+t),\ldots,z_n(x+t))) \ \text{ for } \ x \in \mathbb{R}.$$

It holds, that $z(-\infty) = 1$ and $z(+\infty) = 0$:

$$z(-\infty) = \sup_{x' \in \overline{\mathbb{R}}} T(E(-\infty,x'),A(z_1(x'),\ldots,z_n(x'))) \geq$$

$$\geq T(E(-\infty,-\infty),A(z_1(-\infty),\ldots,z_n(-\infty))) = T(1,A(1,\ldots,1)) = 1;$$

$$z(+\infty) = \sup_{x' \in \overline{\mathbb{R}}} T(E(+\infty,x'),A(z_1(x'),\ldots,z_n(x'))) = 0,$$

because

a) $x' \neq +\infty \Longrightarrow T(E(+\infty,x'),A(z_1(x'),\ldots,z_n(x'))) = T(0,A(z_1(x'),\ldots,z_n(x'))) = 0,$

b) $x' = +\infty \Longrightarrow T(E(+\infty,+\infty),A(z_1(+\infty),\ldots,z_n(+\infty))) = T(1,A(0,\ldots,0)) = 0.$

(N1): If $x_1 \leq x_2$, then

$$z(x_1) = \sup_{t \in \mathbb{R}} T(\phi(|t|),A(z_1(x_1+t),\ldots,z_n(x_1+t))) \geq$$

$$\geq \sup_{t \in \mathbb{R}} T(\phi(|t|), A(z_1(x_2+t), \ldots, z_n(x_2+t))) = z(x_2).$$

(N2): It is easy to see that

$$\sup_{x \in \mathbb{R}} z(x) = \sup_{x \in \mathbb{R}} \sup_{t \in \mathbb{R}} T(\phi(|t|), A(z_1(x+t), \ldots, z_n(x+t))) \geq$$

$$\geq \sup_{x \in \mathbb{R}} A(z_1(x), \ldots, z_n(x)) = 1.$$

It is easy to see that for all $x \in \mathbb{R}$ it holds

$$z(x) = \sup_{t \in \mathbb{R}} T(\phi(|t|), A(z_1(x+t), \ldots, z_n(x+t))) \leq$$

$$\leq \sup_{t \in \mathbb{R}} T(\phi(|t|), A(z_1(x-h), \ldots, z_n(x-h))) \leq A(z_1(x-h), \ldots, z_n(x-h)).$$

Therefore

$$\inf_{x \in \mathbb{R}} z(x) \leq \inf_{x \in \mathbb{R}} A(z_1(x-h), \ldots, z_n(x-h)) = 0.$$

(N3): Let us consider $z(x_0-0) = \inf_{x<x_0} z(x)$ for $x_0 \in \mathbb{R}$. If $z(x_0-0) = 0$, then the equality $z(x_0-0) = z(x_0)$ is obvious. Let us consider the case, when $z(x_0-0) > 0$. We will show, that for all $\delta > 0$ it is true that $z(x_0) \geq z(x_0-0) - \delta$. Thus we will show, that $z(x_0-0) = z(x_0)$. It is enough to consider $\delta < z(x_0-0)$.

Let us take $x_m = x_0 - \frac{1}{m}$, $m \in \mathbb{N}$. Since

$$z(x_m) = \sup_{t \in \mathbb{R}} T(\phi(|t|), A(z_1(x_m+t), \ldots, z_n(x_m+t))) =$$

$$= \sup_{t \geq 0} T(\phi(t), A(z_1(x_m-t), \ldots, z_n(x_m-t))),$$

there exists such $t_m \in [0,h]$ that

$$T(\phi(t_m), A(z_1(x_m-t_m), \ldots, z_n(x_m-t_m))) > z(x_m) - \delta \geq z(x_0-0) - \delta.$$

For sequence $(t_m)_{m \in \mathbb{N}}$ one can find a convergent subsequence $(t_{m_k})_{k \in \mathbb{N}}$. Let us denote $t_0 = \lim_{k \to \infty} t_{m_k}$. Let us form another sequence $(\tau_k)_{k \in \mathbb{N}}$ by taking $\tau_k = \max\{t_{m_k}, t_0\}$ for all $k \in \mathbb{N}$. We have $\lim_{k \to \infty} \tau_k = t_0$ and

$$x_0 - \frac{1}{m_k} - \tau_k \leq x_{m_k} - t_{m_k} \quad \text{and} \quad x_0 - \frac{1}{m_k} - \tau_k \leq x_0 - t_0 \ \text{ for all } k \in \mathbb{N}.$$

Therefore

$$T(\phi(t_{m_k}), A(z_1(x_0 - \frac{1}{m_k} - \tau_k), \ldots, z_n(x_0 - \frac{1}{m_k} - \tau_k))) \geq$$

$$\geq T(\phi(t_{m_k}), A(z_1(x_{m_k} - t_{m_k}), \ldots, z_n(x_{m_k} - t_{m_k}))) \geq z(x_0-0) - \delta.$$

Hence

$$z(x_0) \geq T(\phi(t_0), A(z_1(x_0 - t_0), \ldots, z_n(x_0 - t_0))) =$$

$$= \lim_{k\to\infty} T(\phi(t_{m_k}), A(z_1(x_0 - \frac{1}{m_k} - \tau_k), \ldots, z_n(x_0 - \frac{1}{m_k} - \tau_k))) \geq z(x_0 - 0) - \delta.$$

□

A result of lower general aggregation operator $\tilde{A}_{E,\overrightarrow{T}}$ also fulfils properties of inputs in the form of fuzzy real numbers.

Theorem 4.2.2. *Let $A\colon [0,1]^n \to [0,1]$ be a continuous aggregation operator, T be a continuous t-norm, and E be the T-fuzzy equivalence relation defined on $\overline{\mathbb{R}}$ by function ϕ (see (4.2)) according to (4.3). Then $\tilde{A}_{E,\overrightarrow{T}}(z_1, \ldots, z_n) \in \mathbb{R}([0,1])$ (see formula (4.5)) for all $z_1, \ldots, z_n \in \mathbb{R}([0,1])$.*

Proof. Let us denote $z = \tilde{A}_{E,\overrightarrow{T}}(z_1, \ldots, z_n)$. It holds

$$z(x) = \inf_{x'\in\mathbb{R}} \overrightarrow{T}(E(x,x')|A(z_1(x'), \ldots, z_n(x'))) =$$

$$= \inf_{t\in\mathbb{R}} \overrightarrow{T}(\phi(|t|)|A(z_1(x+t), \ldots, z_n(x+t))) \quad \text{for} \quad x \in \mathbb{R}.$$

It holds, that $z(+\infty) = 0$ and $z(-\infty) = 1$:

$$z(+\infty) = \inf_{x'\in\overline{\mathbb{R}}} \overrightarrow{T}(E(+\infty,x')|A(z_1(x'), \ldots, z_n(x'))) \leq$$

$$\leq \overrightarrow{T}(E(+\infty,+\infty)|A(z_1(+\infty), \ldots, z_n(+\infty))) = A(0, \ldots, 0) = 0;$$

$$z(-\infty) = \inf_{x'\in\overline{\mathbb{R}}} \overrightarrow{T}(E(-\infty,x')|A(z_1(x'), \ldots, z_n(x'))) = 1,$$

because

a) $x' \neq -\infty \Longrightarrow \overrightarrow{T}(E(-\infty,x')|A(z_1(x'), \ldots, z_n(x'))) = \overrightarrow{T}(0|A(z_1(x'), \ldots, z_n(x'))) = 1,$

b) $x' = -\infty \Longrightarrow \overrightarrow{T}(E(-\infty,-\infty)|A(z_1(-\infty), \ldots, z_n(-\infty))) = \overrightarrow{T}(1|A(1, \ldots, 1)) = 1.$

(N1): If $x_1 \leq x_2$, then

$$z(x_1) = \inf_{t\in\mathbb{R}} \overrightarrow{T}(\phi(|t|)|A(z_1(x_1+t), \ldots, z_n(x_1+t))) \geq$$

$$\geq \inf_{t\in\mathbb{R}} \overrightarrow{T}(\phi(|t|)|A(z_1(x_2+t), \ldots, z_n(x_2+t))) = z(x_2)$$

(N2): It is easy to see that

$$\inf_{x\in\mathbb{R}} z(x) = \inf_{x\in\mathbb{R}} \inf_{t\in\mathbb{R}} \overrightarrow{T}(\phi(|t|)|A(z_1(x+t), \ldots, z_n(x+t))) \leq$$

$$\leq \inf_{x\in\mathbb{R}} \overrightarrow{T}(1|A(z_1(x),\ldots,z_n(x))) = \inf_{x\in\mathbb{R}} A(z_1(x),\ldots,z_n(x)) = 0.$$

Taking into account that for all $x \in \mathbb{R}$ it holds

$$z(x) = \inf_{|t|\leq h} \overrightarrow{T}(\phi(|t|)|A(z_1(x+t),\ldots,z_n(x+t))) \geq$$

$$\geq \overrightarrow{T}(1|A(z_1(x+h),\ldots,z_n(x+h))) = A(z_1(x+h),\ldots,z_n(x+h)),$$

we obtain

$$\sup_{x\in\mathbb{R}} z(x) \geq \sup_{x\in\mathbb{R}} A(z_1(x+h),\ldots,z_n(x+h)) = 1.$$

(N3):

$$\inf_{x<x_0} z(x) = \inf_{x<x_0}\inf_{t\in\mathbb{R}} \overrightarrow{T}(\phi(|t|)|A(z_1(x+t),\ldots,z_n(x+t))) =$$

$$= \inf_{t\in\mathbb{R}}\inf_{x<x_0} \overrightarrow{T}(\phi(|t|)|A(z_1(x+t),\ldots,z_n(x+t))) =$$

$$= \inf_{t\in\mathbb{R}} \overrightarrow{T}(\phi(|t|)|A(z_1(x_0+t),\ldots,z_n(x_0+t))) = z(x_0).$$

□

4.3 Upper and lower general aggregation operators based on a t-norm extension and acting on fuzzy real numbers

In this subsection we consider the case when inputs of operators $\tilde{A}^T_{E,T}$ and $\tilde{A}^T_{E,\overrightarrow{T}}$ are fuzzy real numbers. We check whether these operators give us as the result a fuzzy real number as well. Here we take a continuous ordinary aggregation operator A, which acts on the extended real line, i.e. $A\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$. We analyse the results of the following aggregations:

$$\tilde{A}^T_{E,T}(z_1,\ldots,z_n)(x) = \sup_{x'=A(x_1,\ldots,x_n)} T(E(x,x'),T(z_1(x_1),\ldots,z_n(x_n))), \tag{4.6}$$

$$\tilde{A}^T_{E,\overrightarrow{T}}(z_1,\ldots,z_n)(x) = \inf_{x'=A(x_1,\ldots,x_n)} \overrightarrow{T}(E(x,x')|T(z_1(x_1),\ldots,z_n(x_n))), \tag{4.7}$$

where $x,x',x_1,\ldots,x_n \in \overline{\mathbb{R}}$ and $z_1,\ldots,z_n \in \mathbb{R}([0,1])$.

First, we will show that operator $\tilde{A}^T_{E,T}$ preserves properties of inputs in the form of fuzzy real numbers.

Theorem 4.3.1. *Let $A\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}$ be a continuous aggregation operator such that $A(\mathbb{R}^n) = \mathbb{R}$, T be a continuous t-norm, and E be the T-fuzzy equivalence relation defined on $\overline{\mathbb{R}}$ by a continuous function ϕ (see formula (4.2)) according to (4.3). Then $\tilde{A}^T_{E,T}(z_1,\ldots,z_n) \in \mathbb{R}([0,1])$ (see formula (4.6)) for all $z_1,\ldots,z_n \in \mathbb{R}([0,1])$.*

Proof. Let us denote $z = \tilde{A}^T_{E,T}(z_1,\ldots,z_n)$. Taking into account the assumptions regarding fuzzy equivalence relation E provided in subsection 4.2, we obtain that

$$z(x) = \sup_{x+t=A(x_1,\ldots,x_n)} T(\phi(|t|), T(z_1(x_1),\ldots,z_n(x_n))) \ \text{ for } \ x \in \mathbb{R}.$$

It holds, that $z(-\infty) = 1$ and $z(+\infty) = 0$:

$$z(-\infty) = \sup_{x'=A(x_1,\ldots,x_n)} T(E(-\infty,x'), T(z_1(x_1),\ldots,z_n(x_n))) \geq$$

$$\geq T(E(-\infty,-\infty), T(z_1(-\infty),\ldots,z_n(-\infty))) = T(1,T(1,\ldots,1)) = 1;$$

$$z(+\infty) = \sup_{x'=A(x_1,\ldots,x_n)} T(E(+\infty,x'), T(z_1(x_1),\ldots,z_n(x_n))) = 0,$$

because
a) $x' \neq +\infty \Longrightarrow T(E(+\infty,x'), T(z_1(x_1),\ldots,z_n(x_n))) = T(0, T(z_1(x_1),\ldots,z_n(x_n))) = 0$,
b) if $x' = +\infty$ and $A(x_1,\ldots,x_n) = x'$, then there exists $i \in \{1,\ldots,n\}$ such that $x_i = +\infty$, and we obtain

$$T(E(+\infty,+\infty), T(z_1(x_1),\ldots,z_{i-1}(x_{i-1}), z(+\infty), z_{i+1}(x_{i+1}),\ldots,z_n(x_n))) =$$

$$= T(1, T(z_1(x_1),\ldots,z_{i-1}(x_{i-1}), 0, z_{i+1}(x_{i+1}),\ldots,z_n(x_n)))) = 0.$$

(N1): We need to prove that for $x,y \in \mathbb{R}$, such that $x \leq y$, it holds

$$\sup_{x+t=A(x_1,\ldots,x_n)} T(\phi(|t|), T(z_1(x_1),\ldots,z_n(x_n))) \geq$$

$$\geq \sup_{y+t=A(y_1,\ldots,y_n)} T(\phi(|t|), T(z_1(y_1),\ldots,z_n(y_n))).$$

The proof is based on the fact that for all $t \in \mathbb{R}$ and $y_1,\ldots,y_n \in \overline{\mathbb{R}}$ such that $y+t = A(y_1,\ldots,y_n)$ by the continuity and boundary conditions of A we could find such values $x_1,\ldots,x_n \in \overline{\mathbb{R}}$, that $x_1 \leq y_1,\ldots,x_n \leq y_n$ and $A(x_1,\ldots,x_n) = x+t$. Then

$$T(\phi(|t|), T(z_1(x_1),\ldots,z_n(x_n))) \geq T(\phi(|t|), T(z_1(y_1),\ldots,z_n(y_n))).$$

By this the inequality is proved.

(N2): In order to show that $\inf\limits_{x\in\mathbb{R}} z(x) = 0$ and $\sup\limits_{x\in\mathbb{R}} z(x) = 1$ we use the fact that $\tilde{A}^T(z_1,\dots,z_n) \in \mathbb{R}([0,1])$ (see Theorem 4.1.1). Let us denote $\eta = \tilde{A}^T(z_1,\dots,z_n)$.

For all $x \in \mathbb{R}$ it holds

$$z(x) = \sup_{x+t=A(x_1,\dots,x_n)} T(\phi(|t|), T(z_1(x_1),\dots,z_n(x_n))) \leq$$

$$\leq \sup_{x+t=A(x_1,\dots,x_n),\ |t|\leq h} T(z_1(x_1),\dots,z_n(x_n)) =$$

$$= \sup_{x-h\leq y\leq x+h} \sup_{y=A(x_1,\dots,x_n)} T(z_1(x_1),\dots,z_n(x_n)) = \sup_{x-h\leq y\leq x+h} \eta(y) = \eta(x-h).$$

Thus $\inf\limits_{x\in\mathbb{R}} z(x) \leq \inf\limits_{x\in\mathbb{R}} \eta(x-h) = 0$.

Now, taking into account that $\tilde{A}^T \leq \tilde{A}^T_{E,T}$, we obtain

$$\sup_{x\in\mathbb{R}} z(x) = \sup_{x\in\mathbb{R}} \tilde{A}^T_{E,T}(z_1,\dots,z_n)(x) \geq \sup_{x\in\mathbb{R}} \tilde{A}^T(z_1,\dots,z_n)(x) = \sup_{x\in\mathbb{R}} \eta(x) = 1.$$

(N3): Let us consider $z(x_0-0) = \inf\limits_{x<x_0} z(x)$ for $x_0 \in \mathbb{R}$. If $z(x_0-0) = 0$, then the equality $z(x_0-0) = z(x_0)$ is obvious. Let us consider the case, when $z(x_0-0) > 0$. We will show, that for all $\delta > 0$ it is true that $z(x_0) \geq z(x_0-0) - \delta$. Thus we will show, that $z(x_0-0) = z(x_0)$. It is enough to consider $\delta < z(x_0-0)$.

Let us take $x_m = x_0 - \frac{1}{m}$, $m \in \mathbb{N}$. Since

$$z(x_m) = \sup_{x_m+\tau=A(t_1,\dots,t_n)} T(\phi(|\tau|), T(z_1(t_1),\dots,z_n(t_n))),$$

there exist such $\tau_m \in [-h,h]$ and $t^m = (t_1^m,\dots,t_n^m) \in \overline{\mathbb{R}}^n$ that $x_m + \tau_m = A(t_1^m,\dots,t_n^m)$ and

$$T(\phi(|\tau_m|), T(z_1(t_1^m),\dots,z_n(t_n^m))) > z(x_m) - \delta \geq z(x_0-0) - \delta.$$

One could find such subsequences $(\tau_{m_k})_{k\in\mathbb{N}}$ and $(t^{m_k})_{k\in\mathbb{N}}$ that there exist $\lim\limits_{k\to\infty} \tau^i_{m_k} \in \mathbb{R}$ and $\lim\limits_{k\to\infty} t_i^{m_k} \in \overline{\mathbb{R}}$ for all $i = 1,\dots,n$. Let us denote $\tau_0 = \lim\limits_{k\to\infty} \tau_{m_k}$ and $t_i^0 = \lim\limits_{k\to\infty} t_i^{m_k}$, $i = 1,\dots,n$. Let us define new sequences $(\tau_i^k)_{k\in\mathbb{N}}$:

$$\tau_i^k = \min\{t_i^{m_k},\ t_i^0\},\ k \in \mathbb{N},\ \ i = 1,\dots,n.$$

Then $\tau_i^k \leq t_i^{m_k}$ for all $k \in \mathbb{N}$, and $\lim\limits_{k\to\infty} \tau_i^k = t_i^0$, $i = 1,\dots,n$. Therefore

$$T(\phi(|\tau_{m_k}|), T(z_1(\tau_1^k),\dots,z_n(\tau_n^k))) \geq$$

$$\geq T(\phi(|\tau_{m_k}|), T(z_1(t_1^{m_k}),\dots,z_n(t_n^{m_k}))) \geq z(x_0-0) - \delta.$$

Taking into account that $x_0+\tau_0=A(t_1^0,\dots,t_n^0)$, we obtain

$$z(x_0)\geq T(\phi(|\tau_0|),T(z_1(t_1^0),\dots,z_n(t_n^0)))=$$

$$=\lim_{k\to\infty}T(\phi(|\tau_{m_k}|),T(z_1(\tau_1^k),\dots,z_n(\tau_n^k)))\geq z(x_0-0)-\delta.$$

□

For the result $z=\tilde{A}^T_{E,\overrightarrow{T}}(z_1,\dots,z_n)$ (see formula (4.7)) the properties of fuzzy real numbers could be proved analogously to the previous case, except the boundary condition $\sup\limits_{x\in\mathbb{R}}z(x)=1$. In order to this property be fulfilled, ordinary aggregation operator A should satisfy some additional strong conditions, which are not necessary hold in general. For example, the following condition is sufficient:

$$\text{if } A(x_1,\dots,x_n)\to-\infty,\text{ then } \max\{x_1,\dots,x_n\}\to-\infty. \tag{4.8}$$

Theorem 4.3.2. *Let $A\colon\overline{\mathbb{R}}^n\to\overline{\mathbb{R}}$ be a continuous aggregation operator such that (4.8) is true and $A(\mathbb{R}^n)=\mathbb{R}$, T be a continuous t-norm, and E be the T-fuzzy equivalence relation defined on $\overline{\mathbb{R}}$ by a continuous function ϕ (see (4.2)) according to (4.3). Then $\tilde{A}^T_{E,\overrightarrow{T}}(z_1,\dots,z_n)\in\mathbb{R}([0,1])$ (see formula (4.7)) for all $z_1,\dots,z_n\in\mathbb{R}([0,1])$.*

Proof. Let us denote $z=\tilde{A}^T_{E,\overrightarrow{T}}(z_1,\dots,z_n)$. Taking into account the assumptions regarding fuzzy equivalence relation E provided in subsection 4.2, we obtain that

$$z(x)=\inf_{x+t=A(x_1,\dots,x_n)}\overrightarrow{T}(\phi(|t|)|T(z_1(x_1),\dots,z_n(x_n)))\quad\text{for}\quad x\in\mathbb{R}.$$

It holds, that $z(+\infty)=0$ and $z(-\infty)=1$:

$$z(+\infty)=\inf_{x'=A(x_1,\dots,x_n)}\overrightarrow{T}(E(+\infty,x')|T(z_1(x_1),\dots,z_n(x_n)))\leq$$

$$\leq\overrightarrow{T}(1|T(z_1(+\infty),\dots,z_n(+\infty)))=\overrightarrow{T}(1|T(0,\dots,0))=0;$$

$$z(-\infty)=\inf_{x'=A(x_1,\dots,x_n)}\overrightarrow{T}(E(-\infty,x')|T(z_1(x_1),\dots,z_n(x_n)))=1,$$

because

a) $x'\neq-\infty\Longrightarrow\overrightarrow{T}(E(-\infty,x')|T(z_1(x_1),\dots,z_n(x_n)))=\overrightarrow{T}(0|T(z_1(x_1),\dots,z_n(x_n)))=1$,

b) if $x'=-\infty$ and $A(x_1,\dots,x_n)=x'$, then

$$\overrightarrow{T}(E(-\infty,x')|T(z_1(x_1),\dots,z_n(x_n)))=\overrightarrow{T}(1|T(1,\dots,1)=1.$$

(N1): We need to prove that for $x, y \in \mathbb{R}$, such that $x \leq y$, it holds

$$\inf_{x+t=A(x_1,\ldots,x_n)} \overrightarrow{T}(\phi(|t|)|T(z_1(x_1),\ldots,z_n(x_n))) \geq$$

$$\geq \inf_{y+t=A(y_1,\ldots,y_n)} \overrightarrow{T}(\phi(|t|)|T(z_1(y_1),\ldots,z_n(y_n))).$$

The proof is based on the fact that for all $t \in \mathbb{R}$ and $x_1, \ldots, x_n \in \overline{\mathbb{R}}$ such that $x + t = A(x_1, \ldots, x_n)$ by the continuity and boundary conditions of A we could find such values $y_1, \ldots, y_n \in \overline{\mathbb{R}}$, that $x_1 \leq y_1, \ldots, x_n \leq y_n$ and $A(y_1, \ldots, y_n) = y + t$. Then

$$\overrightarrow{T}(\phi(|t|)|T(z_1(x_1),\ldots,z_n(x_n))) \geq \overrightarrow{T}(\phi(|t|)|T(z_1(y_1),\ldots,z_n(y_n))).$$

By this the inequality is proved.

(N2): In order to show that $\inf_{x\in\mathbb{R}} z(x) = 0$ and $\sup_{x\in\mathbb{R}} z(x) = 1$ we use the fact that $\tilde{A}^T(z_1,\ldots,z_n) \in \mathbb{R}([0,1])$ (see Theorem 4.1.1). Let us denote $\eta = \tilde{A}^T(z_1,\ldots,z_n)$. Taking into account, that $\tilde{A}^T_{E,\overrightarrow{T}} \leq \tilde{A}^T$, we obtain

$$\inf_{x\in\mathbb{R}} z(x) = \inf_{x\in\mathbb{R}} \tilde{A}^T_{E,\overrightarrow{T}}(z_1,\ldots,z_n)(x) \leq \inf_{x\in\mathbb{R}} \tilde{A}^T(z_1,\ldots,z_n)(x) = \inf_{x\in\mathbb{R}} \eta(x) = 0.$$

For all $x \in \mathbb{R}$ it holds

$$z(x) = \inf_{x+t=A(x_1,\ldots,x_n)} \overrightarrow{T}(\phi(|t|)|T(z_1(x_1),\ldots,z_n(x_n))) \geq$$

$$\geq \inf_{x+t=A(x_1,\ldots,x_n),\ |t|\leq h} T(z_1(x_1),\ldots,z_n(x_n)) \geq \inf_{A(x_1,\ldots,x_n)\leq x+h} T(z_1(x_1),\ldots,z_n(x_n)).$$

Taking into account, that $\sup_{x\in\mathbb{R}} z(x) = \lim_{x\to-\infty} z(x)$, we obtain

$$A(x_1,\ldots,x_n) \leq x+h \text{ and } x \to -\infty \Longrightarrow A(x_1,\ldots,x_n) \to -\infty \Longrightarrow$$

$$\Longrightarrow \max\{x_1,\ldots,x_n\} \to -\infty \Longrightarrow z_i(x_i) \to 1 \text{ for all } i = 1,\ldots,n \Longrightarrow$$

$$\Longrightarrow T(z_1(x_1),\ldots,z_n(x_n)) \to 1.$$

And finally, we have $\sup_{x\in\mathbb{R}} z(x) = 1$.

(N3): Let us consider $z(x_0 - 0) = \inf_{x<x_0} z(x)$ for $x_0 \in \mathbb{R}$. We will show, that $z(x_0 - 0) = z(x_0)$, where

$$z(x_0) = \inf_{x_0+t=A(x_1,\ldots,x_n)} \overrightarrow{T}(\phi(|t|)|T(z_1(x_1),\ldots,z_n(x_n))).$$

We will prove that for arbitrary $t_0 \in \mathbb{R}$ and $x_1^0, \ldots, x_n^0 \in \overline{\mathbb{R}}$ such that $x_0 + t_0 = A(x_1^0, \ldots, x_n^0)$ it holds

$$z(x_0 - 0) \leq \overrightarrow{T}(\phi(|t_0|)|T(z_1(x_1^0),\ldots,z_n(x_n^0))).$$

By this the equality $z(x_0-0)=z(x_0)$ will be proved.

If $t_0<0$, then we take $\tilde{x}=x_0+t_0$, $\tilde{t}=0$ and obtain

$$z(x_0-0)=\inf_{x<x_0}\inf_{x+t=A(x_1,\dots,x_n)}\overrightarrow{T}(\phi(|t|)|T(z_1(x_1),\dots,z_n(x_n)))\leq$$

$$\leq\overrightarrow{T}(\phi(|\tilde{t}|)|T(z_1(x_1^0),\dots,z_n(x_n^0)))\leq\overrightarrow{T}(\phi(|t_0|)|T(z_1(x_1^0),\dots,z_n(x_n^0))).$$

If $t_0\geq 0$, then we consider two sequences: $\tilde{x}_m=x_0-\frac{1}{m}$ and $\tilde{t}_m=t_0+\frac{1}{m}$, $m\in\mathbb{N}$. We obtain

$$z(x_0-0)\leq\inf_{m\in\mathbb{N}}\overrightarrow{T}(\phi(|\tilde{t}_m|)|T(z_1(x_1^0),\dots,z_n(x_n^0)))=\overrightarrow{T}(\phi(|t_0|)|T(z_1(x_1^0),\dots,z_n(x_n^0))).$$

$\square$

Chapter 5

Approximate systems induced by upper and lower general aggregation operators

In Chapter 3 we described upper and lower general aggregation operators which provide upper and lower approximations of the pointwise extension of an ordinary aggregation operator. The presence of two approximate operators led us to the idea of constructing an approximate system induced by these operators. The construction of such system allows us to generalize the concept of approximation of general aggregation operators and to perform research in connection with other theories like rough sets, fuzzy rough sets and other. The results presented in this chapter could be found in [48].

5.1 $\mathbb{M}$-approximate systems

The concept of an $\mathbb{M}$-approximate system was first introduced by A. Shostak in [34] and further studied in [11, 17, 35]. This concept provides an alternative view on the relations between fuzzy sets, fuzzy topological systems, rough sets, and fuzzy rough sets. This tool gives a framework allowing to generalize these theories.

In the context of $\mathbb{M}$-approximate systems two lattices play the fundamental role. The first one is an infinitely distributive complete lattice $\mathbb{L} = (\mathbb{L}, \leq, \wedge, \vee)$. The bottom and the top elements of $\mathbb{L}$ are denoted by $0_{\mathbb{L}}$ and $1_{\mathbb{L}}$, respectively. The second lattice is denoted by $\mathbb{M}$ and is assumed to be complete. The bottom and the top elements of $\mathbb{M}$ are denoted by $0_{\mathbb{M}}$ and $1_{\mathbb{M}}$, respectively.

Definition 5.1.1. *An upper $\mathbb{M}$-approximate operator on $\mathbb{L}$ is defined as a mapping $u\colon \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ such that*
(1u) $u(0_{\mathbb{L}},\alpha)=0_{\mathbb{L}}$ *for all* $\alpha\in\mathbb{M}$;
(2u) $a\le u(a,\alpha)$ *for all* $a\in\mathbb{L}$ *and for all* $\alpha\in\mathbb{M}$;
(3u) $u(a\vee b,\alpha)=u(a,\alpha)\vee u(b,\alpha)$ *for all* $a,b\in\mathbb{L}$ *and for all* $\alpha\in\mathbb{M}$;
(4u) $u(u(a,\alpha),\alpha)=u(a,\alpha)$ *for all* $a\in\mathbb{L}$ *and for all* $\alpha\in\mathbb{M}$;
(5u) $\alpha\le\beta,\ \alpha,\beta\in\mathbb{M}\Longrightarrow u(a,\alpha)\le u(a,\beta)$ *for all* $a\in\mathbb{L}$;
(6u) if $0_{\mathbb{M}}\ne 1_{\mathbb{M}}$, *then* $u(a,0_{\mathbb{M}})=a$ *for all* $a\in\mathbb{L}$.

Definition 5.1.2. *A lower $\mathbb{M}$-approximate operator on $\mathbb{L}$ is a mapping $l\colon \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ such that*
(1l) $l(1_{\mathbb{L}},\alpha)=1_{\mathbb{L}}$ *for all* $\alpha\in\mathbb{M}$;
(2l) $a\ge l(a,\alpha)$ *for all* $a\in\mathbb{L}$ *and for all* $\alpha\in\mathbb{M}$;
(3l) $l(a\wedge b,\alpha)=l(a,\alpha)\wedge l(b,\alpha)$ *for all* $a,b\in\mathbb{L}$ *and for all* $\alpha\in\mathbb{M}$;
(4l) $l(l(a,\alpha),\alpha)=l(a,\alpha)$ *for all* $a\in\mathbb{L}$ *and for all* $\alpha\in\mathbb{M}$;
(5l) $\alpha\le\beta,\ \alpha,\beta\in\mathbb{M}\Longrightarrow l(a,\alpha)\ge l(a,\beta)$ *for all* $a\in\mathbb{L}$;
(6l) if $0_{\mathbb{M}}\ne 1_{\mathbb{M}}$, *then* $l(a,0_{\mathbb{M}})=a$ *for all* $a\in\mathbb{L}$.

Definition 5.1.3. *A triple $(\mathbb{L},u,l)$, where $u\colon \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ and $l\colon \mathbb{L}\times\mathbb{M}\to\mathbb{L}$ are upper and lower $\mathbb{M}$-approximate operators on $\mathbb{L}$, is called an $\mathbb{M}$-approximate system.*

5.2 Construction of •-approximate system using upper and lower general aggregation operators

There are several works where some particular constructions of $\mathbb{M}$-approximate systems are presented (see, e.g., [17, 35]). In some constructions the one-point lattice $\mathbb{M}$ denoted by • is used (in this case $0_{\mathbb{M}}=1_{\mathbb{M}}$) (see, e.g., [35]). We consider the construction of approximate system in the case of one-point lattice $\mathbb{M}$.

Let us have a given set X. We take lattice $\mathbb{L}$ as the lattice of all general aggregation operators:

$$\mathbb{L}=\{\tilde{A}\colon \bigcup_n([0,1]^X)^n\to[0,1]^X \mid \tilde{A} \text{ is a general aggregation operator}\}. \tag{5.1}$$

The inequality $\le_{\mathbb{L}}$ on $\mathbb{L}$ is defined as follows: for all $\mu_1,\ldots,\mu_n\in[0,1]^X$

$$\tilde{A}_1(\mu_1,\ldots,\mu_n)\le_{\mathbb{L}}\tilde{A}_2(\mu_1,\ldots,\mu_n)\Longleftrightarrow$$

$$\Longleftrightarrow \tilde{A}_1(\mu_1,\ldots,\mu_n)(x) \le \tilde{A}_2(\mu_1,\ldots,\mu_n)(x), \;\; x \in X.$$

For the supremum $\tilde{A}_1 \vee_{\mathbb{L}} \tilde{A}_2$ and infimum $\tilde{A}_1 \wedge_{\mathbb{L}} \tilde{A}_2$ of two elements $\tilde{A}_1, \tilde{A}_2 \in \mathbb{L}$ we have the following formulae for all $\mu_1,\ldots,\mu_n \in [0,1]^X$ and $x \in X$:

$$(\tilde{A}_1 \vee_{\mathbb{L}} \tilde{A}_2)(\mu_1,\ldots,\mu_n)(x) = \tilde{A}_1(\mu_1,\ldots,\mu_n)(x) \vee \tilde{A}_2(\mu_1,\ldots,\mu_n)(x),$$

$$(\tilde{A}_1 \wedge_{\mathbb{L}} \tilde{A}_2)(\mu_1,\ldots,\mu_n)(x) = \tilde{A}_1(\mu_1,\ldots,\mu_n)(x) \wedge \tilde{A}_2(\mu_1,\ldots,\mu_n)(x).$$

The top and the bottom elements $1_{\mathbb{L}}$ and $0_{\mathbb{L}}$ respectively are

$$1_{\mathbb{L}}(\mu_1,\ldots,\mu_n) = \begin{cases} \tilde{1}, & \exists i \in \{1,\ldots,n\}\ \mu_i \ne \tilde{0}, \\ \tilde{0}, & \mu_1 = \ldots = \mu_n = \tilde{0}, \end{cases}$$

and

$$0_{\mathbb{L}}(\mu_1,\ldots,\mu_n) = \begin{cases} \tilde{0}, & \exists i \in \{1,\ldots,n\}\ \mu_i \ne \tilde{1}, \\ \tilde{1}, & \mu_1 = \ldots = \mu_n = \tilde{1}. \end{cases}$$

It is easy to see, that $\mathbb{L}$ is infinitely distributive lattice.

Let us have a left-continuous t-norm T, the corresponding residuum $\overrightarrow{T}$, and a T-fuzzy equivalence relation E. We define operators $u\colon \mathbb{L} \to \mathbb{L}$ and $l\colon \mathbb{L} \to \mathbb{L}$ as follows:

$$(u(\tilde{A}))(\mu_1,\ldots,\mu_n)(x) = \sup_{x' \in X} T(E(x,x'), \tilde{A}(\mu_1,\ldots,\mu_n)(x')), \tag{5.2}$$

$$(l(\tilde{A}))(\mu_1,\ldots,\mu_n)(x) = \inf_{x' \in X} \overrightarrow{T}(E(x,x') | \tilde{A}(\mu_1,\ldots,\mu_n)(x')). \tag{5.3}$$

Theorem 5.2.1. *Let $\mathbb{L}$ be a lattice defined by (5.1), $\mathbb{M}$ be a one-point lattice, $u\colon \mathbb{L} \to \mathbb{L}$ and $l\colon \mathbb{L} \to \mathbb{L}$ be operators defined by (5.2) and (5.3) respectively. Then $(\mathbb{L},u,l)$ is a •-approximate system.*

Proof. We should show that properties $(1u)-(4u)$ and $(1l)-(4l)$ hold:
$(1u)$ We show that $(u(0_{\mathbb{L}}))(\mu_1,\ldots,\mu_n) = 0_{\mathbb{L}}(\mu_1,\ldots,\mu_n)$ for all $\mu_1,\ldots,\mu_n \in [0,1]^X$. If there exists $i \in \{1,\ldots,n\}$ such that $\mu_i \ne \tilde{1}$, then for all $x \in X$ it holds

$$(u(0_{\mathbb{L}}))(\mu_1,\ldots,\mu_n)(x) = \sup_{x' \in X} T(E(x,x'), \tilde{0}(x')) = \tilde{0}(x).$$

If $\mu_i = \tilde{1}$ for all $i \in \{1,\ldots,n\}$, then for all $x \in X$ it holds

$$(u(0_{\mathbb{L}}))(\mu_1,\ldots,\mu_n)(x) = \sup_{x' \in X} T(E(x,x'), \tilde{1}(x')) = \tilde{1}(x).$$

$(2u)$ We prove that $\tilde{A} \leq u(\tilde{A})$ for all $\tilde{A} \in \mathbb{L}$. For all $\mu_1, \dots, \mu_n \in [0,1]^X$ and for all $x \in X$ we have

$$(u(\tilde{A}))(\mu_1, \dots, \mu_n)(x) = \sup_{x' \in X} T(E(x,x'), \tilde{A}(\mu_1, \dots, \mu_n)(x')) \geq$$

$$\geq T(E(x,x), \tilde{A}(\mu_1, \dots, \mu_n)(x)) = \tilde{A}(\mu_1, \dots, \mu_n)(x).$$

$(3u)$ We show that $u(\tilde{A}_1 \vee_{\mathbb{L}} \tilde{A}_2) = u(\tilde{A}_1) \vee_{\mathbb{L}} u(\tilde{A}_2)$ for all $\tilde{A}_1, \tilde{A}_2 \in \mathbb{L}$. For all $x \in X$ and for all $\mu_1, \dots, \mu_n \in [0,1]^X$ it holds

$$(u(\tilde{A}_1 \vee_{\mathbb{L}} \tilde{A}_2))(\mu_1, \dots, \mu_n)(x) = \sup_{x' \in X} T(E(x,x'), (\tilde{A}_1 \vee_{\mathbb{L}} \tilde{A}_2)(\mu_1, \dots, \mu_n)(x')) =$$

$$= \sup_{x' \in X} [T(E(x,x'), \tilde{A}_1(\mu_1, \dots, \mu_n)(x')) \vee T(E(x,x'), \tilde{A}_2(\mu_1, \dots, \mu_n)(x'))] =$$

$$= \sup_{x' \in X} T(E(x,x'), \tilde{A}_1(\mu_1, \dots, \mu_n)(x')) \vee \sup_{x' \in X} T(E(x,x'), \tilde{A}_2(\mu_1, \dots, \mu_n)(x')) =$$

$$= (u(\tilde{A}_1))(\mu_1, \dots, \mu_n)(x) \vee (u(\tilde{A}_2))(\mu_1, \dots, \mu_n)(x).$$

$(4u)$ It is necessary to show that $u(u(\tilde{A})) = u(\tilde{A})$ for all $\tilde{A} \in \mathbb{L}$. In order to prove this property, we will show that inequalities $u(u(\tilde{A})) \leq u(\tilde{A})$ and $u(u(\tilde{A})) \geq u(\tilde{A})$ hold. The second inequality holds by property $(2u)$. Let us prove that for all $\mu_1, \dots, \mu_n \in [0,1]^X$ and for all $x \in X$

$$(u(u(\tilde{A})))(\mu_1, \dots, \mu_n)(x) \leq (u(\tilde{A}))(\mu_1, \dots, \mu_n)(x).$$

Taking into account, that the result of applying u is an extensional fuzzy sets, we obtain

$$(u(u(\tilde{A})))(\mu_1, \dots, \mu_n)(x) =$$

$$= \sup_{x' \in X} T(E(x,x'), (u(\tilde{A}))(\mu_1, \dots, \mu_n)(x')) \leq (u(\tilde{A}))(\mu_1, \dots, \mu_n)(x)$$

for all $\mu_1, \dots, \mu_n \in [0,1]^X$ and for all $x \in X$.

$(1l)$ We show that $(l(1_{\mathbb{L}}))(\mu_1, \dots, \mu_n) = 1_{\mathbb{L}}(\mu_1, \dots, \mu_n)$ for all $\mu_1, \dots, \mu_n \in [0,1]^X$. If there exists $i \in \{1, \dots, n\}$ such that $\mu_i \neq \tilde{0}$, then for all $x \in X$ it holds

$$(l(1_{\mathbb{L}}))(\mu_1, \dots, \mu_n)(x) = \inf_{x' \in X} \overrightarrow{T}(E(x,x') | \tilde{1}(x')) = \tilde{1}(x).$$

If $\mu_i = \tilde{0}$ for all $i \in \{1, \dots, n\}$, then for all $x \in X$ it holds

$$(u(1_{\mathbb{L}}))(\mu_1, \dots, \mu_n)(x) = \inf_{x' \in X} \overrightarrow{T}(E(x,x') | \tilde{0}(x')) = \tilde{0}(x).$$

$(2l)$ We show that $\tilde{A} \geq l(\tilde{A})$ for all $\tilde{A} \in \mathbb{L}$. For all $\mu_1, \ldots, \mu_n \in [0,1]^X$ and for all $x \in X$ we have

$$(l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x) = \inf_{x' \in X} \overrightarrow{T}(E(x,x')|\tilde{A}(\mu_1, \ldots, \mu_n)(x')) \leq$$
$$\leq \overrightarrow{T}(E(x,x)|\tilde{A}(\mu_1, \ldots, \mu_n)(x)) = \tilde{A}(\mu_1, \ldots, \mu_n)(x).$$

$(3l)$ We prove that $l(\tilde{A}_1 \wedge \tilde{A}_2) = l(A_1) \wedge l(\tilde{A}_2)$ for all $\tilde{A}_1, \tilde{A}_2 \in \mathbb{L}$. For all $x \in X$ and for all $\mu_1, \ldots, \mu_n \in [0,1]^X$ it holds

$$(l(\tilde{A}_1 \wedge_{\mathbb{L}} \tilde{A}_2))(\mu_1, \ldots, \mu_n)(x) = \inf_{x' \in X} \overrightarrow{T}(E(x,x')|(\tilde{A}_1 \wedge_{\mathbb{L}} \tilde{A}_2)(\mu_1, \ldots, \mu_n)(x')) =$$
$$= \inf_{x' \in X} [\overrightarrow{T}(E(x,x')|\tilde{A}_1(\mu_1, \ldots, \mu_n)(x')) \wedge \overrightarrow{T}(E(x,x')|\tilde{A}_2(\mu_1, \ldots, \mu_n)(x'))] =$$
$$= \inf_{x' \in X} \overrightarrow{T}(E(x,x')|\tilde{A}_1(\mu_1, \ldots, \mu_n)(x')) \wedge \inf_{x' \in X} \overrightarrow{T}(E(x,x')|\tilde{A}_2(\mu_1, \ldots, \mu_n)(x')) =$$
$$= (l(\tilde{A}_1))(\mu_1, \ldots, \mu_n)(x) \wedge (l(\tilde{A}_2))(\mu_1, \ldots, \mu_n)(x).$$

$(4l)$ We show that $l(l(\tilde{A})) = l(\tilde{A})$ for all $\tilde{A} \in \mathbb{L}$. In order to prove this property, we will show that inequalities $l(l(\tilde{A})) \leq l(\tilde{A})$ and $l(l(\tilde{A})) \geq l(\tilde{A})$ hold. The first inequality holds by property $(2l)$. Let us prove that for all $\mu_1, \ldots, \mu_n \in [0,1]^X$ and for all $x \in X$

$$(l(l(\tilde{A})))(\mu_1, \ldots, \mu_n)(x) \geq (l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x).$$

By the definitions we have

$$(l(l(\tilde{A})))(\mu_1, \ldots, \mu_n)(x) = \inf_{x' \in X} \overrightarrow{T}(E(x,x')|(l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x')) \text{ and}$$
$$\overrightarrow{T}(E(x,x')|(l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x')) =$$
$$= \sup\{\alpha \in [0,1] \mid T(\alpha, E(x,x')) \leq (l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x')\}.$$

Taking $\alpha = (l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x)$, we obtain

$$T((l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x), E(x,x')) \leq (l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x')\},$$

which holds for all $x, x' \in X$ by the fact, that $(l(\tilde{A}))(\mu_1, \ldots, \mu_n)$ is an extensional fuzzy set with respect to E. It means that for all $x, x' \in X$

$$\overrightarrow{T}(E(x,x')|(l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x')) \geq (l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x)$$

and thus

$$\inf_{x' \in X} \overrightarrow{T}(E(x,x')|(l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x')) \geq (l(\tilde{A}))(\mu_1, \ldots, \mu_n)(x).$$

□

5.3 Construction of $\mathbb{M}$-approximate system using upper and lower general aggregation operators

Let us have a given set X and a left-continuous t-norm T. In order to construct this example of $\mathbb{M}$-approximate system, we take lattice $\mathbb{M}$ as the lattice of all T-fuzzy equivalence relations:

$$\mathbb{M} = \{E\colon X \times X \to [0,1] \mid E \text{ is a T-fuzzy equivalence relation}\}. \tag{5.4}$$

We consider the inequality $\leq_{\mathbb{M}}$ on $\mathbb{M}$ defined as follows:

$$E_1 \leq_{\mathbb{M}} E_2 \Longleftrightarrow E_1(x,y) \leq E_2(x,y) \text{ for all } x,y \in X.$$

For the supremum $\tilde{E}_1 \vee_{\mathbb{M}} E_2$ and infimum $E_1 \wedge_{\mathbb{M}} E_2$ of two elements $E_1, E_2 \in \mathbb{M}$ we have the following formulae for all $x,y \in X$:

$$(E_1 \vee_{\mathbb{M}} E_2)(x,y) = \bigwedge\{E(x,y) \mid E \text{ is a T-fuzzy equiv. rel. such that } E \geq_{\mathbb{M}} E_1 \;\&\; E \geq_{\mathbb{M}} E_2\},$$

$$(E_1 \wedge_{\mathbb{M}} E_2)(x,y) = E_1(x,y) \wedge E_2(x,y).$$

The top and the bottom elements $1_{\mathbb{M}}$ and $0_{\mathbb{M}}$ respectively are

$$1_{\mathbb{M}}(x,y) = 1 \text{ and}$$

$$0_{\mathbb{M}}(x,y) = \begin{cases} 0, & x \neq y, \\ 1, & x = y, \end{cases}$$

for all $x,y \in X$. It is easy to see, that $\mathbb{M}$ is complete lattice.
Approximation operators $u\colon \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ and $l\colon \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ in this case are defined as follows:

$$(u(\tilde{A},E))(\mu_1,\ldots,\mu_n)(x) = \sup_{x' \in X} T(E(x,x'), \tilde{A}(\mu_1,\ldots,\mu_n)(x')), \tag{5.5}$$

$$(l(\tilde{A},E))(\mu_1,\ldots,\mu_n)(x) = \inf_{x' \in X} \overrightarrow{T}(E(x,x') | \tilde{A}(\mu_1,\ldots,\mu_n)(x')). \tag{5.6}$$

Theorem 5.3.1. *Let $\mathbb{L}$ be a lattice defined by (5.1), $\mathbb{M}$ be a lattice defined by (5.4), $u\colon \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ and $l\colon \mathbb{L} \times \mathbb{M} \to \mathbb{L}$ be operators defined by (5.5) and (5.6) respectively. Then $(\mathbb{L}, u, l)$ is an M-approximate system.*

Proof. One could prove properties $(1u)-(4u)$ and $(1l)-(4l)$ analogously to the Theorem 5.2.1. Now we will show that properties $(5u)-(6u)$ and $(5l)-(6l)$ hold.
$(5u)$ We prove that for all $E_1, E_2 \in \mathbb{M}$

$$E_1 \leq E_2 \Longrightarrow u(\tilde{A}, E_1) \leq u(\tilde{A}, E_2) \text{ for all } \tilde{A} \in \mathbb{L}.$$

The following holds for all $\mu_1, \ldots, \mu_n \in [0,1]^X$ and for all $x \in X$:

$$E_1(x,x') \leq E_2(x,x'),\;\; x' \in X \Longrightarrow$$

$$\Longrightarrow \sup_{x' \in X} T(E_1(x,x'), \tilde{A}(\mu_1, \ldots, \mu_n)(x')) \leq \sup_{x' \in X} T(E_2(x,x'), \tilde{A}(\mu_1, \ldots, \mu_n)(x')) \Longrightarrow$$
$$\Longrightarrow (u(\tilde{A}, E_1))(\mu_1, \ldots, \mu_n)(x) \leq (u(\tilde{A}, E_2))(\mu_1, \ldots, \mu_n)(x).$$

$(6u)$ Now we show, that $u(\tilde{A}, 0_{\mathbb{M}}) = \tilde{A}$ for all $\tilde{A} \in \mathbb{L}$. For all $\mu_1, \ldots, \mu_n \in [0,1]^X$ and for all $x \in X$ it holds:

$$(u(\tilde{A}, 0_{\mathbb{M}}))(\mu_1, \ldots, \mu_n)(x) = \sup_{x' \in X} T(0_{\mathbb{M}}(x,x'), \tilde{A}(\mu_1, \ldots, \mu_n)(x')) = \tilde{A}(\mu_1, \ldots, \mu_n)(x).$$

$(5l)$ We show that for all $E_1, E_2 \in \mathbb{M}$

$$E_1 \leq E_2 \Longrightarrow l(\tilde{A}, E_1) \geq l(\tilde{A}, E_2) \text{ for all } \tilde{A} \in \mathbb{L}.$$

The following holds for all $\mu_1, \ldots, \mu_n \in [0,1]^X$ and for all $x \in X$:

$$E_1(x,x') \leq E_2(x,x'),\;\; x' \in X \Longrightarrow$$

$$\Longrightarrow \inf_{x' \in X} \overrightarrow{T}(E_1(x,x') | \tilde{A}(\mu_1, \ldots, \mu_n)(x')) \geq \inf_{x' \in X} \overrightarrow{T}(E_2(x,x') | \tilde{A}(\mu_1, \ldots, \mu_n)(x')) \Longrightarrow$$
$$\Longrightarrow (l(\tilde{A}, E_1))(\mu_1, \ldots, \mu_n)(x) \geq (l(\tilde{A}, E_2))(\mu_1, \ldots, \mu_n)(x).$$

$(6l)$ And finally, we prove that $l(\tilde{A}, 0_{\mathbb{M}}) = \tilde{A}$ for all $\tilde{A} \in \mathbb{L}$. For all $\mu_1, \ldots, \mu_n \in [0,1]^X$ and for all $x \in X$ it holds:

$$(l(\tilde{A}, 0_{\mathbb{M}}))(\mu_1, \ldots, \mu_n)(x) = \inf_{x' \in X} \overrightarrow{T}(0_{\mathbb{M}}(x,x') | \tilde{A}(\mu_1, \ldots, \mu_n)(x')) = \tilde{A}(\mu_1, \ldots, \mu_n)(x).$$

□

In this section we described approximate systems using upper and lower general aggregation operators. A different construction of approximate systems for aggregation operators could be obtained, when we specify properties of general aggregation operators and thus describe another lattice $\mathbb{L}$.

Conclusions

The research presented in the thesis shows, that suggested constructions of general aggregation operators are consistent within the theory of aggregation functions. In our work we have obtained the following results:

- an aggregation operator acting on fuzzy sets, which takes into account an equivalence relation between these fuzzy sets, has been constructed and has been applied for analysis of solving parameters, in order to choose optimal solution of bilevel linear programming problems;
- a construction of general aggregation operators based on a fuzzy equivalence relation has been described and the properties of these operators have been investigated;
- the case when inputs of the general aggregation operators are in the form of fuzzy real numbers has been studied;
- an approximate system based on upper and lower general aggregation operators has been constructed.

I believe that the overall goal of the thesis has been achieved. This research is relevant since it deals with aggregation of fuzzy structures, which arises in different research areas. Some of the possible applications have been provided in the thesis. The proposed constructions could potentially contribute to theoretical investigations in the areas of aggregation functions and fuzzy mathematics. The research presented in the theses is considered as completed work, however, it could be continued in several directions. For example, one could consider the case, when instead of the ordinary fuzzy sets $\mathbb{L}$-sets are used, where $\mathbb{L}$ is an arbitrary lattice. Also, different relations between the objects, other than equivalence relation, could be involved in the proposed constructions.

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