University of Latvia
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PhD Thesis

Volterra integral equations on time scales

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Abstract

In this thesis, we introduce a more general result on existence, uniqueness and boundedness for solutions of nonlinear Volterra integral equation and nonstandard Volterra type integral equation on an arbitrary time scales. We use Lipschitz functions, which have an unbounded Lipschitz coefficient, and the Banach fixed point theorem at an appropriate functional space endowed with a suitable Bielecki type norm. Furthermore it allows to get new sufficient conditions for existence, uniqueness, estimate of growth of rates, boundedness of solutions and continuous dependence of solutions. We illustrate new sufficient conditions by typical examples. Nonstandard Volterra integral equations could be very useful in many applications, especially in economic processes.

We also study Hyers-Ulam stability of a nonlinear Volterra integral equation and Hyers–Ulam stability of general Volterra type integral equations on unbounded and bounded time scales. Many integro-differential equations can be reduced to general Volterra type integral equations. We further develop the previous results using Lipschitz function, which has an unbounded Lipschitz coefficient and the Banach fixed point theorem at appropriate functional space with Bielecki type norm. Sufficient conditions are obtained for Hyers-Ulam stability. The results will be useful for numerical approximations of solutions of Volterra integral equations.

MSC: 45D05, 45G10, 34N05

Keywords: Volterra integral equations, time scales, bounded solution, continuous dependence, Hyers-Ulam stability, Bielecki type norm.

Dedication

I dedicate this thesis to God Almighty my creator, my strong pillar, my source of inspiration, wisdom, knowledge and understanding. He has been the source of my strength throughout this doctorate program.

I also dedicate my thesis work to my family and many friends. A special feeling of gratitude to my loving parents, Ramanbhai Christian and Snehprabha Christian, whose words of encouragement and push for tenacity ring in my ears. My sister, Shreya Christian and my brother, Shreyas Christian have never left my side and are very special. I am grateful for all of their endless love and sacrifices that they made on my behalf. Their prayers have sustained me thus far.

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Contents

Introduction

History of the Integral from the 17th century, the foundations for the discovery of the integral were first laid by Bonaventura Cavalieri, an Italian mathematician in around 1635. Cavalieri's work centered around the observation that a curve can be considered to be sketched by a moving point and an area to be sketched by a moving line.

By contrast with differential equations, which got off to a flying start with Isaac Newton's second law of motion in his Principia, 1687. Integral equations arrived late; they made their first appearance sporadically and even furtively during the third and fourth decades of the nineteenth century. Sometimes they were seen only as alternative formulations of differential problems and sometimes not. General recognition of their independent importance came slowly. Even the name "integral equation" was not suggested until the late 1880's and it was adopted only in the early 1900's. In the last four or five years of the nineteenth century, Vito Volterra and Ivar Fredholm succeeded in working out fundamental linear theories of the two types which have since carried their names. Suddenly integral equations blazed forth in the mathematical heavens, a supernova heralding the analysis of the twentieth century. Alternative approaches were discovered, connections found, nonlinear equations were attacked and generalizations were invented. Applications were brought in by the bushel. Then, almost as suddenly interest began to change even to fade. Mathematicians went on energetically to generalize David Hibert's geometrical way of looking at integral equations in "spaces" whose "points" are functions or infinite sequences of numbers led to active study of abstract spaces and their transformations: Hilbert spaces, the spaces Ip and Lp of Frederic Riesz, the complete normed linear vector spaces of Stefan Banach in the 1920's. Results and viewpoints of great power and generality were attained through topological and algebraic approaches. They reshaped the theoretical foundations of physics especially quantum mechanics and statistical mechanics.

The birth of integration and differentiation can hardly be distinguished from that of mechanics. Archimedes found the area of a parabolic segment by thinking in terms of moments, Galileo Galilei in effect differentiated kt2 or integrated a constant twice while groping for the true law of falling bodies, decades before "the calculus" could be said to exist. He died in 1642, less than a year before Newton's birth. Galilei's and Johann Kepler's work in the beginning of the seventeenth century inspired others to study problems of motion, notably Christiaan Huygens and above all. Isaac Newton who, "standing on the shoulders of giants", as he said succeeded magnificently in seeing the correct laws of dynamics and of gravitational attraction along with extending to dynamics Isaac Barrow's insight into the reciprocal nature of the area problem and that of tangents.

In 1823, Abel proposed a generalization of the tautochrone problem whose solution involved the solution of an integral equation which has more recently been designated as an integral equation of the first kind and in 1837, Liouville showed that the determination of a particular solution of a linear differential equation of the second order could be effected by solving an integral equation of a different type called the integral equation of the second kind. The ripple of mathematical interest which had its origin in these investigations increased at first but slowly. Recently, however, stimulated by the researches of Volterra, Fredholm and Hilbert in the period between 1896 and the present time that which seemed at first only a ripple has grown into a formidable wave which bids fair to carry the integral equation theory into a place beside the most important of the mathematical disciplines.

The theory and applications of integral equations or as it is often called of the inversion of definite integrals have come suddenly into prominence and have held during the last half dozen years a central place in the attention of mathematicians. By an integral equation, a term first suggested by du Bois-Reymond in 1888 is understood an equation in which the unknown function occurs under one or more signs of definite integration.

In mathematics, integral equations are equations in which an unknown function appears under an integral sign. There is a close connection between differential and integral equations and some problems may be formulated either way. For example, Green's function, Fredholm theory and Maxwell's equations.

Integral equations in many scientific and engineering problems, a large class of initial and boundary value problems can be converted to Volterra or Fredholm integral equations. The potential theory contributed more than any field to give rise to integral equations. Mathematical physics models, such as diffraction problems, scattering in quantum mechanics, conformal mapping and water waves also contributed to the creation of integral equations.

Integral equations often arise in electrostatic, low frequency electro magnetic problems, electro magnetic scattering problems and propagation of acoustical and elastical waves.

Integral equations appear in many forms. Two distinct ways that depend on the limits of integrations are used to characterize integral equations.

If atleast one limit of integration is a variable, the equation is called a linear Volterra integral equation given in the form

$$x(t) = f(t) + \lambda \int_a^t K(t,s)x(s)ds.$$

Moreover, two other distinct kinds that depend on the appearance of the unknown function $x(t)$ are defined as follows:

1. If the unknown function $x(t)$ appears only under the integral sign of Volterra equation, the integral equation is called a first kind of Volterra integral equation respectively.

2. If the unknown function $x(t)$ appears both inside and outside the integral sign of Volterra equation, the integral equation is called a second kind Volterra integral equation.

In Volterra integral equation presented above, if $f(t)$ is identically zero, the resulting equation

$$x(t) = \lambda \int_a^t K(t,s)x(s)ds$$

is called homogeneous Volterra integral equation respectively. It is interesting to point out that any equation that includes both integrals and derivatives of the unknown function $f(t)$ is called integro-differential equation.

The Volterra integro-differential equation is of the form

$$x^{(k)}(t) = f(t) + \lambda \int_a^t K(t,s)x(s)ds, \quad x^{(k)} = \frac{d^k(x)}{dt^k}.$$

The Volterra integral equations are a special type of integral equation. These equations were introduced by the Italian mathematician and physicist Vito Volterra known for his contributions to mathematical biology and integral equations and one of the founders of functional analysis. Traian Lalescu studied Volterra's integral equations in his 1908 thesis, Paris. Sueles equations de Volterra written under the directions of Emile Picard. Lalescu wrote the first book ever on integral equations in 1911. Volterra showed early promise in mathematics before attending the University of Pisa, where he fell under the influence of Enrico Betti and where he became professor of rational mechanics in 1883. He immediately started work developing his theory of functionals which led to his interest and later contributions in integral and integro-differential equations. His work is summarised in his book Theory of functionals and of Integral and Integro-Differential Equations, 1930. Volterra is the only person who was a plenary speaker in the International Congress of Mathematicians four times (1900, 1908, 1920, 1928).

The Volterra integral equations are divided into two groups. The first and the second kind. A linear Volterra equations of the first kind is of the form

$$f(t) = \int_a^t K(t,s)x(s)ds$$

and the second kind is of the form

$$x(t) = f(t) + \lambda \int_a^t K(t,s)x(s)ds,$$

where f is a given function and x is an unknown function. If $f(t) = 0$ then it is called homogeneous integral equation and if $f(t) \neq 0$ then it is called non homogeneous integral equation. The function K in the integral is called the kernel and λ is a parameter (Which plays the same role as the eigenvalue in linear algebra).

Examples of the Volterra integral equations of the first kind are

$$te^{-t} = \int_0^t e^{s-t}x(s)ds,$$

and

$$5t^2 + t^3 = \int_0^t (5 + 3t - 3s)x(s)ds.$$

However examples of the Volterra integral equations of the second kind are

$$x(t) = 1 - \int_0^t x(s)ds$$

and

$$x(t) = t + \int_0^t (t-s)x(s)ds.$$

The application of Volterra integral equations can be found in demography, viscoelastic material and insurance mathematics. A variety of analytic and numerical methods, such as successive approximations method, Laplace transform method, Spline collocation method, Runge Kutta method and others have been used to handle Volterra integral equations. The recently developed methods, namely, the adomian decomposition method (ADM), the modified decomposition method (MADM) and the variation iteration method (VIM) to handle Volterra integral equations. Some of the traditional methods, namely, successive approximations method, Laplace Transform Method, series solution method. The Volterra's population growth model, biological species living together, propagation of stocked fish in a new lake, the heat transfer and the heat radiation are among may areas that are described by integral equations. There are wide mathematical literature on Volterra integral equations. See, [16, 21].

The nonlinear Volterra integral equations are characterized by at least one variable limit of integration. In the nonlinear Volterra integral equations of the second kind, the unknown function $x(t)$ appears inside and outside the integral sign. The nonlinear Volterra integral equation of the second kind is represented by the form

$$x(t) = f(t) + \int_a^t K(t,s,x(s))ds.$$

However, the nonlinear Volterra integral equations of the first kind contains the nonlinear function $K(t,s,x(s))$ inside the integral sign. The nonlinear Volterra integral equation of the first kind is expressed in the form

$$f(t) = \int_a^t K(t,s,x(s))ds.$$

For these two kinds of equations, the $K(t,s,x(s))$ and the $f(t)$ are given real valued functions. The specific conditions under which a solution exists for the nonlinear Volterra integral equation $x(t) = f(t) + \int_a^t K(t,s,x(s))ds$ are

- The function $f(t)$ is integrable and bounded in $a \leq t \leq b$.
- The function $K(t,s,x(s))$ is integrable and bounded $|K(t,s,x(s))| < M$ in $a \leq t,\ s \leq b$.
- Often assume that the function $K(t,s,x(s))$ satisfy the Lipschitz condition

$$|K(t,s,z) - K(t,s,z')| < M|z - z'|.$$

Historically, two of the most important types of mathematical equations that have been used to mathematically describe various dynamic process are differential and integral equations, and difference and summation equations respectively in continuous time or in discrete time. Traditionally researchers have used either differential and integral equations or difference and summation equations but not a combination of the two areas. However, it is now becoming apparent that certain phenomena do not involve only continuous aspects or not only discrete aspects. Rather they feature elements of both the continuous and the discrete. These types of hybrid processes are seen e.g. in population dynamics where non-overlapping generation [40] occur. But neither difference equations nor differential equations give a good explanation of most population growth.

To effectively treat hybrid dynamical systems, a more modern and flexible mathematical framework is needed to accurately model continuous -discrete processes in a mutually consistent manner. An emerging area that has the potential to effectively manage the above situations is the field of dynamic equations on time scales.

In mathematics, a dynamical system is a system in which a function describes the time dependence of a point in a geometrical space. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe and the number of fish each springtime in a lake. At any given time, a dynamical system has a state given by a tuple of real numbers (a vector) that can be represented by a point in an appropriate state space (a geometrical manifold). The evolution rule of the dynamical system is a function that describes what future states follow from the current state. Often the function is deterministic that is for a given time interval only one future state follows from the current state. However, some systems are stochastic, in that random events also affect the evolution of the state variables.

The time scale calculus is a unification of the theory of difference calculus with differential and integral calculus, offering a formalism for studying hybrid discrete–continuous dynamical systems. It has applications in any field that requires simultaneous modelling of discrete and continuous data. It gives a new definition of a derivative such that if one differentiates a function, which acts on the real numbers then the definition is equivalent to standard differentiation but if one uses a function acting on the integers then it is equivalent to the forward difference operator. The theory of time scales was introduced by German mathematician Stefan Hilger in his PhD thesis [32] in 1988 (supervised by Bernd Aulbach) in order to unify continuous and discrete analysis. However, similar ideas have been used before and go back at least to the introduction of the Riemann–Stieltjes integral which unifies sums and integrals. In mathematics, the Riemann–Stieltjes integral is a generalization of the Riemann integral named after Bernhard Riemann and Thomas Joannes Stieltjes. The definition of this integral was first published in 1894 by Stieltjes. It serves as an instructive and useful precursor of the Lebesgue integral and an invaluable tool in unifying equivalent forms of statistical theorems that apply to discrete and continuous probability, which allows us to generalize a process to deal with both continuous and discrete cases and any combination. Because of its hybrid formalism, the time scales calculus has recently received considerable attention. Hence, dynamic equations on a time scale have a potential for applications. In the population dynamics, the insect population can be better modelled using time scale calculus. The reason behind this is that they evolve continuously while in season, die out in winter while their eggs are incubating or dormant and then hatch in a new season giving rise to a non-overlapping population.

Lots of excellent books, monographs and research papers are available in this field majorally contributed by Martin Bohner, Allan Peterson, Lynn Erbe, Ravi P. Agarwal, Samir H. Saker, Zhenlai Han, Qi Ru Wang, Youssef N. Raffoul and many more. The books [13, 30] present a complete discussion on time scale calculus. A very nice survey article is written by Agarwal et al. [2]. The last 2 decades have seen the application of the theory to control theory as a means for studying and implementing adaptive controls as an alternative to the more classically used optimization methods. This is evidenced by the exponential growth in papers on the topic that have been published by both mathematicians and engineers alike within the last decade alone.

The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale (or time set), which is an arbitrary non empty closed subset of the real. If we choose the time scale to be the set of real numbers, the general result yields a result for an ordinary differential equation and if we choose the time scale to be the set of an integers, the same result yields a result for difference equations. The results apply not only for the set of real numbers or the set of integers but to more general time scales such as a contour set. Unification and extension are the two main aspects of the time scales calculus. We will introduce the delta derivative f^{Δ} for a function f defined on $\mathbb{T}$ and it turns out that

- $f^{\Delta} = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$
- $f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$

I believe that integral equations on time scales have an enormous potential for rich and diverse applications and thus they are most worthy of attention. Studies into the area will not only provide a deeper understanding of traditional integral and summation equations by uncovering the strange distinctions and interesting links between the two areas but will also lead to new discoveries in those dynamic equations on time scales where the delta derivative is present. Time scales have applications in any field where modelling requires both continuous and discrete data simultaneously.

The three most popular examples of calculus on time scales are differential calculus, difference calculus and quantum calculus. Dynamic equations on a time scale have a potential for applications, such as in population dynamics, economics, electrical circuits, heat transfer etc. We refer to [2, 3, 4, 5, 6] for comprehensive study of time scales. The main techniques that we employ are contemporary components of nonlinear analysis including the Banach fixed point theorem, Lipschitz condition and a novel definition of measuring distance in normed space.

D.B. Pachpatte [49] (Assistant Professor, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, Maharashtra, India), consider a nonstandard Volterra type dynamic integral equation

$$x(t) = f(t) + \int_a^t K(t, s, x(s), x^{\Delta}(s))\, \Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}.$$

Using the relation

$$x(\sigma(s)) = x(s) + x^{\Delta}(s)\mu(s).$$

In many cases (above equation) can be reduced to nonstandard Volterra type integral equation

$$x(t) = f(t) + \int_a^t K(t, s, x(s), x(\sigma(s)))\, \Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}$$

where $x\colon I_{\mathbb{T}} \to \mathbb{R}^n$ is the unknown function, $f\colon I_{\mathbb{T}} \to \mathbb{R}^n$ and $K\colon I_{\mathbb{T}}\times I_{\mathbb{T}}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$ are nonlinear functions.

In 1940 Stanislaw Ulam [68] at the University of Wisconsin raised the question when a solution of an equation, differing slightly from a given one must be somehow near to the exact solution of the given equation. In the next year, Donald.H. Hyers [34] gave a partial solution to the question of S.M. Ulam in the context of Banach space in the case of additive mapping. That was the first significant breakthrough and a step toward more solutions in this area. Since then a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers's theorem. In 1978, Th.M. Rassias succeeded in extending Hyers's theorem for mappings between Banach spaces by considering an unbounded Cauchy difference subject to a continuity condition upon the mapping.

The stability concept proposed by S.M. Ulam and D.H. Hyers was named as *Hyers-Ulam stability.* Afterwards Th.M. Rassias [59] introduced new ideas of Hyers-Ulam stability using unbounded right-hand sides in the involved inequalities depending on certain functions introducing therefore the so-called *Hyers-Ulam-Rassias stability.*

In 2007, S.M. Jung [36] proved using a fixed point approach that the Volterra nonlinear integral equation is Hyers-Ulam-Rassias stable on a compact interval under certain conditions. Then several authors [17, 27, 28] generalized the previous result on the Volterra integral equations to infinite interval in the case when the integrand is Lipschitz with a fixed Lipschitz constant. In the near past many research papers have been published about Ulam-Hyers stability of Voltera integral equations of different type including nonlinear Volterra integro-differential equations, mixed integral dynamic system with impulses etc.[18, 19, 63, 75].

T. Kulik and C.C. Tisdell [40] gave the basic qualitative and quantitative results to Volterra integral equations on time scales in the case when the integrand is Lipschitz with a fixed Lipschitz constant. A. Reinfelds and S. Christian [60, 61] generalized previous results using Lipschitz function, which has an unbounded Lipschitz coefficient.

To the best of our knowledge, the first ones who pay attention to Hyers-Ulam stability for Volterra integral equations on time scales are S. Andras, A.R. Meszaros [8] and L. Hua,Y. Li, J. Feng [33]. However they restricted their research to the case when integrand satisfies Lipschitz conditions with some Lipschitz constant. We generalize the results of [8, 33] using Lipschitz type function, which can be an unbounded and the Banach fixed point theorem at appropriate functional space with Bielecki type norm [11]. There are also papers on impulsive integral equations on time scales [64, 65].

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I believe that this research work is catching the attention because its based on the theory of time scales, which has recently received a lot of attention and it opens a door for new researchers. Volterra integral equation on time scales related research in Latvia is conducted by A. Reinfelds and S. Christian. These studies have gained international recognition. It is anticipated that the thesis results will actively continue traditional studies of Latvian mathematics in the theory of Volterra integral equation on time scales and also successfully develop these theoretical mathematics perspective direction.

Goal and objectives

The main objective of the doctoral thesis is to introduce a more general result: existence, uniqueness, boundedness and certain growth rates for solutions of nonlinear Volterra integral equation on time scales using Lipschitz function, which has an unbounded Lipschitz coefficient and the Banach fixed point theorem at appropriate functional space. Furthermore it allows to get new sufficient conditions for boundedness of solutions.

The tasks of the thesis are strongly correlated with its goal:

1. To generalize the results using Lipschitz function that allows to get more general conditions.
2. To find new sufficient conditions for the existence and uniqueness of the solution for non-standard Volterra type integral equation on time scales.
3. To investigate the condition for boundedness of solutions and prove the continuous dependence of solutions.
4. To introduce Hyers-Ulam stability of a nonlinear Volterra integral on unbounded time scales using the Banach fixed point theorem at appropriate functional space with Bielecki type norm.
5. To develop new existence and uniqueness conditions of the solutions and analyse Hyers-Ulam stability of Volterra integral type integral equation on time scales.

Thesis structure

This thesis is structured in the following way.

Chapter 1 presents a general overview of necessary preliminaries. Starting with some basic definitions from time scales calculus theory continuing with the definition of time scales with examples. The concept of forward and backward jump operators are useful for describing the structure of the time scale under consideration as well as in defining the generalized derivative. For function $f\colon \mathbb{T} \to \mathbb{R}$, we introduce a derivative and an integral. Fundamental results. e.g. the product rule and the quotient rule are presented. We use the so called cylinder transformation to introduce the exponential function on time scales with many properties. Most of the dynamical equations considered in [13] with a few expections are linear. For an extensive collection of results on nonlinear problems, we refer to the book Kaymakealan, Lakshmikantham and Sivasundaram [42]. The concept of metric space and Banach fixed point theorem are included in this chapter. We use Banach fixed point theorem in all our research work. Bielecki norm is now used very frequently to obtain global existence and uniqueness results for wide classes of differential and integral equations. In this thesis, we wish to present the basic concept of Hyers-Ulam stability .

Chapter 2 contains a more general result: Existence, uniqueness, boundedness and certain growth rates for solutions of nonlinear Volterra integral equation on time scales. Kulik [40] restricted his research to the case when Lipschitz type function is constant. We generalize the results of [40, 49, 56, 62] using Lipschitz function [60, 61], which has an unbounded Lipschitz coefficient and the Banach fixed point theorem at appropriate functional space. Furthermore it allows to get new sufficient conditions for boundedness of solutions [45]. In addition, we should note articles [1, 38] that are very important in this direction. Dynamic equations on time scales, this new and compelling area of mathematics is more general and versatile than the traditional theories of differential and difference equations. The field of dynamic equations on time scales contains and extends the classical theory of differential, difference, integral and summation equations as special cases. To understand the notation in this chapter some basic definitions are also written [13, 15].

Chapter 3 presents the more general result of existence, uniqueness and boundedness for the solutions of nonstandard Volterra type integral equation on an arbitrary time scales. This chapter considers nonstandard Volterra type integral equation on an arbitrary time scales $\mathbb{T}$.

$$x(t) = f(t) + \int_a^t K(t, s, x(s), x(\sigma(s)))\,\Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T} \tag{1}$$

where $x\colon I_{\mathbb{T}} \to \mathbb{R}^n$ is the unknown function, $f\colon I_{\mathbb{T}} \to \mathbb{R}^n$ and $K\colon I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear functions.

We generalize and the same time simplify the results [40, 49, 56, 62], find the new sufficient conditions for the existence and uniqueness of the solution for nonstandard Volterra type integral equation. We also find condition for boundedness of solutions and prove the continuous dependence of solutions.

Chapter 4 shows Hyers-Ulam stability of a nonlinear Volterra integral equation on unbounded time scales. Sufficient conditions are obtained based on the Banach fixed point theorem and Bielecki type norm.

T. Kulik and C.C. Tisdell [40] gave the basic qualitative and quantitative results to Volterra integral equations on time scales in the case when the integrand is Lipschitz with a fixed Lipschitz constant A. Reinfelds and S. Christian [60, 61] generalized previous results using Lipschitz function, which has an unbounded Lipschitz coefficient.

To the best of our knowledge, the first ones who pay attention to Hyers-Ulam stability for Volterra integral equations on time scales are S. Andras, A.R. Meszaros [8] and L. Hua, Y. Li, J. Feng [33]. However they restricted their research to the case when integrand satisfies Lipschitz conditions with some Lipschitz constant. We generalize the results of [8, 33] using Lipschitz function, which has an unbounded Lipschitz coefficient and the Banach fixed point theorem at appropriate functional space with Bielecki type norm [11]. There are also papers on impulsive integral equations on time scales [64, 65].

Chapter 5 indicates Hyers–Ulam stability of general Volterra type integral equations on unbounded and bounded time scales. We give an existence and uniqueness conditions of the solutions of Volterra type integral equations on time scales using Banach contraction principle, Bielecki type norm [11] and Lipschitz type functions. Furthermore it allows to get sufficient conditions for Hyers-Ulam stability.

D.B. Pachpatte [57] studied qualitative properties of solutions of general nonlinear Volterra integral equation

$$x(t) = f\left(t, x(t), \int_a^t K(t, s, x(s))\right) \Delta s$$

on time scales.

Using the methods developed at [60, 61], we give new existence and uniqueness conditions of solutions and analyse Hyers-Ulam stability for the following class of Volterra type integral equation on an arbitrary time scales $\mathbb{T}$.

$$x(t) = f\left(t, x(t), x(\sigma(t)), \int_a^t K(t, s, x(s), x(\sigma(s)))\, \Delta s\right), \; a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}. \tag{2}$$

This type of integral equations could be very useful for modelling economic processes. For example, a Keynesian-Cross model with "lagged" income [26, 67].

Many integro-differential equations can be reduced to Volterra type integral equations (2).

Approbation

The results obtained in the process of thesis writing have been presented at two international conferences (23rd and 24th international conferences on Mathematical Modelling and Analysis-MMA) in 2018 and 2019 ([C1], [C2]); one (12th Latvian Mathematical conference) in 2018 ([C3]) ; four (75th, 76th, 77th, 78th scientific conferences of University of Latvia) in 2017, 2018, 2019, 2020 ([C7], [C6] ,[C5], [C4]).

The main results of the research have been reflected in the four scientific publications ([P4], [P5], [P6], [P7]); in the two international conferences ([P2], [P3]) and in the Latvian mathematical society conference ([P1]).

Chapter 1

Preliminaries

1.1 Time scales

The time scale calculus is the unification of the theory of difference calculus with differential and integral calculus. The new school of thought was introduced by German mathematician Stefan Hilger in his PhD thesis [32] in 1988. The key concept of the study of dynamic equations on time scales is a way of unifying and extending continuous and discrete cases and any combination. It has applications in any field that requires simultaneous modelling of discrete and continuous data simultaneously. Because of its hybrid formalism, in the past few years, this area of mathematics has received considerable attention. Applications of time scales can be found in various fields such as population dynamics, economics, electrical circuits, heat transfer etc. We refer to [13, 15] for in depth study of time scales.

Definition 1.1.1

A time scale (OR a measure chain) is a non empty closed subset of the real numbers $\mathbb{R}$. The common notation for a general time scale is $\mathbb{T}$.

The calculus of time scales was initiated by Stefan Hilger in his PhD thesis [32] in order to create a theory that can unify discrete and continuous analysis. Indeed below we will introduce the delta f^{Δ} for a function f defined on $\mathbb{T}$, and it turns out that

- $f^{\Delta} = f^{'}$is the usual derivative if $\mathbb{T} = \mathbb{R}$ and
- $f^{\Delta} = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$

Examples of time scales are as follows as:

- $\mathbb{T} = \mathbb{R}$
- $\mathbb{T} = \mathbb{Z}$
- $\mathbb{T} = \mathbb{N}$
- $\mathbb{T} = \mathbb{N}_0 = \mathbb{N} \cup 0$
- $\mathbb{T} = \{hk\colon k \in \mathbb{Z}\}$, $h > 0$
- $\mathbb{T} = [a, b]$

- $\mathbb{T} = \{p^k : k \in \mathbb{N}_0\}$, $p > 1$
- $\mathbb{T} = \{2^k : k \in \mathbb{N}_0\}$
- $\mathbb{T} = \{n^2 : n \in \mathbb{N}_0\}$
- $\mathbb{T} = \{1, 2, 3, 4, 5\}$
- $\mathbb{T} = [1, 2] \cup \{3\} \cup [4, 5]$
- The Cantor set
- $\mathbb{T} = \{0\} \cup \{1, \frac{1}{2}, \frac{1}{3} \ldots.\}$

Examples of not time scales are as follows:

- $\mathbb{T} = \mathbb{Q}$
- $\mathbb{T} = \mathbb{R} \setminus \mathbb{Q}$
- $\mathbb{T} = \mathbb{C}$
- $\mathbb{T} = (a, b), a, b \in \mathbb{R}$.

Since a time scale may or may not be connected, the concept of jump operator is useful for describing the structure of the time scale under consideration and is also used in defining the generalized derivative.

Definition 1.1.2
For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by
$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$.

Definition 1.1.3
For $t \in \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by
$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$.

In the Definition 1.1.2 and 1.1.3, for empty set ϕ, we put $\inf \phi = \sup \mathbb{T}$, i.e. $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t and $\sup \phi = \inf \mathbb{T}$, i.e. $\rho(t) = t$ if $\mathbb{T}$ has a minimum t.

Definition 1.1.4
The graininess function μ is the distance from a point to the closest point on the right and is given by $\mu(t) = \sigma(t) - t$.

For a right dense t, $\sigma(t) = t$ and $\mu(t) = 0$.
For a left dense t, $\rho(t) = t$.

1.2 Classification of points

Here we propose the classification of points, definitions and basic examples, which are important for our future consideration. Detailed information on classification of points can be found [13, 15].

- $\sigma(t) > t$, t is right scattered

- $\rho(t) < t$, t is left scattered
- $\sigma(t) = t$, t is right dense
- $\rho(t) = t$, t is left dense
- $\rho(t) < t < \sigma(t)$, t is isolated
- $\rho(t) = t = \sigma(t)$, t is dense

Definition 1.2.1

If $f\colon \mathbb{T} \to \mathbb{R}$ is a function, then we define the functions $f^{\sigma}\colon \mathbb{T} \to \mathbb{R}$ by $f^{\sigma}(t) = f(\sigma(t))$ for all $t \in T$.

Definition 1.2.2

If $\mathbb{T}$ is a time scale with left scattered maximum m, then the set $\mathbb{T}^k$ is defined by $\mathbb{T} - m$. Otherwise, $\mathbb{T}^k = \mathbb{T}$.

Table 1.1: Some common time scales and their corresponding forward and backward jump operators and graininess

$\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$\mathbb{Z}$	$t+1$	$t-1$	1
$h\mathbb{Z}$	$t+h$	$t-h$	h
$q^{\mathbb{N}_0}$	qt	$\frac{t}{q}$	$(q-1)t$
$2^{\mathbb{N}_0}$	$2t$	$\frac{t}{2}$	t
$\mathbb{N}^2{}_0$	$(\sqrt{t}+1)^2$	$(\sqrt{t}-1)^2$	$(2\sqrt{t}+1)$

Example 1.2.1

Let us briefly consider the two examples $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

1. If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$.

$$\sigma(t) = \inf\{s \in \mathbb{R}\colon s > t\} = \inf(t, \infty) = t$$

and similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be

$$\mu(t) = 0, \forall t \in \mathbb{T}.$$

2. If $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$.

$$\sigma(t) = \inf\{s \in \mathbb{Z}\colon s > t\} = \inf\{t+1, t+2, t+3, ...\} = t+1$$

and similarly $\rho(t) = t - 1$. Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ in this case is

$$\mu(t) = 1, \forall t \in \mathbb{T}.$$

For the two cases discussed above, the graininess function is a constant function. For the general case, many formulas will have some containing factor $\mu(t)$. This term is there in case $\mathbb{T} = \mathbb{Z}$ since $\mu(t) = 1$. However, for the case $\mathbb{T} = \mathbb{R}$ this term disappears since $\mu(t) = 0$ in this case.

1.3 Differentiation

First time the definition of delta derivative f^{Δ} for a function f defined on $\mathbb{T}$ was introduced by Stefan Hilger [32]. In this section, we give the definition of delta (or Hilger) derivative, some of its useful properties and basic examples.
We consider a function $f\colon \mathbb{T} \to \mathbb{R}$ and define the delta derivative of f at a point $t \in \mathbb{T}^k$.

Definition 1.3.1

Let $f\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. We define $f^{\Delta}(t)$ (delta derivative) to be the number (if it exists) with property that given any $\epsilon > 0$, there is a neighbourhood U (i.e $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) of t, such that

$$\left| f\left(\sigma(t)\right) - f(s) - f^{\Delta}(t)\left(\sigma(t) - s\right)\right| \leq \epsilon|\sigma(t) - s|, \forall s \in U.$$

If f is delta differentiable on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta}(t)\colon \mathbb{T}^k \to \mathbb{R}$ is then called delta derivative of f on $\mathbb{T}^k$.

Example 1.3.1

1. If $f\colon \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = \alpha$ for all $t \in \mathbb{T}$. Where $\alpha \in \mathbb{R}$ is constant then $f^{\Delta}(t) = 0$. This is clear because for any $\epsilon > 0$,
$$|f\left(\sigma(t)\right) - f(s) - 0.\left[\sigma(t) - s\right]| = |\alpha - \alpha| = 0 \leq \epsilon|\sigma(t) - s|$$
holds for all $s \in \mathbb{T}$.
2. $f\colon \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then $f^{\Delta}(t) = 1$. This follows since for any $\epsilon > 0$,
$$|f\left(\sigma(t)\right) - f(s) - 1.\left[\sigma(t) - s\right]| = |\sigma(t) - s - (\sigma(t) - s)| = 0 \leq \epsilon|\sigma(t) - s|$$
holds for all $s \in \mathbb{T}$.

Theorem 1.3.1

Assume $f\colon \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following

1. If f is differentiable at t, then f is continuous at t.
2. If f is continuous at t and t is right scattered, then f is differentiable at t with $f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$.
3. If f is right dense, then f is differentiable at t if and only if the $\lim_{s \to t} \frac{f(t)-f(s)}{t-s}$ exists as a finite number. In this case $f^{\Delta}(t) = \lim_{s \to t} \frac{f(t)-f(s)}{t-s}$.
4. If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$.

- If f is delta differentiable for every $t \in \mathbb{T}^k$, then f is said to be differentiable on $\mathbb{T}$ and f^{Δ} is a new function defined on $\mathbb{T}^k$.

  To avoid the seperate discussion of the two cases $\mu(t) = 0$ and $\mu(t) > 0$. We give another formula which holds whenever f is differentiable at $t \in \mathbb{T}^k$.
  $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$. When applying this formula, we do not need to distinguish between the two cases $\mu(t) = 0$ and $\mu(t) > 0$. This formula holds in both cases.

Example 1.3.2

1. If $\mathbb{T} = \mathbb{R}$, then we can deduce from Theorem 1.3.1, result 3 that $f^{\Delta}(t) = \lim_{s \to t} \frac{f(t)-f(s)}{t-s} = f'(t)$ the limit exists. i.e. $f \colon \mathbb{R} \to \mathbb{R}$ is differentiable at t.
2. If $\mathbb{T} = \mathbb{Z}$ then Theorem 1.3.1, result 2 yields that
$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1}$$
$$= f(t+1) - f(t) = \Delta f(t)$$
where Δ is the forward difference operator.

- The derivatives of sums, products and quotients of differential functions.

Theorem 1.3.2
If $f, g \colon \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$, then

1. The sum $f + g \colon \mathbb{T} \to \mathbb{R}$ is differentiable at t with $(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t)$.
2. For any constant α, $\alpha f \colon \mathbb{T} \to \mathbb{R}$ is differentiable at t $(\alpha f)^{\Delta}(t) = \alpha . f^{\Delta}(t)$.
3. The product $fg \colon \mathbb{T} \to \mathbb{R}$ is differentiable at t with $(fg)^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$.
4. If $f(t)f(\sigma(t)) \neq 0$ then $\frac{1}{f}$ is differentiable at t with $(\frac{1}{f})^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}$.
5. If $g(t)g(\sigma(t)) \neq 0$ then $\frac{f}{g}$ is differentiable at t and $(\frac{f}{g})^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}$.

Clearly, $1^{\Delta} = 0$ and $t^{\Delta} = 1$, so we can use Theorem 1.3.2, result 3 to find
$$(t^2)^{\Delta} = (t.t)^{\Delta} = t + \sigma(t)$$
and we can use Theorem 1.3.2, result 4 to find
$$\left(\frac{1}{t}\right)^{\Delta} = -\frac{1}{t\sigma(t)}.$$

One word of caution: The forward jump operator σ is not necessarily a differentiable function. Clearly, at a point which is left dense and right scattered at the same time, σ is not continuous. Hence σ is not differentiable at such a point since we know.

Theorem 1.3.3 Every differentiable function is continuous.
However, σ is an example of a function which we call rd-continuous. A function f defined on $\mathbb{T}$ is rd-continuous if it is continuous at every right dense point and if the left sided limit exists in every left-dense point. The importance of rd-continuous functions is revealed by the following result by Hilger [31].

1.4 Integration

In this section, we introduce definitions, properties and examples of integration on time scales. Definitions and properties are important to introduce a more general result on existence, uniqueness and boundedness for solutions of nonlinear Volterra integral equation and nonstandard Volterra type integral equation on an arbitrary time scales [60, 61]. More information on integration on time scales can be found in [13, 15]

Definition 1.4.1

A function $f\colon \mathbb{T} \to \mathbb{R}$ is said to be if its regulated if its left sided and right sided limits exist at all left dense and right dense points in $\mathbb{T}$ respectively.

Definition 1.4.2

A function $f\colon \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right dense points in $\mathbb{T}$ and its left sided limits exist (finite) at left dense points in $\mathbb{T}$. The set of rd-continuous function is denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. Some results related to rd-continuous and regulated functions are obtained in the following theorem.

Theorem 1.4.1

Let $f\colon \mathbb{T} \to \mathbb{R}$

1. If f is continuous, then f is rd - continuous.
2. If f is rd-continuous, then f is regulated.
3. The jump operator σ is rd-continuous.
4. If f is regulated or rd-continuous, then so is f^σ.
5. Assume f is continuous. If $g\colon \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property too.

Definition 1.4.3

A continuous function $f\colon \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with (region of differentiable) D, provided $D \subset \mathbb{T}^k$ is countable and contains no right scattered elements of $\mathbb{T}$ and f is differentiable at each $t \in D$.

Theorem 1.4.2

Every regulated function on a compact interval is bounded.

Theorem 1.4.3 Existence of Pre-Antiderivatives

Let f be regulated. Then there exists a function F which is pre-differentible with region of differentiation $\mathbb{T}^k$, such that

$$F^\Delta(t) = f(t), \forall t \in \mathbb{T}^k.$$

Definition 1.4.4

If $F\colon \mathbb{T} \to \mathbb{R}$ is an anti-derivative of $f\colon \mathbb{T} \to \mathbb{R}$ then we define the Cauchy integral by $\int_a^b f(t)\Delta t = F(b) - F(a)$, $\forall a, b \in \mathbb{T}$.

Example 1.4.1

If $\mathbb{T} = \mathbb{Z}$, evaluate the indefinite integral $\int a^t \Delta t$, where $a \neq 1$ is a constant. Since $\left(\frac{a^t}{a-1}\right)^\Delta = \Delta\left(\frac{a^t}{a-1}\right) = \frac{a^{t+1}-a^t}{a-1} = a^t$. We get $\int a^t \Delta t = \frac{a^t}{a-1} + C$, where C is an arbitrary constant.

Theorem 1.4.4 Existence of Anti-derivatives

Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$ then F defined by $F(t) = \int_{t_0}^{t} f(\tau)\Delta\tau$ for $t \in \mathbb{T}$ is an antiderivative of t.

Theorem 1.4.5

If $f \in C_{rd}$ and $t \in \mathbb{T}^k$ then $\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)$.

Proof:

By Theorem 1.4.4, there exists an antiderivative of F of f and

$$\begin{aligned}\int_t^{\sigma(t)} f(\tau)\Delta\tau &= F(\sigma(t)) - F(t) \\ &= \mu(t)F^\Delta(t) \\ &= \mu(t)f(t)\end{aligned}$$

Some basic properties of integration on time scales are given in the following theorem.

Theorem 1.4.6

If $a, b, c \in \mathbb{T}$, $\beta \in \mathbb{R}$ and $f, g \in C_{rd}$ then

1. $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$
2. $\int_a^b \beta f(t)\Delta t = \beta \int_a^b f(t)\Delta t$
3. $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$
4. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$
5. $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t$
6. $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$
7. $\int_a^a f(t)\Delta t = 0$
8. $|f(t)| \leq g(t)$ on $[a, b)$, then $|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t$
9. If $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t)\Delta t \geq 0$

Note that the formulas (5) and (6) in the above theorem are called integration by parts formulas. Also note that all of the formulas given in theorem also hold for the case that f and g are only regulated functions.

Theorem 1.4.7

Let $a, b \in \mathbb{T}$ and $f \in C_{rd}$.

1. If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt$$

2. If $[a,b]$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t), & \text{if } a<b \\ 0, & \text{if } a=b \\ -\sum_{t\in[a,b)} \mu(t)f(t), & \text{if } a>b \end{cases}$$

3. If $\mathbb{T} = h\mathbb{Z} = hk\colon k\in\mathbb{Z}$, where $h>0$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h, & \text{if } a<b \\ 0, & \text{if } a=b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h, & \text{if } a>b \end{cases}$$

4. If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t), & \text{if } a<b \\ 0, & \text{if } a=b \\ -\sum_{t=b}^{a-1} f(t), & \text{if } a>b. \end{cases}$$

Definition 1.4.5
If $a\in\mathbb{T}$, $\sup\mathbb{T}=\infty$ and f is rd-continuous on $[a,\infty)$ then we define the improper integral by

$$\int_a^\infty f(t)\Delta t = \lim_{b\to\infty}\int_a^b f(t)\Delta t$$

provided this limit exists and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

1.5 The exponential function

We use the cylinder transformation [13] introduced to define a generalized exponential function for an arbitrary time scale $\mathbb{T}$. The exponential function is then shown to satisfy an initial value problem involving a first order linear dynamic equation. Many properties are useful to solve an initial value problem.

Definition 1.5.1
If a function $p\colon\mathbb{T}\to\mathbb{R}$ is regressive provided $1+\mu(t)p(t)\neq 0$ for all $t\in\mathbb{T}^k$ holds. The set of all regressive and rd-continuous functions $f\colon\mathbb{T}\to\mathbb{R}$ will be denoted by $R = R(\mathbb{T}) = R(\mathbb{T},\mathbb{R})$.

Definition 1.5.2
If $p\in R$ then we define the exponential function by $e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$ for $s,t\in\mathbb{T}$, where ξ is a cylindrical function

$$\xi_h(z) = \begin{cases} z, & h=0 \\ \frac{1}{h}\ln(1+hz), & h>0. \end{cases}$$

and ln is the principal logarithm function.

Lemma 1.5.1
If $p \in R$ then the semi-group property $e_p(t,r)e_p(r,s) = e_p(t,s)$ for all for $r,s,t \in \mathbb{T}$ is satisfied.

Definition 1.5.3
If $p \in R$ then the first order linear dynamic equation $y^{\Delta} = p(t)y$ is called regressive.

Theorem 1.5.1
Suppose $y^{\Delta} = p(t)y$ is regressive and fix $t_0 \in \mathbb{T}$. Then $e_p(\cdot, t_0)$ is a solution of the initial value problem $y^{\Delta} = p(t)y$, $y(t_0) = 1$ on $\mathbb{T}$.

Theorem 1.5.2 Properties of exponential function
If $p \in R$, then

1. $e_0(t,s) = 1$ and $e_p(t,t) = 1$
2. $e_p(\sigma(t),s) = (1 + \mu(t)p(t))e_p(t,s)$
3. $e_p(t,s) = \frac{1}{e_p(s,t)}$
4. $e_p(t,s)e_p(s,r) = e_p(t,r)$
5. $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$

Definition 1.5.4
If $q > 1$, then any dynamic equation on either of the time scales $q^{\overline{z}}$ or q^{N_0} is called a q-difference equation.

Many properties of the exponential function derived and some of them are contained in Bohner and Peterson book [13]. Several time scales the form of the corresponding exponential function is rather unexpected and involves factorials, binomial coefficients or the Wallis product.

Observe that we have important Bernoulli type estimate [14].

$$1 \le 1 + \int_a^t \beta(s)\,\Delta s \le e_{\beta}(t,a) \le \exp\left(\int_a^t \beta(s)\,\Delta s\right)$$

for all $t \in I_{\mathbb{T}}$.

1.6 Banach fixed point theorem

Stefan Banach was a Polish mathematician who is generally considered one of the world's most important and influential 20th century mathematicians. He was the founder of modern functional analysis. His major work was the 1932 book, Theory of Linear Operations, the first monograph on the general theory of functional analysis.

Some of the notable mathematical concepts that bear Bancha's name include Banach's spaces, Banach algebras, Banach measures, the Banach-Tarski paradox, the Hahn-Banach theorem, the Banach-Steinhaus theorem, the Banach-Mazur game, the Banach-Alaoglu theorem, and the Banach fixed point theorem.

Let X be an arbitrary set, which could consist of vectors in $\mathbb{R}^n$, functions, sequences, matrices, etc. We want to endow this set with a metric; i.e. a way to measure distances between elements of X. A distance or metric is a function $d: X \times X \to \mathbb{R}$ such that if we take two elements $x, y \in X$ the number $d(x, y)$ gives us the distance between them. However, a distance needs to satisfy certain properties.

For metric space and fixed point theory, see [20, 24].

Example 1.6.1

Define the distance between two elements $x, y \in \mathbb{R}$ to be

$$d(x, y) = |x - y|$$

Define the distance between two elements $x, y \in \mathbb{R}^2$ to be

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

Where $x = (x_1, x_2), y = (y_1, y_2)$. The properties of these distance functions suggest the following general definition.

Definition 1.6.1 Metric space

A metric space is a pair (X, d), where X is a set and $d: X \times X \to \mathbb{R}$ is map such that for any $x, y, z \in X$

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0 \Leftrightarrow x = y$,
3. $d(x, y) = d(y, x)$ $(symmetry)$,
4. $d(x, y) \leq d(x, y) + d(y, z)$ $(triangleinequality)$.

The elements of X are called points of X. The map d is called a metric (or a distance function).

Property (1) just states that a distance is always a non-negative number. Property (2) tells us that the distance identifies points; i.e. if the distance between x and y is zero, it is because we are considering the same point. Property (3) states that a metric must measure distances symmetrically; i.e. it does not matter where we start measuring it.

Definition 1.6.2

Two metrics d_1 and d_2 are equivalent if and only if there exist positive constant α and β such that for every $x, y \in X$.

$$\alpha d_1(x, y) \leq d_2(x, y) \leq \beta d_2(x, y).$$

Equivalent metrics d_1 and d_2 generate the same topology on X.

Finally, the triangular inequality is a generalization of the famous result that holds for the Euclidean distance in the plane.

Example 1.6.2

The Euclidean metric on $\mathbb{R}^n$ is defined by

$$d_1(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$$

We also may define on $\mathbb{R}^n$ taxicab or Manhattan metric.

$$d_2(x,y) = \sum_{i=1}^{n} |x_i - y_i| = |x_1 - y_1| + |x_2 - y_2| + ... + |x_n - y_n|.$$

or maximum metric

$$d_\infty(x,y) = \max_{1 \le i \le n} |x_i - y_i|.$$

Metrics d_1, d_2 and d_∞ are equivalent on $\mathbb{R}^n$.

Definition 1.6.3 Convergent sequence

Let (X,d) be a metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in X is a collection of elements $x_1, x_2, ...$ in X.

Definition 1.6.4

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in X is said to converges to $x \in X$ if for any $\varepsilon > 0$ there exists $N > 0$ such that

$$d(x_n, x) < \varepsilon, \forall n \ge N.$$

The point x is called the limit of the sequence and is denoted by $\lim_{n\to\infty} x_n$. We write also $x_n \to x$ as $n \to \infty$.

Remark 1.6.1

The definition of convergence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ to x is equivalent to the statement that $d(x_n, x) \to 0, n \to \infty$.

Lemma 1.6.1

Every convergent sequence in a metric space has a unique limit.

Proof:

Assume that $\{x_n\}_{n\in\mathbb{N}}$ converges to x and y. For any $\varepsilon > 0$ there exist $M > 0$ and $N > 0$ such that $d(x_n, x) < \varepsilon$ for $n \ge M$ and $d(x_n, y) < \varepsilon$ for $n \ge N$. Without loss of generality we can assume that $M \le N$. Then for any $n \ge N$ we have

$$d(x,y) \le d(x, x_n) + d(x_n, y) < \varepsilon + \varepsilon = 2\epsilon.$$

This inequality is true for any $\varepsilon > 0$. $\Rightarrow d(x,y) = 0 \to x = y$.

Complete metric space

Definition 1.6.5

Let (X,d) be a metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called a Cauchy sequence if for any $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for any $m \ge n_\varepsilon$, $n \ge n_\varepsilon$.

Theorem 1.6.1

Any convergent sequence in a metric space is a Cauchy sequence.

Proof:

Assume that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence which converges to x. Let $\varepsilon > 0$ be given. Then there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N$. Let $m, n \in N$ be such that $m \geq N$, $n \geq N$. Then

$$d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Then converse of this theorem is not true. For example, let $X = (0, 1]$. Then $(\frac{1}{n})$ is a Cauchy sequence which is not convergent in X.

Definition 1.6.6

A metric space (X, d) is said to be complete if every Cauchy sequence in X converges (to a point in X).

Fixed point theorem

Fixed point theorem is known as the *Banach contraction principle.*

Definition 1.6.7

Let X be a metric space equipped with a distance d. A map $f : X \to X$ is said to be *Lipschitz continuous* if there is $\lambda \geq 0$ such that

$$d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2), \forall x_1, x_2 \in X.$$

The smallest λ for which the above inequality holds is the *Lipschitz constant* of f. If $\lambda \leq 1$, f is said to be *non-expensive*, if $\lambda < 1$, f is said to be a *contraction.*

Definition 1.6.8 Fixed point

The point $\bar{x} \in X$ is fixed point of the map f if $f(\bar{x}) = \bar{x}$.

Theorem 1.6.2 Banach fixed point theorem

Let f be contraction on a complete metric space X. Then f has a unique fixed point $\bar{x} \in X$.

Proof:

Notice that if $x_1, x_2 \in X$ are fixed points of f, then

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$$

which imply $x_1 = x_2$. Choose now any $x_0 \in X$, and define the iterate sequence $x_{n+1} = f(x_n)$. By induction on n,

$$d(x_{n+1}, x_n) \leq \lambda^n d(f(x_0), x_0)$$

If $n \in \mathbb{N}$ and $m \geq 1$,

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(f(x_{n+m}, x_{n+m-1}) + ... + d(x_{n+1}, x_n) \\ &\leq (\lambda^{n+m} + ... + \lambda^n) d(f(x_0), x_0) \\ &\leq \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0). \end{aligned}$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and admits a limit $\bar{x} \in X$, for X is complete. Since f is continuous, we have $f(\bar{x}) = \bar{x}$.

Remark 1.6.2

Note that letting $m \to \infty$ in proof of Theorem 1.6.2 we find the relation

$$d(x_n, \bar{x}) \leq \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0)$$

which provides a control on the convergence rate of x_n to the fixed point $\bar{x}$. The completeness of X plays here crucial role. Indeed, contractions on incomplete metric spaces may fail to have fixed points.

1.7 Bielecki norm

Adam Bielecki was born on February 13, 1910, in Borysław, an eastern Galicia centre of oil mining.

In 1935 Adam got his Ph.D. from the Jagiellonian University upon presentation of the Ph.D. thesis [11] written under the supervision of professor Witold Wilkosz. From 1936 till 1939 Adam Bielecki held an instructorship (equivalent to assistant professorship) at the Chair of Theoretical Physics of UJ.

We note that in 1956 Bielecki [12] first used the norm defined for proving global existence and uniqueness of solutions of ordinary differential equations. It is now used very frequently to obtain global existence and uniqueness results for wide classes of differential and integral equations. For developments related to the topic, see [22] and the references cited therein. The well known Banach fixed point theorem [21], coupled with a Bielecki type norm [11] and an integral inequality with an explicit estimate given in [51] are used to establish the results.

Bielecki's method [11, 12] of weighted norm has been used very frequently to obtain global existence and uniqueness results for wide classes of differential, differential-delay, integral, integro-differential, integro-functional, and many other functional equations. For a review of the results obtained by the mentioned method, see C. Corduneanu's paper [22]. For some extension of Bielecki's method, see also [41]. At the present time, there is a huge number of papers which make use of Bielecki's method frequently not quoting its author. In fact the method became a standard technique in dealing with the mentioned problems. It is important to observe that up to now Bielecki's method was used, as a rule, for fixed point equations considered in spaces of continuous or bounded and measurable functions.

A. Reinfelds and S. Christian used Bielecki type norm in their results [60, 61].

1.8 Hyers-Ulam-Rassias stability

Stanislaw Marcin Ulam (3 April 1909–13 May 1984) was a polish scientist in the field of mathematics and nuclear physics. He participated in the Manhattan Project, originated the Teller–Ulam design of thermonuclear weapons, discovered the concept of the cellular automaton, invented the Monte Carlo method of computation and suggested nuclear pulse propulsion. In pure and applied mathematics, he proved some theorems and proposed several conjectures.

The stability problem of functional equations originated from a question of S.M. Ulam. In 1940 S.M. Ulam [68] at the University of Wisconsin raised the question when a solution of an equation differing slightly from a given one must be somehow near to the exact solution of the given equation. In the next year, Donald.H. Hyers [34] gave an affirmative answer to the question of S.M. Ulam in the context of Banach space in the case of additive mapping. That was the first significant breakthrough and a step toward more solutions in this area. Since then, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers's theorem.

In 1978, Themistocles M. Rassias succeeded in extending Hyers's theorem for mappings between Banach spaces by considering an unbounded Cauchy difference subject to a continuity condition upon the mapping. He is a Greek mathematician and a Professor at the National Technical University of Athens is among the oldest higher education institute of Greece and the most prestigious among engineering schools. He has published more than 300 papers, 10 research books and 45 edited volumes in research Mathematics as well as 4 textbooks in Mathematics for University students. His research work has received more than 15,000 citations according to Google Scholar and more than 5,000 citations according to MathSciNet. His h-index is 45. He serves as a member of the Editorial Board of several international mathematical journals. He has contributed a number of results in the stability of minimal submanifolds, in the solution of Ulam's Problem for approximate homomorphisms in Banach spaces, in the theory of isometric mappings in metric spaces and in Complex analysis. He was the first to prove the stability of the linear mapping. This result of Rassias attracted several mathematicians worldwide who began to be stimulated to investigate the stability problems of functional equations.

So the stability concept proposed by S.M. Ulam and D.H. Hyers, was named as *Hyers-Ulam stability.* Afterwards Th.M. Rassias [59] introduced new ideas of Hyers-Ulam stability using unbounded right-hand sides in the involved inequalities, depending on certain functions, introducing therefore the so-called *Hyers-Ulam-Rassias stability.*

The study on Hyers-Ulam stability of differential equations was announced by Obloza in 1993 [47, 48]. Since then many scholars focused on the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of differential equations, see [47, 71] and the references therein.

It should be pointed out that, the Hyers-Ulam stability is different from Lyapunov stability of differential equations. The Hyers-Ulam stability means that a differential equation has a close exact solution generated by the approximate solution and the error of the approximate solution can be estimated. However, the Lyapunov stability means that if a solution which starts out near an equilibrium point, it will stay near the equilibrium point forever. The Hyers-Ulam stability can be widely used to the fields where it is difficult to find the exact solutions. e.g. numerical analysis, optimization, biology and economics and so on [7].

In 2007, S.M. Jung [36] proved using a fixed point approach that the Volterra nonlinear integral equation is Hyers-Ulam-Rassias stable on a compact interval under certain conditions. Then several authors [17, 27, 28] generalized the previous result on the Volterra integral equations to infinite interval in the case when the integrand is Lipschitz with a fixed Lipschitz constant. In the near past many research papers have been published about Hyers-Ulam stability of Voltera integral equations of different type including nonlinear Volterra integro-differential equations, mixed integral dynamic system with impulses etc.[18, 19, 63, 75].

To the best of our knowledge, the first ones who pay attention to Hyers-Ulam stability for Volterra integral equations on time scales are S. Andras, A.R. Meszaros [8] and L. Hua, Y. Li, J. Feng [33]. However they restricted their research to the case when integrand satisfies Lipschitz conditions with some Lipschitz constant. We generalize the results of [8, 33] using Lipschitz function, which has an unbounded Lipschitz coefficient, and the Banach's fixed point theorem at appropriate functional space with Bielecki type norm. There are also papers on impulsive integral equations on time scales [64, 65].

Chapter 2

Volterra integral equations on unbounded time scales

2.1 Existence and uniqueness of solutions

Our main purpose is to fill the gap in the literature by furnishing the basic theory of linear and nonlinear Volterra equations on time scales.

We believe that integral equations on time scales have an enormous potential for rich and diverse applications and thus they are most worthy of attention. Studies into the area will not only provide a deeper understanding of traditional integral and summation equations by uncovering the strange distinctions and interesting links between the two areas but will also lead to new discoveries in those dynamic equations on time scales where the delta derivative is present.

Our view here is based on two simple facts: in the investigation of dynamic equations on time scales, the analysis most often turns to that of a related integral equation on time scales; and the area of integral equations on time scales by its general nature will enjoy at least as many applications to science, engineering and technology as the field of dynamic equations on time scales.

We focus on the qualitative and quantitative properties of solutions to Volterra integral equations on time scales. Some important questions that this work addresses are

- Under what conditions do the integral equations actually have (possibly unique) solutions?
- If solutions do exist then what are their nature; and how can we find them; or closely approximate them?

The main techniques those are employ to answer the above questions are from contemporary areas of nonlinear analysis including the fixed point theorem of Banach space; inequality theory and a novel definition of measuring distance in metric spaces and normed spaces.

The results contained herein complement those of Stefan Hilger's seminal paper of 1990 [31] and more recently, those of Tisdell and Zaidi [67] and Kulik and Tisdell [40].

Nonlinear problems on unbounded intervals

Here we consider nonlinear Volterra integral equation on time scales.

$$x(t) = f(t) + \int_a^t K(t, s, x(s))\,\Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T} \tag{2.1}$$

and $\mathbb{T}$ is unbounded above. So that we are interested in solutions to these problems that are defined on the unbounded interval $[a, +\infty)_{\mathbb{T}}$. We apply our results to some initial value problems with unbounded domains.

T. Kulik and C. Tisdell [40] restricted his research to the case when Lipschitz type function is constant. We generalize the results of [40, 49, 56, 62] using Lipschitz function which has an unbounded Lipschitz coefficient and the Banach fixed point theorem at appropriate functional space. Furthermore, it allows to get new sufficient conditions for boundedness of solutions [45]. In addition, we should note articles [1, 38] those are very important in this direction.

For $p \in R$, We define the exponential function $e_p(\cdot, a)$ on the time scales $\mathbb{T}$ as the unique solution to the scale IVP

$$x^{\Delta} = p(t)x, \quad x(a) = 1$$

If $p \in R^+$, then $e_p(t, a) > 0$ for all $t \in \mathbb{T}$.

Consider the nonlinear integral equation (2.1) with $I_{\mathbb{T}} = [a, +\infty)_{\mathbb{T}}$. We now construct the appropriate Banach space for our analysis. Let $\beta\colon I_{\mathbb{T}} \to \mathbb{R}$ be a regressive and rd-continuous scalar function and let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^n$. We will consider the linear space of continuous functions $C(I_{\mathbb{T}}; \mathbb{R}^n)$, such that

$$\sup_{t \in I_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t, a)} < \infty$$

and denote this special space by $C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$. We couple the linear space $C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$ with a suitable metric, namely

$$d_\beta(x, y) = \sup_{t \in I_{\mathbb{T}}} \frac{|x(t) - y(t)|}{e_\beta(t, a)}.$$

We will also consider $C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$ coupled with a suitable Bielecki type norm, expressly

$$\|x\|_\beta = \sup_{t \in I_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t, a)}.$$

We generalize and at the same time simplify the results [40, 49, 56, 62] assuming L can be an unbounded rd-continuous function.

The above definitions of d_β and $\|x\|_\beta$ are generalisation of Bielecki's metric and [11, 24, 25, 66] in the time scale environment and complement those introduced in [67].

Observe that we have important Bernoulli type estimate [14].

$$1 \le 1 + \int_a^t \beta(s)\,\Delta s \le e_\beta(t, a) \le \exp\left(\int_a^t \beta(s)\,\Delta s\right) \tag{2.2}$$

for all $t \in I_{\mathbb{T}}$.

We now present our first major result of this section concerning the existence and uniqueness.

Theorem 2.1.2

Consider the integral equation (2.1). Let $K\colon I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be jointly continuous in its first and third variables and rd-continuous in its second variable, $f\colon I_{\mathbb{T}} \to \mathbb{R}^n$ be continuous, $L\colon I_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, $\gamma > 1$ and $\beta(s) = L(s)\gamma$. If

$$|K(t,s,p) - K(t,s,q)| \leq L(s)|p-q|, \quad p,q \in \mathbb{R}^n, \quad s < t, \tag{2.3}$$

$$m = \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| f(t) + \int_a^t K(t,s,0)\,\Delta s \right| < \infty, \tag{2.4}$$

then the integral equation (2.1) has a unique solution $x \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$.

Proof:

Consider the following equivalent formulation of (2.1), namely

$$x(t) = \left(f(t) + \int_a^t K(t,s,0)\,\Delta s \right) + \int_a^t \left(K(t,s,x(s)) - K(t,s,0) \right)\Delta s. \tag{2.5}$$

We will show that (2.5) has a unique solution and thus (2.1) must also have a unique solution.

Let $L\colon I_{\mathbb{T}} \to \mathbb{R}$ be the function defined in (2.3) and let $\beta(s) = L(s)\gamma$, where $\gamma > 1$. Consider the Banach space $C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ and let operator F be defined by

$$[Fx](t) = \left(f(t) + \int_a^t K(t,s,0)\,\Delta s \right) + \int_a^t \left(K(t,s,x(s)) - K(t,s,0) \right)\Delta s.$$

Fixed point of F will be solution to (2.5). Thus we want to prove that there exists a unique x such that $Fx = x$. We show that $F\colon C_\beta(I_{\mathbb{T}};\mathbb{R}^n) \to C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. Let $x \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. Taking norms in (2.5), we obtain

$$\begin{aligned}
\|Fx\|_\beta &= \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| f(t) + \int_a^t K(t,s,0)\,\Delta s + \int_a^t \left(K(t,s,x(s)) - K(t,s,0) \right)\Delta s \right| \\
&\leq m + \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t |K(t,s,x(s)) - K(t,s,0)|\Delta s \\
&\leq m + \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s)|x(s)|\,\Delta s \\
&= m + \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s) e_\beta(s,a) \frac{|x(s)|}{e_\beta(s,a)}\,\Delta s \\
&\leq m + \|x\|_\beta \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s) e_\beta(s,a)\,\Delta s \\
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \gamma L(s) e_\beta(s,a)\,\Delta s \\
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \beta(s) e_\beta(s,a)\,\Delta s \\
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta^\Delta(s,a)\,\Delta s
\end{aligned}$$

$$
\begin{aligned}
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} [e_\beta(s,a)]_a^t \\
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} [e_\beta(t,a) - 1] \\
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \left(1 - \frac{1}{e_\beta(t,a)}\right) = m + \frac{\|x\|_\beta}{\gamma} < \infty.
\end{aligned}
$$

Hence, we see that $F\colon C_\beta(I_{\mathbb{T}};\mathbb{R}^n) \to C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. We show that F is a contraction map with contraction constant $\alpha = 1/\gamma < 1$ and then Banach fixed point theorem will apply. For any $u, v \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$

$$
\begin{aligned}
\|Fu - Fv\|_\beta &= \sup_{t \in I_{\mathbb{T}}} \frac{|[Fu](t) - [Fv](t)|}{e_\beta(t,a)} \\
&\leq \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t |K(t,s,u(s)) - K(t,s,v(s))| \, \Delta s \\
&\leq \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s)|u(s) - v(s)| \, \Delta s \\
&= \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s) e_\beta(s,a) \frac{|u(s) - v(s)|}{e_\beta(s,a)} \, \Delta s \\
&\leq \|u - v\|_\beta \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s) e_\beta(s,a) \, \Delta s \\
&= \frac{\|u - v\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \gamma L(s) e_\beta(s,a) \, \Delta s \\
&= \frac{\|u - v\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \beta(s).e_\beta(s,a) \Delta s \\
&= \frac{\|u - v\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta^\Delta(s,a) \, \Delta s \\
&= \frac{\|u - v\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} [e_\beta(s,a)]_a^t \\
&= \frac{\|u - v\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \left[1 - \frac{1}{e_\beta(t,a)}\right] \\
&\leq \frac{\|u - v\|_\beta}{\gamma} = \alpha \|u - v\|_\beta.
\end{aligned}
$$

As $\alpha < 1$, we see that F is a contraction map and so Banach fixed point theorem applies, yielding the existence of a unique fixed point x of F.

Here we give an example of Volterra integral equation for which the problem of existence and uniqueness is not possible to solve using Lipschitz constant. In this example we used unbounded Lipschitz function.

Example 2.1.1

Consider the scalar integral equation

$$
x(t) = t^2 + \int_a^t (s + \sigma(s))[x(s)^2 + 1]^{\frac{1}{2}} \, \Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}, a \geq 0.
$$

We claim that this integral equation has a unique solution for arbitrary $\mathbb{T}$.

Proof:

We will use the above Theorem 2.1.2 and make use of the fact that $K(t,s,p) = (s+\sigma(s))(p^2+1)^{\frac{1}{2}}$ has a bounded partial derivative with respect to p everywhere. Consider,

$$\begin{aligned} |K(t,s,p) - K(t,s,q)| &= (s+\sigma(s)) \left|(p^2+1)^{\frac{1}{2}} - (q^2+1)^{\frac{1}{2}}\right| \\ &\le (s+\sigma(s)) \sup_{r\in\mathbb{R}} \left|\frac{r}{(r^2+1)^{\frac{1}{2}}}\right| |p-q| \\ &\le (s+\sigma(s))\,|p-q|. \end{aligned}$$

Here we used the mean value theorem. So (2.3) holds with $L(s) = s+\sigma(s)$. For a choice of, say, $\gamma = 2$. We then have $\beta(s) = 2(s+\sigma(s))$ and considering that

$$\int_a^t \beta(s)\Delta s = 2\int_a^t (s+\sigma(s))\,\Delta s = 2(t^2-a^2)$$

and $e_\beta(t,a) \ge 1+t^2-a^2$. It is not difficult to verify (2.4) holds. The result now follows from the Theorem 2.1.2.

2.2 Bounded solutions

We now present results concerning boundedness of solutions of Volterra integral equations. Our result gives a nice rate of growth of solutions on the entire interval $[a,\infty)_{\mathbb{T}}$ and we believe that this idea has many potential applications concerning boundedness, asymptotic behaviour and stability of solutions.

In this section we investigate the boundedness of the solution to (2.1) when the forcing term $f(t)$ is bounded on $[t_0,+\infty)_{\mathbb{T}}$. Here our purpose to prove stability result for (2.1) under bounded perturbations, according to the following definition.

Theorem 2.2.1

Consider the Volterra integral equation (2.1) satisfying conditions of Theorem 2.1.2. If in addition

$$\sup_{t\in I_{\mathbb{T}}} \int_a^t L(s)\,\Delta s = \nu < \infty \tag{2.6}$$

and

$$m_1 = \sup_{t\in I_{\mathbb{T}}} \left| f(t) + \int_a^t K(t,s,0)\,\Delta s\right| < \infty, \tag{2.7}$$

then the unique solution of Volterra integral equation (2.1) on time scales that are unbounded above is bounded

$$\sup_{t\in I_{\mathbb{T}}} |x(t)| < +\infty.$$

Proof:

In our case $\beta(s) = \gamma L(s)$. Using the estimate

$$1 \le 1 + \int_a^t \beta(s)\,\Delta s \le e_\beta(t,a) \le \exp\left(\int_a^t \beta(s)\,\Delta s\right) \tag{2.8}$$

and condition

$$\sup_{t\in I_{\mathbb{T}}}\int_a^t L(s)\,\Delta s=\nu<\infty \tag{2.9}$$

We get that

$$1\le e_\beta(t,a)\le \exp\left(\int_a^t \beta(s)\,\Delta s\right)\le \exp(\int_a^t \beta(s)\Delta s).$$

$$1\le e_\beta(t,a)\le \exp\left(\int_a^t \beta(s)\,\Delta s\right)\le \exp(\int_a^t \gamma L(s)\Delta s).$$

$$1\le e_\beta(t,a)\le \exp\left(\int_a^t \beta(s)\,\Delta s\right)\le \exp(\gamma\nu).$$

So norm $\|x\|_\beta$ and supremum norm $\sup_{t\in I_{\mathbb{T}}}|x(t)|$ are equivalent at space $C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. This means that Volterra integral equation solution is bounded if additional conditions (2.6) and (2.7) fulfiled and $\nu>0$.

Example 2.2.1

Consider the scalar integral equation

$$x(t)=2+\int_a^t \frac{(x(s)^2+1)^{\frac{1}{2}}}{s\sigma(s)}\,\Delta s,\quad a,t\in I_{\mathbb{T}}=[a,+\infty)\cap\mathbb{T}, a>0.$$

We claim that this integral equation has bounded solution for arbitrary $\mathbb{T}$.

Proof:

Let us note that

$$\int_a^t \frac{\Delta s}{s\sigma(s)}=a^{-1}-t^{-1}<a^{-1}.$$

The result now follows from the Theorem 2.2.1.

Let us consider Volterra integral equation (2.1) with more general Lipschitz conditions. This allows us to supplemented conditions of existence of bounded solution [45] and get more general sufficient condition for the boundedness of solution of Volterra integral equations.

Theorem 2.2.2

Consider the integral equation (2.1) satisfying conditions of Theorem 2.1.2 and condition (2.7).

Let $L_1\colon I_{\mathbb{T}}\times I_{\mathbb{T}}\to\mathbb{R}$ be continuous in its first and rd-continuous in its second variable, such that

$$|K(t,s,p)-K(t,s,q)|\le L_1(t,s)|p-q|,\quad p,q\in\mathbb{R}^n,\quad s<t,$$

and assume that there exists $t_1\in I_{\mathbb{T}}$, such that

$$m_2=\sup_{t\in I_{\mathbb{T}}}\int_a^{t_1} L_1(t,s)\,\Delta s<\infty$$

$$\sup_{t\in[t_1,+\infty)\cap\mathbb{T}}\int_{t_1}^t L_1(t,s)\,\Delta s\le\lambda<1.$$

Then the integral equation (2.1) has a unique bounded solution.

Proof:

According the Theorem 2.1.2, Volterra integral equation (2.1) has a solution on time scales that are unbounded above. Consider the following equivalent formulation of (2.1),

$$x(t) = \left(f(t) + \int_a^t K(t,s,0)\,\Delta s \right) + \int_a^t \left(K(t,s,x(s)) - K(t,s,0)\right)\,\Delta s.$$

Let

$$M = m_1 + m_2 \sup_{s\in[a,t_1]\cap\mathbb{T}} |x(s)|$$

and suppose that

$$\sup_{t\in I_{\mathbb{T}}} |x(t)| \le \frac{M}{1-\lambda}.$$

Then

$$\begin{aligned}
|F[x(t)]| &\le \sup_{t\in I_{\mathbb{T}}} \left| f(t) + \int_a^t K(t,s,0)\,\Delta s \right| \\
&+ \int_a^{t_1} |K(t,s,x(s)) - K(t,s,0)|\,\Delta s \\
&+ \int_{t_1}^{t} |K(t,s,x(s)) - K(t,s,0)|\,\Delta s \\
&\le m_1 + \int_a^{t_1} L_1(t,s)|x(s)|\,\Delta s + \int_{t_1}^{t} L_1(t,s)|x(s)|\,\Delta s \\
&\le m_1 + m_2 \sup_{s\in[a,t_1]\cap\mathbb{T}} |x(s)| + \int_{t_1}^{t} L_1(t,s)|x(s)|\,\Delta s \\
&\le M + \int_{t_1}^{t} L_1(t,s)|x(s)|\,\Delta s \\
&\le M + \lambda\frac{M}{1-\lambda} = \frac{M}{1-\lambda}.
\end{aligned}$$

So, we have that

$$\sup_{t\in I_{\mathbb{T}}} |F[x(t)]| \le \frac{M}{1-\lambda}.$$

It follows that for unique fixed point of operator F

$$\sup_{t\in I_{\mathbb{T}}} |x(t)| \le \frac{M}{1-\lambda}.$$

This completes the proof.

Chapter 3

A nonstandard Volterra integral equations on time scales

3.1 A nonstandard Volterra integral equation

Many physical systems can be modeled via dynamical systems on time scales. As a response to the needs of the diverse applications, recently many authors have studied the quantitative properties of solutions of Volterra type integral equations on time scales, see [13, 40, 56, 67]. In [10] the authors have studied the Fredholm integral equation in which the functions involved under the integral sign contains the derivative of an unknown functions using Pervo's fixed point theorem, the method of successive approximation and trapezoidal quadrature rule. In view of the importance of the equation studied in [10, 53, 54, 55] has studied the existence, uniqueness and other properties of solutions of more general integral equations using Banach fixed point theorem and suitable integral inequalities with explicit estimates. Motivated by the result in [10, 53, 54, 55], D.B. Pachpatte consider the nonstandard Volterra type dynamic equation on time scales of the form,

$$x(t) = f(t) + \int_a^t K(t, s, x(s), x^{\Delta}(s))\,\Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}. \tag{3.1}$$

Using the relation

$$x(\sigma(s)) = x(s) + x^{\Delta}(s)\mu(s)$$

in many cases equation (3.1) can be reduced to nonstandard Volterra type integral equation.

Where f and K are given functions and x is the unknown function to be found $g\colon I_{\mathbb{T}} \to \mathbb{R}^n$ and $f\colon I_{\mathbb{T}}^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. t is from a time scale $\mathbb{T}$, which is non empty closed subset of $\mathbb{R}$, the set of real numbers. $s \leq t$ and $I_{\mathbb{T}} = I \cap \mathbb{T}, I = [a, \infty)$ then given subset of $\mathbb{R}$, $\mathbb{R}^n$ the real n dimensional Euclidean space with appropriate norm defined by $|\cdot|$. The integral sign represents the general type of operation known as delta integral (for more details see [13]).

Remark 3.1.1

If we choose $f(t, s, x, u) = E(t, s, x) - u$ then by simple calculation it is easy to observe that the equation (3.1) reduces to the equation of the following form

$$x(t) = f(t) + \int_a^t K(t, s, x(s))\,\Delta s. \tag{3.2}$$

Which is recently studied in [40] by Kulik and Tisdell using Banach and Schafer fixed point theorems (see also in [67]). In [10] the authors have studied the continuous case of a variant of equation (3.1) by using Pervo's fixed point theorem and successive approximations.

We introduce the more general result on existence, uniqueness and boundedness for solutions of nonstandard Volterra type integral equation on an arbitrary time scales.

The field of dynamic equations on time scales is an emerging area that has more potential created by Hilger in 1990 [31]. This new and compelling area of mathematics is more general and versatile than the traditional theories of differential and difference equations. The field of dynamic equations on time scales contains and extends the classical theory of differential, difference, integral and summation equations as special cases.

A nonstandard Volterra type integral equation on an arbitrary time scales $\mathbb{T}$.

$$x(t) = f(t) + \int_a^t K(t, s, x(s), x(\sigma(s)))\,\Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}, \quad (3.3)$$

Where $x\colon I_{\mathbb{T}} \to \mathbb{R}^n$ is the unknown function, $f\colon I_{\mathbb{T}} \to \mathbb{R}^n$ and $K\colon I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are nonlinear functions. We generalized and the same time simplified the results [40, 49, 56, 60, 62], found the new sufficient conditions for the existence and uniqueness of solution for nonstandard Volterra type integral equation. We also found condition for boundedness of solutions and prove the continuous dependence of solutions.

T. Kulik and C.C. Tisdell [40] believe that it might be interesting to study the qualitative and quantitative properties of Volterra type integral equations on time scales

$$x(t) = f(t) + \int_a^t K(t, s, x(\sigma(s)))\,\Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}.$$

This type of integral equations could be very useful for modelling economic process, for example, a Keynesian-Cross model with "lagged" income [26, 67].

John Maynard Keynes was a British economist, whose ideas fundamentally changed the theory and practice of macroeconomics and the economic policies of governments. Originally trained in mathematics, he built on and greatly refined earlier work on the causes of business cycles and was one of the most influential economists of the 20th century. Widely considered the founder of modern macroeconomics, his ideas are the basis for the school of thought known as Keynesian economics.

The Economist (An international weekly newspaper) has described Keynes as Britain's most famous 20th century economist. In this section, we formulate and analyse a simple model from economics in the time scale setting, known as a Keynesian–Cross model with "lagged" income.

In a simple closed economy, the dynamics of: aggregate demand, D; aggregate income, y; aggregate consumption, C; aggregate investment, I; government spending, G; are given by three simple equations, namely:

$$D(t) = C(t) + I + G; \quad (3.4)$$

$$C(t) = C_0 + cy(t); \tag{3.5}$$

$$y^{\Delta} = [D^{\sigma} - y], t \geq a; \tag{3.6}$$

Where $\delta < 1$ is a positive constant known as the "speed of adjustment term" and C_0, c are non-negative constants. To keep the model very simple, G and I are assumed to be constant in (3.4) and current consumption is assumed to depend on current income in (3.5). Eq.(3.6) means that the change in income is a fraction of excess demand at $\sigma(t)$ over income at t. The above example is a generalisation of the classical Keynesian–Cross model involving difference equations given in ([26], p. 23) for the special case $\mathbb{T} = \mathbb{Z}$. If we substitute (3.4) and (3.5) into (3.6), then we obtain

$$y^{\Delta} = \delta[C_0 + cy^{\sigma} + I + G - y] = h(t, y, y^{\sigma}).$$

Using the simple, useful formula $y^{\sigma} = y + \mu y^{\Delta}$ and if $1 - \delta c\mu(t) \neq 0$ for $t \geq a$ then a substitution and rearrangement leads to

$$y^{\Delta} = \frac{\delta(c-1)}{1 - \delta c\mu(t)}.y + \frac{\delta(c_0 + I + G)}{1 - \delta C\mu(t)} = f(t)y + g(t). \tag{3.7}$$

It is easy to verify that the dynamic equation in (3.7). So a unique solution y to our problem exists on any compact interval of the type $[a, \sigma(b)]_{\mathbb{T}}$. However, we can go further and define the solution for all $t \geq a$ by solving the linear dynamic equation (3.7) directly. Using the techniques in ([13], Chap.2)

$$y(t) = e_f(t, a)\left[y(a) + \int_a^t \frac{g(s)}{e_f(\sigma(s), a)}\Delta\right], t \geq a.$$

3.2 Existence and uniqueness of solution on time scales

Consider the integral equation (3.3). We now construct the appropriate Banach space for our analysis. Let $\beta\colon I_{\mathbb{T}} \to \mathbb{R}$ be a nonnegative and rd-continuous scalar function, where $I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}$. The Cauchy initial value problem for scalar linear equation

$$x^{\Delta} = \beta(t)x, \quad x(a) = 1$$

has the unique solution $e_{\beta}(\cdot, a)\colon I_{\mathbb{T}} \to \mathbb{R}$ [13]. More explicitly, using the cylinder transformation the generalized exponential function $e_{\beta}(\cdot, a)$ is given by

$$e_{\beta}(t, a) = \begin{cases} \exp\left(\int_a^t \beta(s)\, ds\right), & \text{for } \mu(s) = 0 \\ \exp\left(\int_a^t \frac{\ln(1+\mu(s)\beta(s))}{\mu(s)}\, \Delta s\right), & \text{for } \mu(s) > 0 \end{cases}$$

where ln is the principal logarithm function. We will use the following property of exponential function [13],

$$e_{\beta}(\sigma(t), a) = (1 + \mu(t)\beta(t))e_{\beta}(t, a). \tag{3.8}$$

Observe that we also have Bernoulli's type inequality [14]

$$1 \leq 1 + \int_a^t \beta(s)\,\Delta s \leq e_\beta(t,a) \leq \exp\left(\int_a^t \beta(s)\,\Delta s\right) \tag{3.9}$$

for all $t \in I_{\mathbb{T}}$.

Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^n$. We will consider the linear space of continuous functions $C(I_{\mathbb{T}};\mathbb{R}^n)$ such that

$$\sup_{t\in I_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t,a)} < \infty$$

and denote this special space by $C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. The space $C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ can be endowed with a suitable Bielecki type norm [11], first used in 1956 while proving existence and uniqueness of solutions of ordinary differential equations, expressly

$$\|x\|_\beta = \sup_{t\in I_{\mathbb{T}}} \frac{|x(t)|}{e_\beta(t,a)}.$$

We generalize and at the same time simplify the results [40, 49, 56, 60, 62] assuming L_1 and L_2 can be an unbounded rd-continuous functions. In the case of unbounded time scale, the Lipschitz coefficients L_1 and L_2 could be unbounded. The use of Bielecki type norms related to Lipschitz type functions allows to choose a suitable functional space to prove the following theorem.

Theorem 3.2.1

Consider the integral equation (3.3). Let $K\colon I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in its first, third and fourth variables and rd-continuous in its second variable, $f\colon I_{\mathbb{T}} \to \mathbb{R}^n$ be continuous, $L_1, L_2\colon I_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, $\sup_{s\in I_{\mathbb{T}}} |L_2(s)\mu(s)| = r < 1$, $1 < \gamma < r^{-1}$ and $\beta(s) = \frac{[L_1(s)+L_2(s)]\gamma}{1-r\gamma}$. If

$$|K(t,s,p,\bar{p}) - K(t,s,q,\bar{q})| \leq L_1(s)|p-q| + L_2(s)|\bar{p}-\bar{q}|,\ p,q,\bar{p},\bar{q} \in \mathbb{R}^n,\ s<t, \tag{3.10}$$

$$m = \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| f(t) + \int_a^t K(t,s,0,0)\,\Delta s \right| < \infty, \tag{3.11}$$

then the integral equation (3.3) has a unique solution $x \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$.

We note that in condition of theorem appears graininess.

Proof:

Consider the following equivalent formulation of (3.3), namely

$$x(t) = \left(f(t) + \int_a^t K(t,s,0,0)\,\Delta s\right) + \int_a^t \left(K(t,s,x(s),x(\sigma(s))) - K(t,s,0,0)\right)\Delta s. \tag{3.12}$$

We will show that (3.12) has a unique solution and thus (3.3) must also have a unique solution.

Let $L_1, L_2 \colon I_{\mathbb{T}} \to \mathbb{R}$ be the functions defined in (3.10) and let $\beta(s) = \frac{[L_1(s)+L_2(s)]\gamma}{1-\gamma r}$, where $1 < \gamma < r^{-1}$. From (3.8) follows

$$\begin{aligned}
\text{Ł}_1(s) + L_2(s)(1+\mu(s)\beta(s)) &= L_1(s) + L_2(s) + L_2(s)\mu(s)\beta(s) \\
&\leq L_1(s) + L_2(s) + \frac{r\,[L_1(s)+L_2(s)]\,\gamma}{1-r\gamma} \\
&= \frac{L_1(s)+L_2(s)}{1-r\gamma} = \frac{\beta(s)}{\gamma}.
\end{aligned}$$

Consider the Banach space $C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ and let operator F be defined by

$$\begin{aligned}
[Fx](t) = &\left(f(t) + \int_a^t K(t,s,0,0)\,\Delta s \right) \\
&+ \int_a^t \left(K(t,s,x(s),x(\sigma(s))) - K(t,s,0,0)\right)\,\Delta s.
\end{aligned}$$

Fixed point of F will be solution to (3.12). Thus we want to prove that there exists a unique x such that $Fx = x$. We show that $F\colon C_\beta(I_{\mathbb{T}};\mathbb{R}^n) \to C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. Let $x \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. Taking norms in (3.12), we obtain

$$\begin{aligned}
\|Fx\|_\beta &= \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| f(t) + \int_a^t K(t,s,0,0)\,\Delta s \right. \\
&+ \left. \int_a^t \left(K(t,s,x(s),x(\sigma(s))) - K(t,s,0,0)\right)\Delta s \right| \\
&\leq m + \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t [L_1(s)|x(s)| + L_2(s)|x(\sigma(s))|]\,\Delta s \\
&= m + \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \left[L_1(s)e_\beta(s,a)\frac{|x(s)|}{e_\beta(s,a)} \right. \\
&+ \left. L_2(s)e_\beta(\sigma(s),a)\frac{|x(\sigma(s))|}{e_\beta(\sigma(s),a)} \right]\,\Delta s \\
&\leq m + \|x\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t [L_1(s) + L_2(s)(1+\mu(s)\beta(s))]\,e_\beta(s,a)\,\Delta s \\
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \beta(s)e_\beta(s,a)\,\Delta s \\
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta^\Delta(s,a)\,\Delta s \\
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} [e_\beta(s,a)]_a^t \\
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} (e_\beta(t,a) - 1) \\
&= m + \frac{\|x\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \left(1 - \frac{1}{e_\beta(t,a)}\right) \leq m + \frac{\|x\|_\beta}{\gamma} < \infty.
\end{aligned}$$

Hence, we see that $F\colon C_\beta(I_{\mathbb{T}};\mathbb{R}^n) \to C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. We show that F is a contraction map with contraction constant $\alpha = 1/\gamma < 1$ and then Banach fixed point theorem will apply.

For any $u, v \in C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$

$$\begin{aligned}
\|Fu - Fv\|_\beta &= \sup_{t\in I_{\mathbb{T}}} \frac{|[Fu](t) - [Fv](t)|}{e_\beta(t,a)} \\
&\leq \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t |K(t,s,u(s),u(\sigma(s))) - K(t,s,v(s),v(\sigma(s)))|\,\Delta s \\
&\leq \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t [L_1(s)|u(s)-v(s)| + L_2(s)|u(\sigma(s)) - v(\sigma(s))|]\,\Delta s \\
&= \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \left[L_1(s)e_\beta(s,a)\frac{|u(s)-v(s)|}{e_\beta(s,a)}\right. \\
&+ \left. L_2(s)e_\beta(\sigma(s),a)\frac{|u(\sigma(s)) - v(\sigma(s))|}{e_\beta(\sigma(s),a)}\right]\Delta s \\
&\leq \|u-v\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t [L_1(s) + L_2(s)(1+\mu(s)\beta(s))]\,e_\beta(s,a)\,\Delta s \\
&= \frac{\|u-v\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \beta(s)e_\beta(s,a)\,\Delta s \\
&= \frac{\|u-v\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} [e_\beta(s,a)]_a^t \\
&= \frac{\|u-v\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta^\Delta(s,a)\,\Delta s \\
&= \frac{\|u-v\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} (e_\beta(t,a) - 1) \\
&= \frac{\|u-v\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \left(1 - \frac{1}{e_\beta(t,a)}\right) \leq \frac{\|u-v\|_\beta}{\gamma} = \alpha\|u-v\|_\beta
\end{aligned}$$

As $\alpha < 1$, we see that F is a contraction map and so Banach fixed point theorem applies, yielding the existence of a unique fixed point x of F.

Example 3.2.1

Consider the scalar integral equation

$$x(t) = t^2 + \int_a^t (s+\sigma(s))[x(\sigma(s))^2 + 1]^{\frac{1}{2}}\,\Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}, a \geq 0.$$

We claim that this integral equation has a unique solution if

$$\sup_{s\in I_{\mathbb{T}}} |(s+\sigma(s))\mu(s)| = r < 1$$

for an arbitrary time scales $\mathbb{T}$.

Proof:

We will use the above Theorem (3.2.1) and make use of the fact that $K(t,s,p) = (s+\sigma(s))(p^2+1)^{\frac{1}{2}}$ has a bounded partial derivative with respect to p everywhere, consider

$$\begin{aligned}
|K(t,s,p) - K(t,s,q)| &= \left|(s+\sigma(s))(p^2+1)^{\frac{1}{2}} - (s+\sigma(s))(q^2+1)^{\frac{1}{2}}\right| \\
&\leq (s+\sigma(s)) \sup_{r\in\mathbb{R}} \left|\frac{r}{(r^2+1)^{\frac{1}{2}}}\right| |p-q| \\
&\leq (s+\sigma(s))\,|p-q|.
\end{aligned}$$

Here we used the mean value theorem. So (3.10) holds with $L_2(s) = s+\sigma(s) = 2s + \mu(s)$. It is not difficult to verify that (3.11) holds. Using the Bernoulli's inequality (3.9), we get

$$e_\beta(t,a) \geq 1 + \frac{\gamma(t^2 - a^2)}{1 - r\gamma}$$

is followed by the estimate $m < \infty$. The result now follows from the Theorem 3.2.1.

3.3 Bounded solution

We now present results concerning boundedness of solutions of nonstandard Volterra type integral equations.

Theorem 3.3.1

Consider the nonstandard Volterra type integral equation (3.3) satisfying conditions of Theorem 3.2.1. If in addition

$$\sup_{t \in I_{\mathbb{T}}} \int_a^t (L_1(s) + L_2(s))\,\Delta s = \nu < \infty \tag{3.13}$$

and

$$m_1 = \sup_{t \in I_{\mathbb{T}}} \left| f(t) + \int_a^t K(t,s,0,0)\,\Delta s \right| < \infty, \tag{3.14}$$

then the unique solution of nonstandard Volterra type integral equation (3.3) on unbounded above time scales is bounded

$$\sup_{t \in I_{\mathbb{T}}} |x(t)| < +\infty.$$

Proof:

In our case $\beta(s) = \frac{[L_1(s)+L_2(s)]\gamma}{1-r\gamma}$. Using the estimate (3.8) and condition (3.13), we get that

$$1 \leq e_\beta(t,a) \leq \exp\left(\int_a^t \beta(s)\,\Delta s\right) \leq \exp\left(\frac{\gamma\nu}{1 - r\gamma}\right).$$

So, norm $\|x\|_\beta$ and supremum norm $\sup_{t\in I_{\mathbb{T}}} |x(t)|$ are equivalent at space $C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$. This means that nonstandard Volterra's type integral equation solution is bounded if additional conditions (3.13) and (3.14) are fulfilled and $\nu > 0$.

Example 3.3.1

Consider the scalar integral equation

$$x(t) = 2 + \int_a^t \frac{(x(\sigma(s))^2 + 1)^{\frac{1}{2}}}{s\sigma(s)}\,\Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}, a > 0.$$

We claim that this integral equation has bounded solution if

$$\sup_{s \in I_{\mathbb{T}}} \left| \frac{\mu(s)}{s\sigma(s)} \right| = r < 1$$

for arbitrary unbounded above $\mathbb{T}$.

Proof:

Let us note that for $0 < a \leq t$

$$\int_a^t \frac{\Delta s}{s\sigma(s)} = a^{-1} - t^{-1} < a^{-1}.$$

The result now follows from the Theorem 3.3.1.

3.4 Continuity of solutions on initial conditions

We will generalize and simplify the results [56, 62] for nonstandard Volterra integral equation (3.3) on time scales $\mathbb{T}$. Let $x\colon I_{\mathbb{T}} \to \mathbb{R}^n$ and $y\colon I_{\mathbb{T}} \to \mathbb{R}^n$ be the solutions of the nonstandard Volterra integral equations

$$x(t) = f_1(t) + \int_a^t K(t, s, x(s), x(\sigma(s)))\,\Delta s, \tag{3.15}$$

and

$$y(t) = f_2(t) + \int_a^t K(t, s, y(s), y(\sigma(s)))\,\Delta s. \tag{3.16}$$

Theorem 3.4.1

Consider the nonstandard Volterra integral equations (3.15) and (3.16) satisfying conditions of Theorem 3.2.1. Let $x(t)$ and $y(t)$ be the solutions of appropriate equations (3.15), (3.16) and let $\delta = \|f_1 - f_2\|_\beta$. Then

$$\|x(t) - y(t)\|_\beta \leq \frac{\gamma}{\gamma - 1}\|f_1 - f_2\|_\beta.$$

Proof:

Analogous to the Theorem 3.2.1, we estimate the difference $\|f_1 - f_2\|_\beta$.

We get,

$$\|x - y\|_\beta \leq \|f_1 - f_2\|_\beta + \frac{\|x - y\|_\beta}{\gamma}.$$

So, we have

$$\|x - y\|_\beta \leq \frac{\gamma}{\gamma - 1}\|f_1 - f_2\|_\beta.$$

Example 3.4.1

Let us consider scalar Volterra integral equation

$$x(t) = f_1(t) + \int_1^t \frac{(x(\sigma(s))^2 + 1)^{\frac{1}{2}}}{s\sigma(s)}\,\Delta s$$

and

$$y(t) = f_2(t) + \int_1^t \frac{(x(\sigma(s))^2 + 1)^{\frac{1}{2}}}{s\sigma(s)}\,\Delta s,\ t \in \mathbb{T},$$

where the time scale $\mathbb{T}$ is the union of closed intervals

$$\mathbb{T} = \bigcup_{n\in\mathbb{N}} [n, n + 1/2]$$

and $f_1, f_2\colon \mathbb{T} \to \mathbb{R}$ are continuous and bounded functions. Then the graininess function μ for right dense points is $\mu(t) = 0$ and $\mu(t) = 1/2$ for right scattered points. Note that Lipschitz type function is

$$L_2(s) = \frac{1}{s\sigma(s)}$$

and observe that

$$r = \sup_{s\in\mathbb{T}} |L_2(s)\mu(s)| = \frac{2}{9}.$$

Note that we put $\gamma = 2$ and $\beta(s) = 3,6L_2(s)$, then

$$1 + \mu(s)\beta(s) \le 1,8.$$

We obtain

$$\begin{aligned}
|x(t) - y(t)| &\le |f_1(t) - f_2(t)| + \int_1^t \frac{|x(\sigma(s)) - y(\sigma(s))|}{s\sigma(s)}\,\Delta s \\
&\le |f_1(t) - f_2(t)| + \|x - y\|_\beta \int_1^t \frac{e_\beta(\sigma(s),1)}{s\sigma(s)}\,\Delta s \\
&= |f_1(t) - f_2(t)| + \|x - y\|_\beta \int_1^t \frac{(1+\mu(s)\beta(s))e_\beta(s,1)}{s\sigma(s)}\,\Delta s \\
&\le |f_1(t) - f_2(t)| + \frac{\|x - y\|_\beta}{2} \int_1^t \beta(s)e_\beta(s,1)\,\Delta s \\
&\le |f_1(t) - f_2(t)| + \frac{\|x - y\|_\beta}{2}(e_\beta(t,1) - 1).
\end{aligned}$$

Hence

$$\frac{|x(t) - y(t)|}{e_\beta(t,1)} \le \frac{|f_1(t) - f_2(t)|}{e_\beta(t,1)} + \frac{\|x - y\|_\beta}{2}.$$

It follows

$$\|x(t) - y(t)\|_\beta \le \|f_1(t) - f_2(t)\|_\beta + \frac{\|x(t) - y(t)\|_\beta}{2}$$

or

$$\|x - y\|_\beta \le 2\|f_1 - f_2\|_\beta.$$

Let us note that

$$1 \le e_\beta(t,1) \le \exp\left(\int_a^t \beta(s)\,\Delta s\right) \le e^{3,6}.$$

It follows that

$$e^{-3,6}\sup_{t\in\mathbb{T}} |x(t) - y(t)| \le \|x - y\|_\beta \le \sup_{t\in\mathbb{T}} |x(t) - y(t)|.$$

So, we get estimate

$$\sup_{t\in\mathbb{T}} |x(t) - y(t)| \le 2e^{3,6}\sup_{t\in\mathbb{T}} |f_1(t) - f_2(t)|.$$

Chapter 4

Hyers-Ulam stability of Volterra integral equations

S.M. Ulam [68] at the University of Wisconsin in 1940 raised the question when the solutions of an equations differing slightly from a given one must be somehow near to the solution of the given equation. D.H. Hyers [34] gave an affirmative answer to the question of S.M. Ulam for Cauchy additive functional equation in Banach space. So stability concept proposed by S.M. Ulam and D.H. Hyers was named as *Hyers-Ulam stability.* Afterwards, Th.M. Rassias [59] introduced new ideas of the initial concept of Hyers-Ulam stability using unbounded right-hand sides in the involved inequalities, depending on certain functions, introducing therefore the so-called *Hyers-Ulam-Rassias stability.*

In 2007, S.M. Jung [36] proved using fixed point method that the Volterra nonlinear integral equation is Hyers-Ulam-Rassias stable on a compact interval under certain conditions. Then several authors [17, 27, 28] extended previous results on the Volterra integral equations to infinite intervals in the case when integrand is Lipschitz with a fixed Lipschitz constant. In the near past many research papers have been published about Ulam-Hyers stability of Voltera integral equations of different type including nonlinear Volterra integro-differential equations, mixed integral dynamics system with impulses etc.[18, 19, 63, 75].

The theory of time scales analysis has been rising fast and has acknowledged a lot of interest. The pioneer of this theory was S.Hilger [31]. He introduced this theory in 1988 with the inspiration to unify continuous and discrete calculus. For the introduction to the calculus on time scales and to the theory of dynamic equations on time scales we recommend the books [13, 15] by M. Bohner and A. Peterson.

T. Kulik and C.C. Tisdell [40, 67] gave the basic qualitative and quantitative results to Volterra integral equations on time scales in the case when integrand is Lipschitz with a fixed Lipschitz constant. A. Reinfelds and S. Christian [60, 61] generalized previous results using Lipschitz function, which has an unbounded Lipschitz coefficient.

To the best of our knowledge, the first ones who pay attention to Hyers-Ulam stability for Volterra integral equations on time scales are S.Andras, A.R. Meszaros [8] and L. Hua, Y. Li, J. Feng [33]. However they restricted their research to the case when integrand satisfies Lipschitz conditions with some Lipschitz constant. We generalize the results of [8, 33] using Lipschitz function, which has an unbounded Lipschitz coefficient and the Banach fixed point theorem at appropriate functional space with Bielecki type norm. There are also papers on impulsive integral equations on time scales [64, 65].

4.1 Hyers-Ulam stability on unbounded time scales

Consider the nonlinear Volterra integral equation

$$x(t) = f(t) + \int_a^t K(t,s,x(s))\,\Delta s, \quad a,t \in I_{\mathbb{T}} = [a,+\infty) \cap \mathbb{T}. \tag{4.1}$$

In paper [60], we proved the existence and uniqueness of solution of equation (4.1) using Lipschitz type functions which can be unbounded.

Theorem 4.1.1

Let $K\colon I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \to \mathbb{R}^n$ be jointly continuous in its first and third variables and rd-continuous in its second variable, $f\colon I_{\mathbb{T}} \to \mathbb{R}^n$ be continuous, $L\colon I_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, $\gamma > 1$ and $\beta(s) = L(s)\gamma$. If

$$|K(t,s,x) - K(t,s,x')| \le L(s)|x - x'|, \quad x,x' \in \mathbb{R}^n, \quad s < t, \tag{4.2}$$

$$m = \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| f(t) + \int_a^t K(t,s,0)\,\Delta s \right| < \infty,$$

then the nolinear Volterra integral equation (4.1) has a unique solution $x \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$.

Proof:

Consider the Banach space $C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. To prove the Theorem 4.1.1, we defined an operator $F\colon C_\beta(I_{\mathbb{T}};\mathbb{R}^n) \to C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ by expression

$$[Fx](t) = \int_a^t (K(t,s,x(s)) - K(t,s,0))\,\Delta s.$$

Here $L\colon I_{\mathbb{T}} \to \mathbb{R}$ is the Lipschitz type function defined by (4.2) and $\beta(s) = L(s)\gamma$, where $\gamma > 1$. Analogously to the Theorem 4.1.1 [60].

We can verify that for any $x, x' \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$

$$\begin{aligned}
\|[Fx](t) - [Fx'](t)\|_\beta &= \sup_{t \in I_{\mathbb{T}}} \frac{|[Fx](t) - [Fx'](t)|}{e_\beta(t,a)} \\
&\le \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t |K(t,s,x(s)) - K(t,s,x'(s))|\,\Delta s \\
&\le \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s)|x(s) - x'(s)|\,\Delta s \\
&= \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s)e_\beta(s,a)\frac{|x(s) - x'(s)|}{e_\beta(s,a)}\,\Delta s \\
&\le \|x - x'\|_\beta \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t L(s)e_\beta(s,a)\,\Delta s \\
&= \frac{\|x - x'\|_\beta}{\gamma} \sup_{t \in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \gamma L(s)e_\beta(s,a)\,\Delta s
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\|x-x'\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta^\Delta(s,a)\,\Delta s \\
&= \frac{\|x-x'\|_\beta}{\gamma} \sup_{t\in I_{\mathbb{T}}} \left[1-\frac{1}{e_\beta(t,a)}\right] \\
&\le \frac{\|x-x'\|_\beta}{\gamma}.
\end{aligned}
$$

So, we get

$$
\left\|\int_a^t K(t,s,x(s))\,\Delta s - \int_a^t K(t,s,x'(s))\,\Delta s\right\|_\beta \le \frac{\|x(t)-x'(t)\|_\beta}{\gamma}.
$$

Definition 4.1.1

We say that integral equation (4.1) is Hyers-Ulam stable if there exists a constant $C>0$, such that for each real number $\varepsilon>0$ and for each solution $x\in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ of the inequality

$$
\|x(t)-f(t)-\int_a^t K(t,s,x(s))\,\Delta s\|_\beta \le \varepsilon,
$$

there exists a solution $x_0\in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ of the integral equation (4.1) with the property

$$
\|x(t)-x_0(t)\|_\beta \le C\varepsilon.
$$

Let us find sufficient conditions for the Hyers-Ulam stability of nonlinear Volterra integral equation on time scales.

Theorem 4.1.2

Consider the nonlinear Volterra integral equation (4.1) satisfying conditions of Theorem 4.1.1. Suppose $x\in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ is such that satisfies the inequality

$$
\|x(t)-f(t)-\int_a^t K(t,s,x(s))\,\Delta s\|_\beta \le \varepsilon.
$$

Then nonlinear Volterra integral equation (4.1) is Hyers-Ulam stable.

Proof:

According to the Theorem 4.1.1 [60], there is a unique solution x_0 of the Volterra integral equation (4.1) in Banach space $x_0\in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. Therefore we get the estimate

$$
\|x(t)-x_0(t)\|_\beta \le \left\|x(t)-f(t)-\int_a^t K(t,s,x(s))\,\Delta s\right\|_\beta
$$

$$
+\left\|\int_a^t K(t,s,x(s))\,\Delta s - \int_a^t K(t,s,x_0(s))\,\Delta s\right\|_\beta \le \varepsilon+\gamma^{-1}\|x(t)-x_0(t)\|_\beta.
$$

Hence

$$
\|x(t)-x_0(t)\|_\beta \le C\varepsilon, \tag{4.3}
$$

where $C=(1-\gamma^{-1})^{-1}$.

Example 4.1.1

Consider the scalar Volterra integral equation for an arbitrary $\mathbb{T}$.

$$x(t) = t^2 + \int_a^t (s+\sigma(s))[x(s)^2+1]^{\frac{1}{2}}\,\Delta s, \quad a, t \in I_{\mathbb{T}} = [a, +\infty) \cap \mathbb{T}, a \geq 0. \quad (4.4)$$

According to Theorem 4.1.1 [60], there is a unique solution of the Volterra integral equation (4.4) in Banach space $C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$, where $\beta(t) = L(s)\gamma$ and $\gamma > 1$. Let us note that according to [14], we have estimate

$$1 + \gamma(t^2 - a^2) \leq e_\beta(t, a) \leq \exp(\gamma(t^2 - a^2))$$

which ensures the existence of a solution. It follows from the Theorem 4.1.2 that integral equation (4.4) is Hyers-Ulam stable in Banach space $C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$.

4.2 Hyers-Ulam stability on bounded time scales

In the case of a bounded (compact) time scales $a, b \in I_{\mathbb{T}} = [a, b] \cap \mathbb{T}$, we have

$$1 \leq \sup_{t \in I_{\mathbb{T}}} e_\beta(t, a) \leq \sup_{t \in I_{\mathbb{T}}} \exp \int_a^t \beta(s)\,\Delta s = M < \infty.$$

Let us note that every rd-continuous function on a compact interval is bounded. Therefore supremum norm and Bielecki type norm [11] at Banach space $C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$ are equivalent

$$\sup_{t \in I_{\mathbb{T}}} |x(t)| \leq M\|x\|_\beta \leq M \sup_{t \in I_{\mathbb{T}}} |x(t)|.$$

We can take also $\gamma = 1$. Then $\beta(t) = L(t)$ and we get estimate

$$\|[Fx](t) - [Fx'](t)\|_\beta \leq (1 - M^{-1})\|x - x'\|_\beta.$$

From Theorem 4.1.2, we get

$$\|x(t) - x_0(t)\|_\beta \leq M \left\| x(t) - f(t) - \int_a^t K(t, s, x(s))\,\Delta s \right\|_\beta.$$

It follows

$$\begin{aligned}
\sup_{t \in I_{\mathbb{T}}} |x(t) - x_0(t)| &\leq M\|x(t) - x_0(t)\|_\beta \\
&\leq M^2 \left\| x(t) - f(t) - \int_a^t K(t, s, x(s))\,\Delta s \right\|_\beta \\
&\leq M^2 \sup_{t \in I_{\mathbb{T}}} \left| x(t) - f(t) - \int_a^t K(t, s, x(s))\,\Delta s \right|.
\end{aligned}$$

Here $C = M^2$.

It follows that integral equation (4.1) on bounded time scales is also Hyers-Ulam stable in Banach space with supremum norm.

Chapter 5

Hyers-Ulam stability of Volterra type integral equations

In 1940, S.M. Ulam [68] at the University of Wisconsin raised the question when the solutions of an equations differing slightly from a given one must be somehow near to the exact solution of the given equation. In the following year, D.H. Hyers [34] gave an affirmative answer to the question of S.M. Ulam for Cauchy additive functional equation in Banach space. So stability concept proposed by S.M. Ulam and D.H. Hyers was named as *Hyers-Ulam stability.* Afterwards Th.M. Rassias [59] proposed generalization of the initial concept of Hyers-Ulam stability using unbounded right-hand sides in the involved inequalities, depending on certain functions, introducing therefore the so-called *Hyers-Ulam-Rassias stability.*

In 2007, S.M. Jung [36] proved using fixed point method that the Volterra nonlinear integral equation is Hyers-Ulam-Rassias stable on a compact interval under certain conditions. Then several authors [17, 27, 28] extended previous results on the Volterra integral equations to infinite intervals in the case when integrand is Lipschitz with a fixed Lipschitz constant. In the near past many research papers have been published about Ulam-Hyers stability of Voltera integral equations of different type including nonlinear Volterra integro-differential equations, mixed integral dynamics system with impulses etc.[18, 19, 63, 75].

The theory of time scales analysis has been rising fast and has acknowledged a lot of interest. The pioneer of this theory was S.Hilger [31]. He introduced this theory in 1988 with the inspiration to unify continuous and discrete calculus. For the introduction to the calculus on time scales and to the theory of dynamic equations on time scales we recommend the books [13, 15] by M. Bohner and A. Peterson.

T. Kulik and C.C. Tisdell [40, 67] gave the basic qualitative and quantitative results to Volterra integral equations on time scales in the case when integrand is Lipschitz with a fixed Lipschitz constant. A. Reinfelds and S. Christian [60, 61] generalized previous results using Lipschitz function, which has an unbounded Lipschitz coefficient.

To the best of our knowledge, the first ones who pay attention to Hyers-Ulam stability for Volterra integral equations on time scales are S. Andras, A.R. Meszaros [8] and L. Hua, Y. Li, J. Feng [33]. However, they restricted their research to the case when integrand satisfies Lipschitz conditions with some Lipschitz constant. We generalize the results of [8, 33] using Lipschitz type function, which can be an unbounded and the Banach fixed point theorem at appropriate functional space with

Bielecki type norm. There are also papers on impulsive integral equations on time scales [64, 65].

D.B. Pachapatte [56] studied qualitative properties of solutions of general nonlinear Volterra integral equation on time scales.

$$x(t) = f\left(t, x(t), \int_a^t K(t,s,x(s)\right) \Delta s. \tag{5.1}$$

In the present paper we give new existence and uniqueness conditions of solutions and analyse Hyers-Ulam stability for the following class of Volterra type integral equation on an arbitrary time scales $\mathbb{T}$.

$$x(t) = f\left(t, x(t), x(\sigma(t)), \int_a^t K(t,s,x(s),x(\sigma(s)))\right) \Delta s, \quad a,t \in I_{\mathbb{T}} = [a,+\infty) \cap \mathbb{T}. \tag{5.2}$$

Remark:

Many integro-differential equations can be reduced to Volterra type integral equations for example

$$x^{\Delta}(t) = f(t, x(t), x^{\Delta}(t)) + \int_a^t K(t,s,x(s),x^{\Delta}(s))ds,\ x(a) = \alpha.$$

Define a new function

$$y(t) = x^{\Delta}(s).$$

So we have Volterra type integral equation

$$\begin{aligned} x(t) &= \alpha + \int_a^t y(s)ds \\ y(t) &= f(t,x(t),y(t)) + \int_a^t K(t,s,x(s),y(s))ds. \end{aligned}$$

5.1 Hyers-Ulam stability on unbounded time scales

Lets us give new existence and uniqueness conditions of solutions and analysis. Hyers-Ulam stability for the following class of Volterra type integral equation (5.2). We assume that Lipschitz coefficients L_1 and L_2 can be an unbounded rd-continuous functions. The use of Bielecki type norms related to Lipschitz type functions allow to choose a suitable functional space to prove the following theorem.

Theorem 5.1.1

Consider the integral equation (5.2). Let $K\colon I_{\mathbb{T}} \times I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous in its first, third and fourth variables and rd-continuous in its second variable, $f\colon I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, $L_1, L_2\colon I_{\mathbb{T}} \to \mathbb{R}$ be rd-continuous, $\sup_{s\in I_{\mathbb{T}}} |L_1(s)\mu(s)| = q < \infty$, $\sup_{s\in I_{\mathbb{T}}} |L_2(s)\mu(s)| = r < 1$, $1 < \gamma < r^{-1}$ and $\beta(s) = \frac{[L_1(s)+L_2(s)]\gamma}{1-r\gamma}$. If

$$|f(t,x,x',x") - f(t,\bar{x},\bar{x}',\bar{x}")| \le M(|x-\bar{x}| + |x'-\bar{x}'| + |x" - \bar{x}"|),$$

where

$$M\left(1+\frac{1+p\gamma}{1-r\gamma}+\frac{1}{\gamma}\right)<1,$$

$$|K(t,s,x,\bar{x})-K(t,s,x',\bar{x}')|\le L_1(s)|x-x'|+L_2(s)|\bar{x}-\bar{x}'|,\ s<t,$$

and

$$m=\sup_{t\in I_{\mathbb{T}}}\frac{1}{e_\beta(t,a)}\left|f\left(t,0,0,\int_a^t K(t,s,0,0)\,\Delta s\right)\right|<\infty,$$

then the integral equation (5.2) has a unique solution $x\in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$.

Proof:

Let $L_1, L_2\colon I_{\mathbb{T}}\to\mathbb{R}$ be the Lipschitz functions and let $\beta(s)=\frac{[L_1(s)+L_2(s)]\gamma}{1-\gamma r}$, where $1<\gamma<r^{-1}$. It follows

$$1+\mu(s)\beta(s)=1+\frac{\mu(s)(L_1(s)+L_2(s))\gamma}{1-\gamma r}=1+\frac{(q+r)\gamma}{1-\gamma r}=\frac{1+q\gamma}{1-r\gamma}$$

and

$$\begin{aligned}
Ł_1(s)+L_2(s)(1+\mu(s)\beta(s)) &= L_1(s)+L_2(s)+L_2(s)\mu(s)\beta(s)\\
&\le L_1(s)+L_2(s)+\frac{r\,[L_1(s)+L_2(s)]\,\gamma}{1-r\gamma}\\
&= \frac{L_1(s)+L_2(s)}{1-r\gamma}=\frac{\beta(s)}{\gamma}.
\end{aligned}$$

Consider the Banach space $C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. To prove the Theorem 5.1.1, we defined an operator $F\colon C_\beta(I_{\mathbb{T}};\mathbb{R}^n)\to C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ by expression

$$[Fx](t)=f\left(t,x(t),x(\sigma(t)),\int_a^t K(t,s,x(s),x(\sigma(s)))\,\Delta s\right).$$

We showed that for any $u,v\in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$

$$\begin{aligned}
\|Fu-Fv\|_\beta &= \sup_{t\in I_{\mathbb{T}}}\frac{|[Fu](t)-[Fv](t)|}{e_\beta(t,a)}\\
&= \sup_{t\in I_{\mathbb{T}}}\frac{1}{e_\beta(t,a)}\left|f\left(t,u(t),u(\sigma(t)),\int_a^t K(t,s,u(s),u(\sigma(s)))\,\Delta s\right)\right.\\
&- \left.f\left(t,v(t),v(\sigma(t)),\int_a^t K(t,s,v(s),v(\sigma(s)))\,\Delta s\right)\right|\\
&\le M\sup_{t\in I_{\mathbb{T}}}\frac{1}{e_\beta(t,a)}\left(|u(t)-v(t)|+|u(\sigma(t))-v(\sigma(t))|\right.\\
&+ \left.\left|\int_a^t K(t,s,u(s),u(\sigma(s)))\,\Delta s-\int_a^t K(t,s,v(s),v(\sigma(s)))\,\Delta s\right|\right)\\
&= I_1+I_2+I_3.
\end{aligned}$$

We get,

$$I_1=M\sup_{t\in I_{\mathbb{T}}}\frac{1}{e_\beta(t,a)}|u(t)-v(t)|=M\|u-v\|_\beta$$

$$I_2 = M \sup_{t\in I_{\mathbb{T}}} \frac{1+\mu(t)\beta(t)}{e_\beta(\sigma(t),a)} |u(\sigma(t)) - v(\sigma(t))| = M\frac{1+q\gamma}{1-r\gamma}\|u-v\|_\beta$$

$$\begin{aligned}
I_3 &= M \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| \int_a^t K(t,s,u(s),u(\sigma(s)))\,\Delta s - \int_a^t K(t,s,v(s),v(\sigma(s)))\,\Delta s \right| \\
&\le M \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \left| \int_a^t [L_1(s)|u(s)-v(s)| + L_2(s)|u(\beta(s)) - v(\beta(s))|]\,\Delta s \right| \\
&\le M\|u-v\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t [L_1(s) + L_2(s)(1+\mu(s)\beta(s))]\, e_\beta(s,a)\,\Delta s] \\
&\le \frac{M}{\gamma}\|u-v\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t \beta(s) e_\beta(s,a)\,\Delta s \\
&= \frac{M}{\gamma}\|u-v\|_\beta \sup_{t\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} \int_a^t e_\beta^\Delta(s,a)\,\Delta s \\
&= \frac{M}{\gamma}\|u-v\|_\beta \sup_{x\in I_{\mathbb{T}}} \frac{1}{e_\beta(t,a)} (e_\beta(t,a) - 1) \\
&= \frac{M}{\gamma}\|u-v\|_\beta \sup_{t\in I_{\mathbb{T}}} \left(1 - \frac{1}{e_\beta(t,a)}\right) = \frac{M}{\gamma}\|u-v\|_\beta.
\end{aligned}$$

It follows

$$\|Fu - Fv\|_\beta \le M\|u-v\|_\beta \left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right).$$

We show that $F\colon C_\beta(I_{\mathbb{T}};\mathbb{R}^n) \to C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. Let $x \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$. Taking norms, we obtain

$$\begin{aligned}
\|Fx\|_\beta = \|Fx - F0 + F0\|_\beta &\le \|Fx - F0\|_\beta + \|F0\|_\beta \\
&\le M\|x\|_\beta \left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right) + m.
\end{aligned}$$

As $M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right) < 1$, we see that F is a contraction map and so Banach fixed point theorem applies, yielding the existence of a unique fixed point x of map F.

Definition 5.1.1

We say that integral equation (5.2) is Hyers-Ulam stable if there exists a constant $C > 0$, such that for each real number $\varepsilon > 0$ and for each solution $x \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ of the inequality

$$\sup_{t\in I_{\mathbb{T}}} \frac{\left|x(t) - f\left(t, x(t), x(\sigma(t)), \int_a^t K(t,s,x(s),x(\sigma(s)))\,\Delta s\right)\right|}{e_\beta(t,a)} = \|x - Fx\|_\beta \le \varepsilon$$

there exists a solution $x_0 \in C_\beta(I_{\mathbb{T}};\mathbb{R}^n)$ of the integral equation (5.2) with the property

$$\|x - x_0\|_\beta \le C\varepsilon.$$

Let us find sufficient condition for the Volterra type integral equation (5.2) to be Hyers-Ulam stable.

Theorem 5.1.2

If $x_0 \in C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$ is solution of Volterra type integral equation (5.2) and

$$M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right) < 1,$$

then Volterra type integral equation (5.2) is Hyers-Ulam stable.

Proof:

According to the Theorem 5.1.1, there is unique solution $x_0 \in C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$ to the Volterra type integral equation (5.2)in Banach space. Therefore we get the estimate

$$\begin{aligned}\|x - x_0\|_\beta &\leq \|x - Fx\|_\beta + \|Fx - Fx_0\|_\beta \\ &\leq \varepsilon + M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right)\|x - x_0\|_\beta.\end{aligned}$$

Hence

$$\|x - x_0\|_\beta \leq C\varepsilon, \tag{5.3}$$

Where $C = \left(1 - M\left(1 + \frac{1+q\gamma}{1-r\gamma} + \frac{1}{\gamma}\right)\right)^{-1}$.

5.2 Hyers-Ulam stability on bounded time scales

In the case of a bounded (compact) time scales $a, b \in I_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. We have

$$1 \leq \sup_{t \in I_{\mathbb{T}}} e_\beta(t, a) \leq \sup_{t \in I_{\mathbb{T}}} \exp \int_a^t \beta(s)\, \Delta s = N < \infty.$$

Let us note that every rd-continuous function on a compact interval is bounded. Therefore supremum norm and Bielecki type norm [11] at Banach space $C_\beta(I_{\mathbb{T}}; \mathbb{R}^n)$ are equivalent

$$\sup_{t \in I_{\mathbb{T}}} |x(t)| \leq N\|x\|_\beta \leq N \sup_{t \in I_{\mathbb{T}}} |x(t)|.$$

We can take also $\gamma = 1$. Then $\beta(s) = \frac{L_1(t)+L_2(t)}{1-r}$ and we get estimate

$$\|Fx - Fx_0\|_\beta \leq M\left(1 + \frac{1+q}{1-r} + (1 - N^{-1})\right)\|x - x_0\|_\beta$$

and

$$\|x - Fx\|_\beta \leq \sup_{t \in I_{\mathbb{T}}} \left| x(t) - f\left(t, x(t), x(\sigma(t)), \int_a^t K(t, s, x(s), x(\sigma(s)))\, \Delta s\right)\right| \leq \varepsilon.$$

From Theorem 5.1.2, we get

$$\begin{aligned}\|x - x_0\|_\beta &\leq \|x - Fx\|_\beta + \|Fx - Fx_0\|_\beta \\ &\leq \varepsilon + M\left(1 + \frac{1+q\gamma}{1-r\gamma} + 1 - N^{-1}\right)\|x - x_0\|_\beta.\end{aligned}$$

It follows

$$\sup_{t \in I_{\mathbb{T}}} |x(t) - x_0(t)| \leq N\|x - x_0\|_\beta \leq C\varepsilon.$$

Here $C = N\left(1 - M\left(2 + \frac{1+q\gamma}{1-r\gamma} - N^{-1}\right)\right)^{-1}$.

It follows that integral equation (5.2) on bounded time scales is also Hyers-Ulam stable in Banach space with supremum norm.

Conclusions

To conclude I would like to say that in my belief the overall goal of the thesis has been achieved :

- The more general result on existence, uniqueness and boundedness for solutions of nonlinear Volterra integral equation and nonstandard Volterra type integral equation on time scales.
- Hyers-Ulam stability of a nonlinear Volterra integral equation and general Volterra type integral equations on unbounded and bounded time scales.
- New sufficient conditions are obtained based on the Banach fixed point theorem, Bielecki type norm and Lipschitz type functions.

The motivation of the first part (categorical) of the work was to generalize the results of T. Kulik and C. Tisdell [40] using Lipschitz function, which has an unbounded Lipschitz coefficient and the Banach fixed point theorem at appropriate functional space endowed with a suitable Bielecki type norm. The ideas for the second part of the work was to generalize the results of S. Andras, A.R. Meszaros [8] and L. Hua, Y. Li, J. Feng [33] using Lipschitz function, which has an unbounded Lipschitz coefficient and the Banach fixed point theorem at appropriate functional space with Bielecki type norm. Each part was also motivated by the examples.

Each part of the research has some own concluding remarks. So, here we just want to mention that, although each part is a accomplished research, it could be also further developed in the future and the ideas of future research are mentioned in the concluding remarks in each chapter or section.

List of conferences

C1 24th International Conference on Mathematical Modelling and Analysis (MMA 2019), "A nonstandard Volterra integral equation on unbounded above time scales", Tallinn, Estonia.

C2 23rd International Conference on Mathematical Modelling and Analysis(MMA 2018), "Volterra integral equation on unbounded above time scales", Sigulda, Latvia.

C3 12th Latvian Mathematical Conference (2018), "Existence, uniqueness and boundedness of solutions of Volterra equation on unbounded above time scales", Ventspils, Latvia.

C4 78th scientific conference of University of Latvia (2020), "Hyers-Ulam stability of Volterra integral equation on time scales", Riga, Latvia.

C5 77th scientific conference of University of Latvia (2019), "A nonstandard Volterra integral equation on unbounded above time scales", Riga, Latvia.

C6 76th scientific conference of University of Latvia (2018), "Existence and uniqueness of solutions of Volterra equation on time scales", Riga, Latvia.

C7 75th scientific conference of University of Latvia (2017), "Dynamical system on time scales", Riga, Latvia.

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