On Grinbergs’ differential geometry and finite fields

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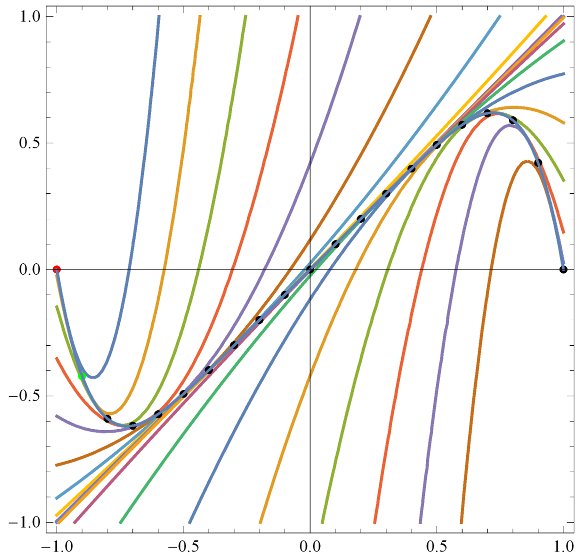
Abstract

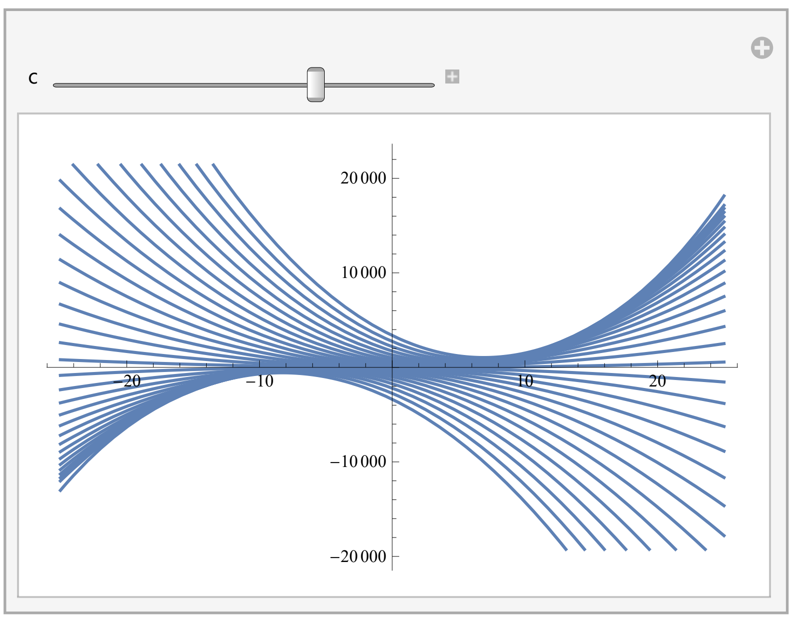
Emanuels Grinbergs, in his youth, during ten years, from 1933 until 1943, wrote three dissertations on one subject, namely, differential geometry [1, 2, 3]. We think that his work in this direction has been neglected for many years, and it is the last time to try to understand the significance of these works. Here, in this short article we touch only one aspect of this work, and compare and put together two approaches, one from thesis of Grinbergs [3], and another, of the author’s, [5, 6], where we show close relation between both.

Introduction

Emanuels Grinbergs, in his youth, during ten years, from 1933 until 1943, wrote three dissertations on one subject, namely, differential geometry [1, 2, 3]. We think that his work in this direction has been neglected for many years, and, what concerns mathematicians, it is the last time to try to understand the significance of his works in these subjects.

Here, in this short article we touch only one aspect of this work, and compare and put together two approaches, one from thesis of Grinbergs [3], and another, of the author’s, [5, 6], where we show close relation between both. Grinbergs puts in the basis of his analysis of osculations dual structure of curves, points on curves and osculating surfaces, forcing to work this duality with respect to points on curves and touching/-contacting surfaces. Next to Grinbergs work, we consider authors attempt to build theory of foliations in finite field, because of the amazing similarity in both these approaches, Grinbergs’ dual structure and D. Zeps permatrices in finite fields. Actually, the only aspect of being this thing amazing is that D. Zeps found his structure in his scientific leader’s Emanuels Grinbergs work of youth.





The basic structure in Grinbergs' differential geometry. Short insight

Let us consider [4]. First, f(x, a) = 0 in formula (3) on page 4. In our example it is $f(x) + (r - x)^3$, with r in place of a, equations of osculations. Further, parametric equations of curve L, equations x=t, and y=a+b t+c $t^2 + t^3$, that would correspond to formula (4) from [4].

$$\begin{cases} F(t_1, a) = 0 \\ F(t_2, a) = 0 \\ \ldots \, .. \\ F(t_p, a) = 0 \end{cases}$$

To get these values, $F(t_i, a)$, Grinbergs uses formula F(t, a) =f [x(t), a]. But, what this means? Parametric values are inserted by fixed parameters a_j, or somewhat else? In first case we were to move along fixed surface, corresponding to values of these a_j. But this situation might suit Grinbergs, because he wants to work in infinitesimally small neighborhood, and then he captures in this single surface small neighborhood, and asks whether it has in limit point X both on curve L and in surface A. For that purpose he takes one more system of equations, formula (7) in [4]:

$$\begin{cases} F(t_0, a) = 0 \\ F'(t_0, a) = 0 \\ \dots .. \\ F^{(p-1)}(t_0, a) = 0 \end{cases}$$

Following G.Julia argument, G. Julia, Éléments de géométrie infinitésimale (Paris, Gauthier-Villars), 1927, 19. 20.pp, he argues that solution of these systems with p giving $F^{(p)}(t_0, \text{a}) \neq 0$ is sufficient condition to conclude that surface A and curve L has contact/tangency of order p-1. To look into approach of Grinbergs, we cite him here:

- The fact that expressions (7) hold and besides (8) $F^{(p)}(t_0, \text{a}) \neq 0$, we are to express in three different ways: a) surface A goes through p infinitely closed points (of curve L), that coincide with the point X_0; b) the surface A and curve L has in the point X_0 contact/tangency of order p-1; c) solving the system of equations of (3) and (4) with respect to unknown x_i and t, just p systems of solutions coincide with the system composed from coordinates of X_0 and t_0, i.e., this system is p-fold solution of the system of equations. The last statement we are to use also in the case when all $\frac{\partial f}{\partial x_i}$ vanish.

Grinbergs wants to work in infinitesimally small neighborhoods. But we want to look wider, to consider to what extent this Grinbergs approach, (maybe somewhat modified?), might work also in case not only confined to infinitesimallity.
Surprisingly, the first look shows that Grinbergs cited argument works well not only considering point series in limit having one single value, but for any series of points. We simply drop the condition to connect number p with order of contact/tangency. Of course, in any case, we may return to arguments of Grinbergs in case we wanted to look on tangency too, and connecting then this with value of p in this case.

Grinbergs' approach in the case of not infinite smallness

Let us check cited argument of Grinbergs for our case dropping infinite smallness. We want to do, as with function f(x,a), that connects points X_i of curve L, the same for surfaces A_j too, to force to work in other direction too, namely, surfaces A_j being parametrized similarly as points X_i previously. We demonstrate our approach on the example from [5].
Once more we are to consider citation from [4], page 5, and point c):
c) solving the system of equations of (3) and (4) with respect to unknown x_i and t, just p systems of
solutions coincide with the system composed from coordinates of X_0 and t_0,
i.e., this system is p-fold solution of the system of equations.
It is notably that here we don't have mentioning of infinite smallness, and quite justified. It would work in any case. Now it isn't necessary to confine number of points p to something in this picture, it may be any.

How Grinbergs' argument works?
To understand this we are to build correctly f from f(x,a)=0. Following [5], this might be $f(x) + (r - x)^3$ with respect to f(x) as curve, or (a+d r^3)+(b-3 d r^2)x+(c+3 d r) x^2, that we are to demonstrate further, in another article. Lower we show this with *Mathematica* example, where we use *Mathematica* Solve. In first line of Solve we left condition Reals, giving imaginary solutions too. But, for what we do here, this is not needed, of course:

```
Clear[r, a, b, c, x, y, t]
f[x_] := a + b x + c x^2 + x^3
g[x_, r_] := f[x] + (r - x) ^3
f::usage = "Solution following point c): all solutions "
Solve[{a + b x + c x^2 + x^3 + (r - x) ^3 == 0 && x == t && y == a + b t + c t^2 + t^3}, {x, y, r}]
f::usage = "Only real solutions left "
Solve[{a + b x + c x^2 + x^3 + (r - x) ^3 == y && x == t && y == a + b t + c t^2 + t^3}, {x, y, r}, Reals]
```

Solution following point c): all solutions

$$\left\{\left\{x \to t,\ y \to a + b\,t + c\,t^2 + t^3,\ r \to t + \left(-a - b\,t - c\,t^2 - t^3\right)^{1/3}\right\},\ \left\{x \to t,\ y \to a + b\,t + c\,t^2 + t^3,\ r \to t - \frac{1}{2}\left(1 - i\sqrt{3}\right)\left(-a - b\,t - c\,t^2 - t^3\right)^{1/3}\right\},\ \left\{x \to t,\ y \to a + b\,t + c\,t^2 + t^3,\ r \to t - \frac{1}{2}\left(1 + i\sqrt{3}\right)\left(-a - b\,t - c\,t^2 - t^3\right)^{1/3}\right\}\right\}$$

Only real solutions left

$$\{\{x \to t,\ y \to a + b\,t + c\,t^2 + t^3,\ r \to t\}\}$$

We receive solution r→t, and $y \to a + b\,t + c\,t^2 + t^3$, i.e., parametric equations of surfaces from family S.

For our small example from [5] we could do all matter without using function Solve, but with simple replacements.

```
Clear[r, y, x, t, a, b, c]
f::usage = "Excluding x from f[x,r] "
a + b x + c x^2 + x^3 + (r - x) ^3 == y /. x → t
f::usage = "Excluding, thus,both x[t] and y[t] from f[x,r] "
%% /. y -> a + b x + c x^2 + x^3 /. x → t
% // Simplify
```

Excluding x from f[x,r]

$$a + (r - t)^3 + b\,t + c\,t^2 + t^3 == y$$

Excluding, thus,both x[t] and y[t] from f[x,r]

$$a + (r - t)^3 + b\,t + c\,t^2 + t^3 == a + b\,t + c\,t^2 + t^3$$

$$r == t$$

Here, action performed by Grinbergs looks like this, giving desired result, r=t, parametric equation, or remnant of it, of surfaces from family S.

Grinbergs’ bundle of osculating surfaces

In §2 Grinbergs shows that points and surfaces may be exchanged in their places in his basis equation: i.e., in formula (37) $f(x_1, x_2, ..., x_n, a_1, , a_2, ..., a_N) = 0$ quantities a_j may be parametrized, similarly as x_i before. Now all picture turns around, surfaces take place of points and reversely. Next, to see that in this way we receive bundle structure in osculations, we are to introduce Lie group action in corresponding differential geometric invariants. Grinbergs is doing in the §6, The simplification of computations in case of fundamental group.

Trimes, perms and permatrices

In [5, 6] we worked in finite fields with order p that gives one to one mapping of x^3. These numbers we called trimes, as opposed to primes, these numbers are 2, 3, 12k+5, 12k+11 and their multiplications without repetitions, integer sequence A074243 [10].

In [5, 6] we generated permutations of integers of trime length from cubic functions, and these permutations we called perms. Along with cubic functions we received their osculating functions, introducing parameter r, following pattern as follows, for curve f(x) we built its osculating curves $f(x) + (r - x)^3$. For a fixed function f(x) we received p differing perms. Altogether, at p as a trime, we received p perms of length p, thus, p×p matrix of integers. Such matrix, where its rows are perms, we called permatrix.
Further we discovered that in this permatrix two groups are acting, along rows and columns, correspondingly. This fact may be interpreted that in the osculating functions of the given origin function bundle structure is working, where as base manifold some/any osculating function is to be taken.

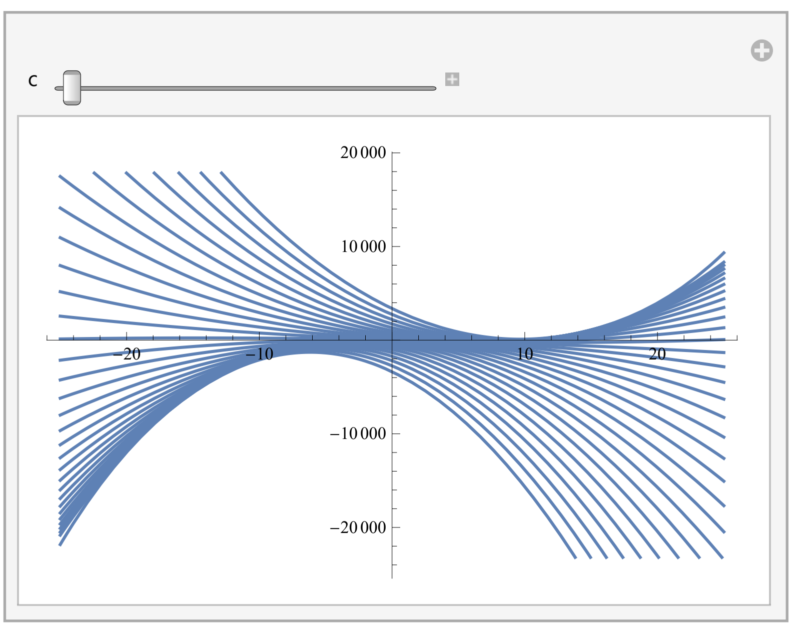
Higher we tried to show that Grinbergs developed just this bundle structure of osculating surfaces, demonstrating duality between surfaces and points of touch/contact/tangency on curves.

The idea of matrices I call permatrices I took from Norman J. Wildberger [11], when I was listening to his lectures in algebraic topology and differential geometry. Behind all this is Lagrange approach to calculus, but about this in some other article.

Grinbergs differential geometry and Tait-Kneser theorem

Next question is about Tait-Kneser theorem [8, 9], did Grinbergs know of some analogue of it? To find answer, we are to analyze Grinbergs work [4] further.
Let us look on some pictures we may easily display in our time, and that was not possible in, say, year 1943 and many years later during the life of E. Grinbergs.

```
Clear[c];
Manipulate[
 Plot[Append[Table[1 - 2 x + c x^2 + x^3 + (k - x)^3, {k, -15, 15}], 1 - 2 x + c x^2 + x^3],
  {x, -25, 25}], {c, -10, 10}]
```



```
Clear[t, x, y]
f[x_] := x - x^5
f::usage = "Equation of  envelope to Osculating surfaces"
eq = y == f[x]
f::usage = "Number of points on curve L"
m = 10; n = m; Δ = 0.01 m;  2 m + 1
p1 = Table[{x0 = Δ k; x0, f[x0]}, {k, -m, m}];
f::usage = "Order of osculating surfaces: p"
p = 2
t1 = Table[Normal[Series[y == f[x], {x, Δ k, p}]], {k, -m, m}];
f::usage = "A sample of osculating surface: x0=Δ"
t1[[m + 2]] // Simplify // N
Append[t1, eq];
Show[ContourPlot[%, {x, -m / n, m / n}, {y, -m / n, m / n}, Axes → True],
Graphics[{PointSize[0.015], Point[p1, VertexColors → {Red, Green}]}], ContourPlot[
  y == f[x], {x, -m / n, m / n}, {y, -m / n, m / n}, Axes → True, ContourStyle → Thickness[0.005]]]
```

Equation of envelope to Osculating surfaces

$y == x - x^5$

Number of points on curve L

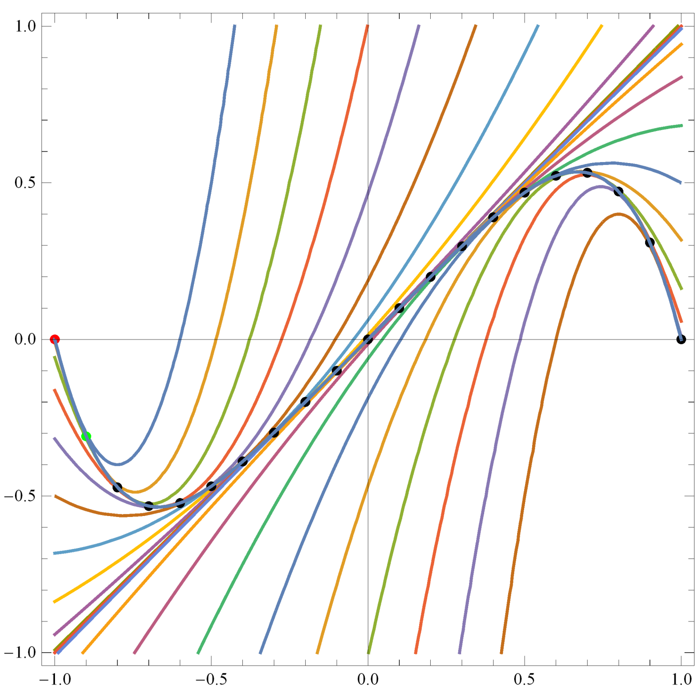
21

Order of osculating surfaces: p

2

A sample of osculating surface: x0=Δ

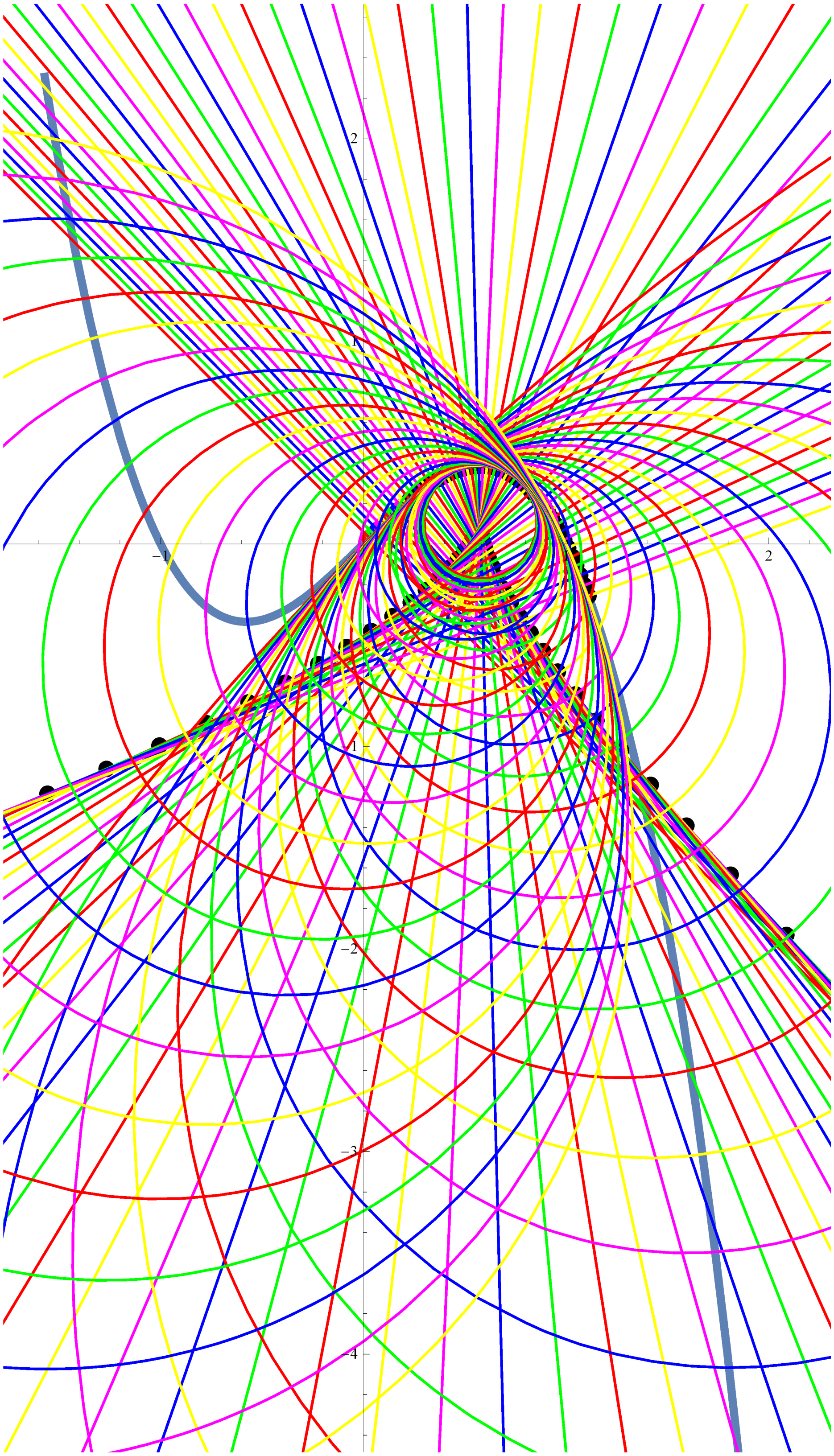
$y == -0.00006 + 1.0015\,x - 0.01\,x^2$



```
x1[t_] := t; x2[t_] := -t^3 + t;
f::usage = "The given curve:"
γ[t]
f::usage = "Curvature:"
cc = κ[t] // Simplify // N;
f::usage = "Series of m Points X with step Δ: cooridinate x"
Δ = 0.02; mm = 55; tdx = Table[Δ (k + 1 / 2), {k, 1, mm}];
tdxy = Table[{tdx[[k]], x2[tdx[[k]]]}, {k, 1, Length[tdx]}];
f::usage = "Equations of normal lines via points X"
nnn = Table[x10 = tdx[[k]]; x20 = x2[tdx[[k]]];
   tFamily[-y, x, x10] - tFamily[-x20, x10, x10] == 0, {k, 1, Length[tdx]}];
f::usage = "The centers of curvature at points X"
tc = Q /@ tdx // Simplify // N;
f::usage = "Equations of circles at points X"
ttt = (N[(x - Q[#][[1]]) ^2 + (y - Q[#][[2]]) ^2] == N[1/(κ[#] ^2)]) & /@ tdx;
```

```
The given curve:
{t, t - t^3}
Curvature:
Series of m Points X with step Δ: cooridinate x
Equations of normal lines via points X
The centers of curvature at points X
Equations of circles at points X
```

```
m = 5;
Show[ParametricPlot[{{x1[t], x2[t]}}, {t, - Pi / 2, 2 Pi / 3}, PlotStyle → Thickness[0.01]],
 Graphics[{PointSize[0.02], Point[Join[tdxy, tc],
    VertexColors → {Red, Green, Blue, Magenta, Yellow, Red, Green, Blue, Magenta, Yellow}]},
  PlotRange → All], ContourPlot[nnn, {x, -m, m}, {y, -m, m},
  ContourStyle → {Red, Green, Blue, Magenta, Yellow}],
 ContourPlot[ttt, {x, -m, m}, {y, -m, m},
  ContourStyle → {Red, Green, Blue, Magenta, Yellow}], AspectRatio → Automatic]
```


2
1
−1
2
−1
−2
−3
−4

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```
κ[t_] := (x1'[t] x2''[t] - x1''[t] x2'[t]) / (x1'[t]^2 + x2'[t]^2)^(3/2)
x1[t_] := t; x2[t_] := - t^5 / (3 + 4 t^2);
γ[t_] := {x1[t], x2[t]}
τ[t_] := x2'[t]
tFamily[x_, y_, t_] := x τ[t] - y;

Q[t_] := γ[t] + 1 / (κ[t] (x1'[t]^2 + x2'[t]^2)^(1/2)) {-x2'[t], x1'[t]}
```