UNIQUELY HAMILTONIAN GRAPHS

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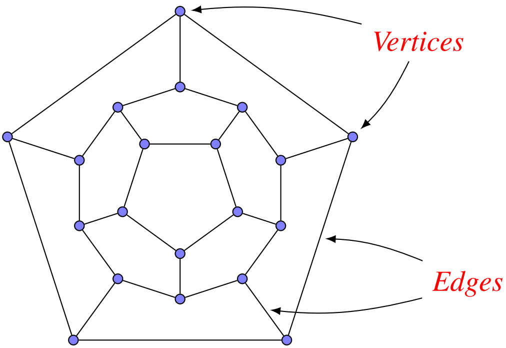
A talk in three parts

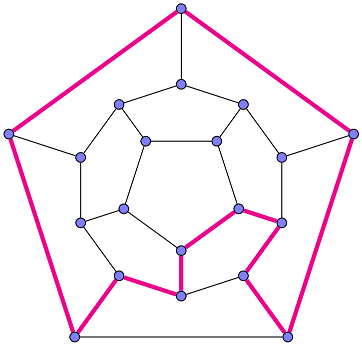
1. Definitions and Prehistory
2. Sheehan's Conjecture
3. UH3 graphs

THE DODECAHEDRON



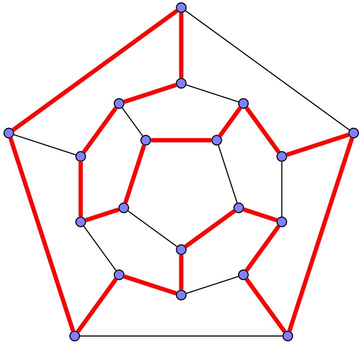
The dodecahedron is a *cubic* and *planar* graph.

CYCLES



A *cycle* is a *circular* sequence of vertices $(v_0, v_1, \ldots, v_{k-1})$, each adjacent to the next.

CYCLES



A *Hamilton cycle* is a cycle that uses all of the vertices of the graph.

Sir William Rowan Hamilton (1805–1865)

Famously invented *quaternions*, but also the "*Icosian Game*".

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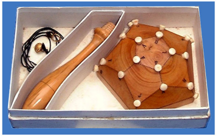
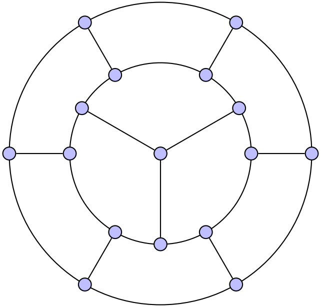
(*Also available in handy portable travel-set*)



4CC – THE FOUR-COLOUR CONJECTURE

CONJECTURE (Guthrie,1850s)
The *faces* of a *cubic planar graph* can be coloured with 4 colours, so that neighbouring “countries” never have the same colour.



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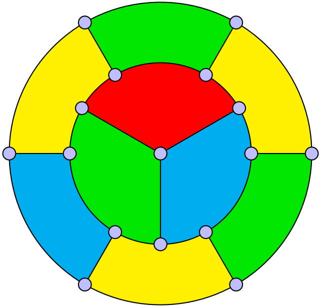
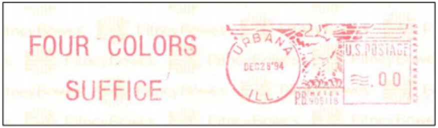


THE FOUR-COLOUR CONJECTURE

The four-colour conjecture:

- dominated graph theory until the 1970s
- consumed numerous academic careers
- catalysed the introduction of a vast range of tools
- caused a furore when resolved in 1976



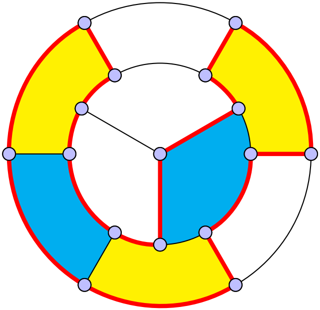
PETER GUTHRIE TAIT (1831–1901)

TAIT'S CONJECTURE

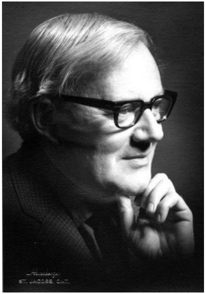
Every 3-connected *cubic planar* graph has a Hamilton cycle.

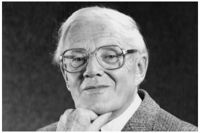
This conjecture is *stronger than* the 4CC — a Hamilton cycle can be used to *find* a 4-colouring of the faces.

Hamilton cycle to face-colouring

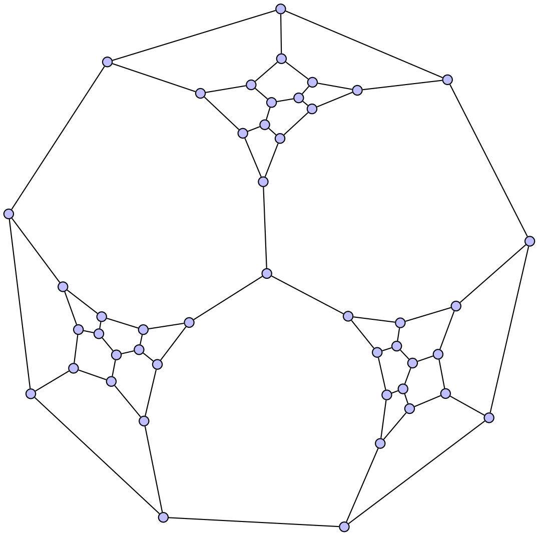


TUTTE — THE MODEST GIANT OF COMBINATORICS



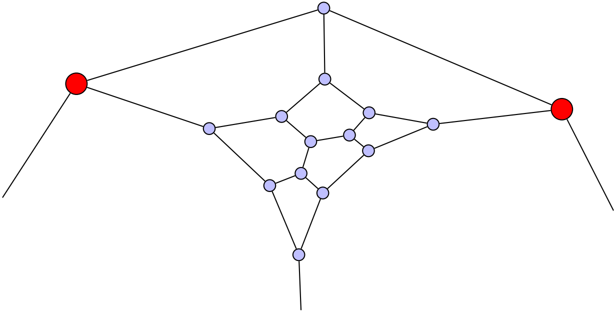


Tutte disproves Tait



But why is this non-hamiltonian?

Look at one of the three identical pieces of the graph.

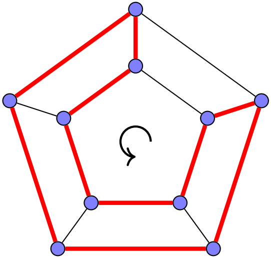


Case analysis shows no Hamilton *path* connecting the red vertices.

Smith's result

Theorem (Smith)

Any edge in a cubic graph lies in an *even number* of Hamilton cycles.



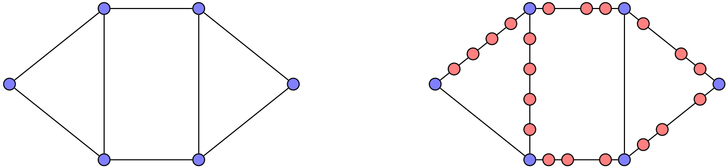
In this example, each *rim edge* lies in 4 hamilton cycles, and each *spoke edge* lies in 2.

So a Hamiltonian cubic graph has *at least three* Hamilton cycles.

Part II: Sheehan’s Conjecture

Uniquely Hamiltonian graphs

A graph is *uniquely Hamiltonian* if it has *exactly one* Hamilton cycle.



Vertices of degree 2 are cheating (at least, uninteresting).

We want *uniquely hamiltonian* graphs with *minimum degree* at least 3, or *UH3 graphs* for short.

Fundamental question

Question

Which graphs *can*, or *cannot*, be UH3 graphs?

Over the last decades, a steady trickle of papers have provided partial answers . . ..

. . . but major questions remain unresolved.

Sheehan's Conjecture

The most famous is *Sheehan's conjecture*.

Conjecture (John Sheehan, 1975)

There are no uniquely hamiltonian 4-*regular* graphs

Andrew and Carsten

Progress to date is largely due to these two eminent mathematicians.

Carsten Thomassen

Andrew Thomason

Thomason's lollipops

A wonderful result that takes the most *modest of ingredients*:

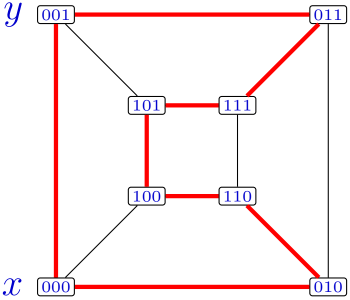
"Any graph has an even number of vertices of odd degree"

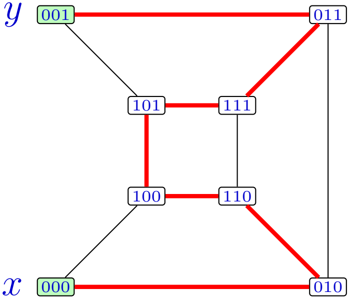
and turns it into something with far-reaching consequences.

Image: Steve Greenberg

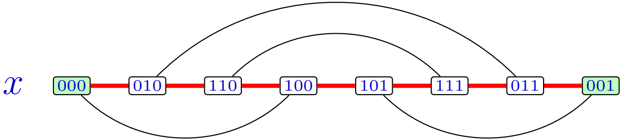
Start with an edge

Take a cubic graph with a Hamilton cycle through an edge $e = xy$, and then delete e.

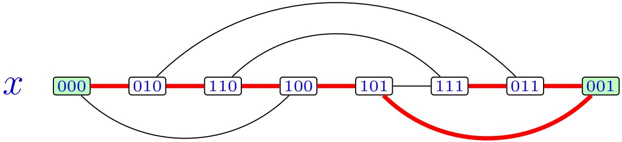




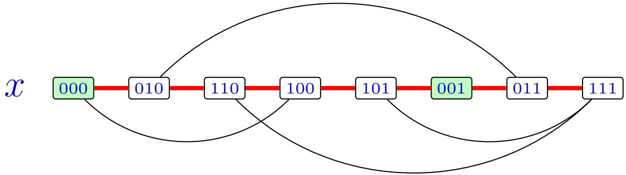
From path to path



Change the Hamilton path by using the “other edge” through y.



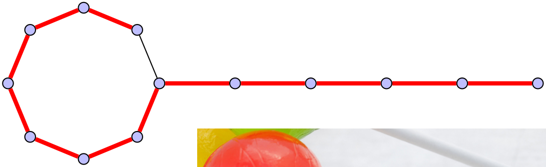
Redraw and repeat

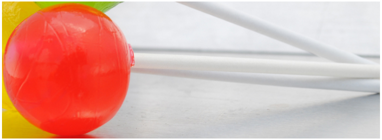


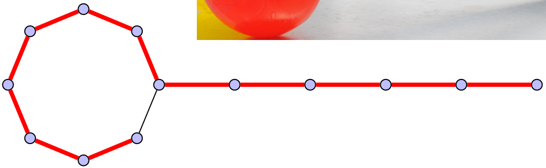
Now two choices — one reverses what we just did, the other moves to a new Hamilton path.

When can this process end?

Why lollipop?







REGULAR UH GRAPHS

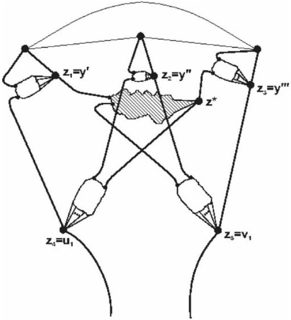
Thomassen adapted Thomason's lollipop method to a

- (*Thomason*) No odd-regular UH-graphs
- (*Thomassen*) No $\geqslant 300$-regular UH-graphs
- (*Haxell, Seamone, Verstraëte*) No $\geqslant 22$ regular UH-graph

Also *no solution* to Sheehan's conjecture!

A UH4 GRAPH!

Sheehan's conjecture seems *almost* self-evidently true . . .



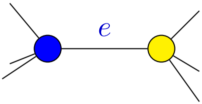
. . . but Herb Fleischner has constructed a *non-regular* UH4 graph.

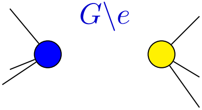
Part III: UH3 Graphs

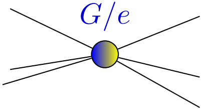
Another Thomassen Conjecture

Conjecture (Thomassen)

In a *hamiltonian graph* G of minimum degree at least 3, there exists at least one edge e such that both the *deletion* $G\backslash e$ and *contraction* G/e are hamiltonian.







Our interest

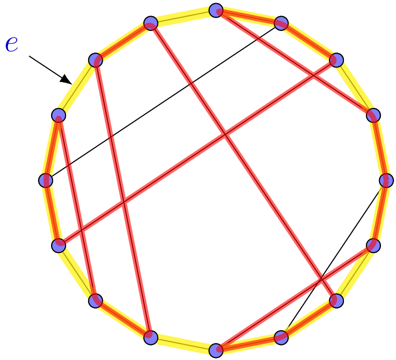
Conjecture (Thomassen)

A Hamiltonian graph has *no chromatic roots* in the interval $(1, 2)$

Fengming and I have several *more precise* conjectures about exactly which graphs have no chromatic roots in $(1, 2)$.

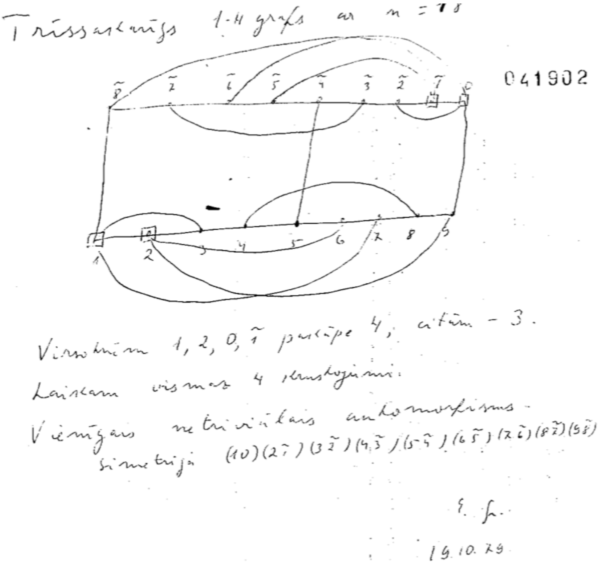
More than one Hamilton cycle

Pick e lying in the yellow cycle, but not the red.



- The red Hamilton cycle is a Hamilton cycle for $G \backslash e$.
- The yellow Hamilton cycle is a Hamilton cycle for G/e.

EXAMPLES OF UH3 GRAPHS



Document generously supplied by Dainis Zeps, academic “grandson” of Grinbergs

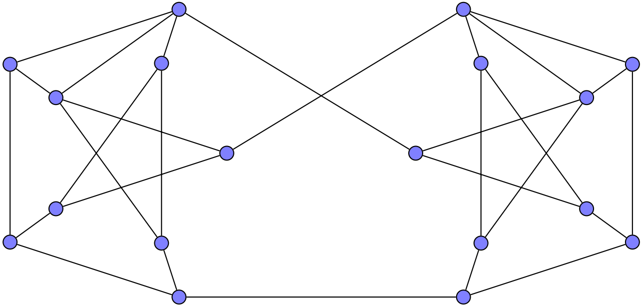
EMANUELS GRINBERGS (1912–1982)

- Latvian polymath with 2 PhDs
- Famous for *Grinberg's theorem*
- Graph found 1979, published 1986

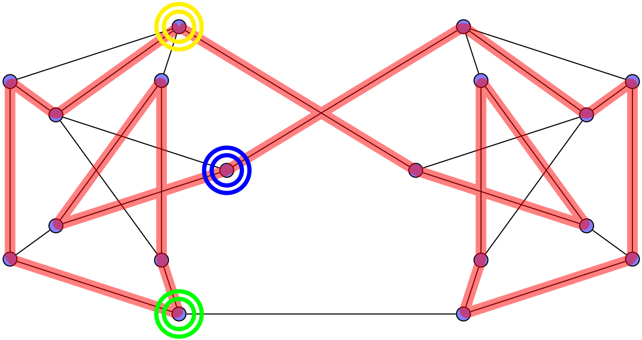
Often known by "westernized" name *Grinberg*.

But really Grinberg's theorem *should be* Grinbergs' theorem.

Redrawing Grinbergs' graph we get something rather familiar.



Why does it work?



In the *half-graph*, there is

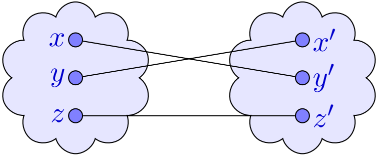
- *exactly one* Hamilton *path* from the yellow to blue,
- *exactly zero* Hamilton paths from yellow to green

PREPARATIONS

Dainis Zeps rescued this construction from Grinbergs' archives.

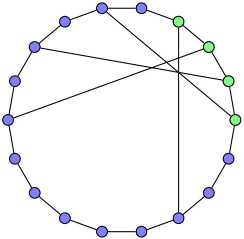
A 3-*preparation* is a triple of vertices (x, y, z) with

- A *unique* Hamilton path from x to y, and
- *No* Hamilton path from x to z.



Computer investigation

Start with a cycle of desired length.

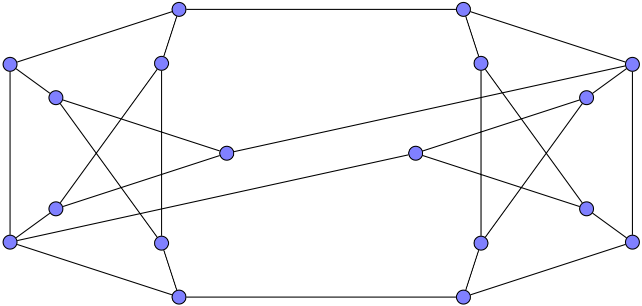


Systematically add *chords* (edges not in the cycle) so that

- No *additional* Hamilton cycles are created
- No chord joins two vertices of degree *greater than* 3

The smallest

Amazingly, working by hand, Grinbergs missed out by *just one edge* on finding the *unique smallest* UH3 graph.

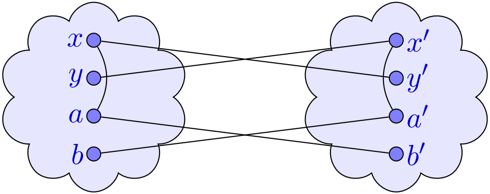


More preparations

Dainis Zeps has worked out what the conditions are in this case.

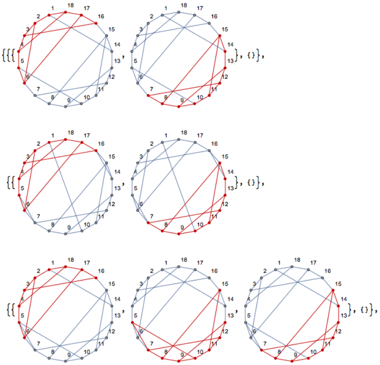
A 4-*preparation* is a quadruple of vertices (x, y, a, b) with:

- A unique Hamilton path from x to y.
- An edge between x and a.
- No Hamilton paths between any two of $\{y, a, b\}$.



Many more small UH3 graphs

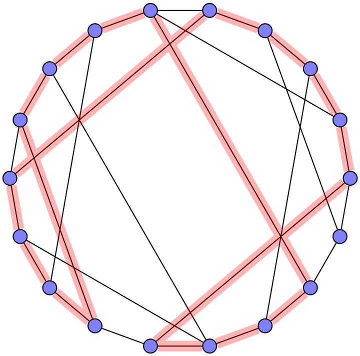
Dainis Zeps is classifying the k-preparations involved.



The graph $P \backslash v$ is a recurring theme.

So how's Thomassen's conjecture?

Our potential counterexample has a *near-Hamilton* cycle (that is, missing just one vertex).



Here!

A *chord* from the *missed vertex* satisfies the conjecture.

GETTING HARDER

So now we are asking for more — we need

- A UH3 graph on n vertices $\ldots$
- $\ldots$ but *also* with no $n-1$ cycles.

First idea: A *bipartite* graph on an *even* number of vertices has no odd cycles, so no $n-1$ cycles.

FOILED AGAIN

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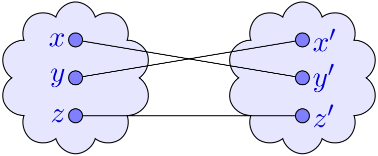
Received 28 August 1995; revised 30 November 1995

We prove that a bipartite uniquely Hamiltonian graph has a vertex of degree 2 in each color class. As consequences, every bipartite Hamiltonian graph of minimum degree d has at least $2^{1-d}d!$ Hamiltonian cycles, and every bipartite Hamiltonian graph of minimum degree at

Where to next?

Can we find a graph and 3 vertices (x, y, z) such that

- Unique Hamilton path from x to y
- No Hamilton path from x to z
- No *near-Hamilton* path from x to y



... but none found so far.

Planar graphs

- Kratochvil & Zeps prove that Hamiltonian *planar triangulations* have *at least* 4 Hamilton cycles.

- Hakimi, Schmeichel & Thomassen find *infinite family* of planar triangulations with *exactly* 4 Hamilton cycles.

Conjecture (Bondy / Jackson)

There are no planar UH3 graphs.

THANK YOU FOR LISTENING