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GENERALIZATIONS OF A GROBMAN-HARTMAN THEOREM FOR NONAUTONOMOUS DYNAMICAL SYSTEM

PhD Thesis

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Abstract

In this thesis, more general results comparing to previously known results for the Grobman-Hartman theorem are obtained, even for R^n, by relaxing the condition on the linear part while strengthening the condition on the nonlinear part. In the study of local qualitative theory of dynamical system, the Grobman-Hartman theorem, also called the linearization theorem, is a very important result. The theorem is about the local behavior of a dynamical system in the neighborhood of a hyperbolic equilibrium point. Many further results have been explored since the initial proof of the theorem in 1959. In our research we use a Green type map and the integral functional equation technique to substantially simplify the proof of the theorem, moreover, the method used in this paper to prove the dynamical equivalence is completely different from previous papers. Furthermore, for a more general point of view we consider nonautonomous differential and difference equations in an arbitrary Banach space, also on time scales, and impulsive differential equations in an arbitrary Banach space. The use of the integral functional equation technique led us not only to the simpler proof, but also to a more general sufficient condition for the existence of a bounded solution on the time scales, also for periodic solution. Moreover, we derive a new sufficient condition for the Hyers-Ulam stability of a linear dynamic equation in the case when the integral of a Green's type map is uniformly bounded. To highlight our improvement, in comparison to previous results, this thesis has been supplemented with several examples.

Keywords: Quasi-linear differential equation, invertible difference equation, Grobman's-Hartman's linearization theorem, impulsive quasi-linear equations, dynamical equivalence, dynamic equations on time scale, Green type map, bounded solution, periodic solution, Hyers-Ulam stability.

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Introduction

Differential equations began with the invention of calculus, which was introduced by Newton and Leibniz. English mathematician Isaac Newton (1642 - 1727) worked on different types of equations and, in 1671, in work [42] he classified three cases of first order differential equations: $\frac{dy}{dx} = f(x)$, $\frac{dy}{dx} = f(x, y)$ and $x\frac{du}{dx} + y\frac{du}{dy} = u$. In 1695 Jacob Bernoulli introduced another type of differential equation, which was named after him: $y' + P(x)y = Q(x)y^n$ [11]. Leibniz obtained solutions for Bernoulli's ordinary differential equation by simplifying it. In 1746, the one-dimensional wave equation was introduced by d'Alembert. Several years later, the three-dimensional equation was discovered by Euler [63]. Since then, many mathematicians have made great contributions to differential equations. Nowadays, differential equations are not the only subject of study in pure mathematics - where the primary focus is on the existence and uniqueness of the solution - but also applied mathematics, with a wide application in physics, engineering, biology, finance, economics, and chemistry.

Although difference equations were discovered much earlier than differential equations, little research has been done on them, compared to differential equations. During the last few decades computer programming, application ranges and visualization possibilities have changed a lot, this have given us the opportunity to compute several mathematical problems, for example, difference equations. In order to solve difference equations with computers, it requires not only a formulation of approximate difference, but also a formulation of the overall difference equation theory. Therefore, difference equations have recently again come to light.

A dynamical system consists of an abstract phase space or state space, whose coordinates describe the state of any instant, and a dynamical rule that specifies, the immediate future of all state variables, given only the present values of those same state variables. In other words, a dynamical system is any system, physical, or biological, that changes in time. For example, the state of a pendulum is its angle and angular velocity, and the evolution rule is Newton's equation $F = ma$. The qualitative theory of dynamical systems originated in Poincare's work *celestial mechanics* (Poincare 1899), and especially in a 270-page work, prize winning, and initially flawed paper (Poincare 1890). The methods developed in his work laid the basis for the local and global analysis of nonlinear differential equations, including the stability theory for fixed points and periodic orbits, stable and unstable manifolds, and the Poincare recurrence theorem. Following Poincare and Hadamard's studies of geodesic flows, G.D.Birkhoff showed that - near any homoclinic point of a two-dimensional map, there is an infinite sequence of periodic orbits whose periods approach infinity (Birkhoff 1927).

In the study of local qualitative theory of dynamical system, the Grobman-Hartman theorem, also called the linearization theorem, is a very important result. The theorem is about the local behavior of dynamical system in the neighborhood of a hyperbolic equilibrium point. The Grobman-Hartman theorem was proved independently by an American mathematician Philip Hartman [27] and a Russian mathematician D.M. Grobman [25] in 1959. H. Poincaré, in 1879, proved the theorem under the assumptions that the elementary divisors of A are simple and that the eigenvalues $\lambda_1, ..., \lambda_n$ of A lie in a half plane in C and satisfy $\lambda_j \neq m_1\lambda_1 + ... + m_n\lambda_n$ for all sets of non-negative integers $(m_1, ..., m_n)$ satisfying $m_1 + ... + m_n > 1$. A similar result for smooth function was established by S. Sternberg [64] in 1957. Further, the linearization problem in the theory of ordinary differential equations was explored by D.M. Grobman [26], P. Hartman [28], K.J. Palmer [46] and other mathematicians [51], [52].

Goals and objectives

The main objectives of this thesis are to introduce more general results for - Grobman-Hartman's theorem of quasi-linear equations, impulsive equations, even on time scales and the existence of bounded solutions, and periodic solutions, as well as Hyers-Ulam stability. Furthermore, the generalization of Grobman-Hartman's theorem is obtained by the use of a different method of proof. To highlight the class of equations applicable for the extended Grobman-Hartman theorem, several examples should be represented.
The tasks undertaken in the thesis are as follows.

- To generalize the Grobman-Hartman theorem using a Green type map and an integral functional equation technique that allows to obtain more general conditions for quasi-linear and impulsive equations.

- To investigate the time-scale calculus of dynamical systems to generalize the Grobman-Hartman theorem on time scales.

- To investigate the existence of bounded solutions, periodic solutions and Hyers-Ulam stability, and to generalize the result of quasi-linear dynamic equations on time scale.

- To introduce several examples to represent the extensions of previously known results.

Thesis structure

The thesis is structured as follows.
Chapter 1 presents basic preliminaries and creates the background for the thesis.

Starting with an introduction to difference and differential equations, continuing with the introduction of time scales calculus, including several examples, the concept of differentiation and integration, and explanation of exponential function. This chapter also contains well know results, such as metric spaces, Banach fixed point theorem and Green's function, as well as the lesser known concept - the Green type map.

Chapter 2 presents the Grobman-Hartman theorem and its Palmer's generalization. Afterwards, by relaxing conditions on the linear part and, at the same time, strengthening conditions on the nonlinear part, more general results of the quasi-linear equations are obtained. We use the Green type map and integral functional equation technique [52], [53], [54] to substantially simplify the proof. Moreover, we use a different method to prove the dynamical equivalence. Furthermore, for a more general point of view, we will consider nonautonomous differential equations and nonautonomous difference equations in an arbitrary Banach space. To highlight our improvement in comparison to previous results, we will use an example where the linear part of the differential equation does not even possess an ordinary dichotomy.

Chapter 3 provides an introduction to impulsive equations and its solution which is complemented with example. The equivalence problem, involving the impulse effect, was first considered by A. Reinfelds and L. Sermone [56], [57], [62]. Other generalizations of the linearization problem are reported in [55], [67], [69] and [70]. At the end of the chapter, proof, and an example, is represented for the generalization theorem.

Chapter 4 shows the generalization of the Grobman-Hartman theorem on time scales. The theorem and proof are complemented with an example.

Chapter 5 formulates the sufficient condition for the existence of bounded or periodic solution to the quasi-linear dynamic equation, where the weaker condition of the nonlinear part and its Lipschitz coefficient play an important role. This chapter also indicates Hyers-Ulam stability, its new sufficient condition of a linear dynamic equation in the case when the integral of Green's type map is uniformly bounded. This chapter contains several interesting examples.

Approbation

The results obtained in the process of writing the thesis have been presented at five international conferences (Mathematical Modelling and Analysis 2013, ICDEA 2015, ICDEA 2016, PODE 2016 and Conference on Differential and Difference Equations and Applications 2014) and five domestic conferences. See the full list of attended conferences on page 100.

The main results of the research have been reflected in the three scientific publications (2 SCOPUS, 2 Web of Science) and abstracts on international or Latvian conferences. See the full list of publications on page 100.

1 Preliminaries and Background

1.1 Introduction to differential equations

Newton and Leibniz invented calculus, which led to beginning of differential equations. English mathematician Isaac Newton (1642-1727) worked on different type of equations and in 1671 in work *Methodus of fluxionum et Serierum Infinitraum* [42] he classified three cases of first order differential equations: $\frac{dy}{dx} = f(x)$, $\frac{dy}{dx} = f(x, y)$ and $x\frac{du}{dx} + y\frac{du}{dy} = u$. Another type of differential equation was introduced in 1695 [11] by Jacob Bernoulli, and the equation was called after his name: $y' + P(x)y = Q(x)y^n$. Leibniz obtain solutions for Bernoullis‘ ordinary differential equation by simplifying it. In 1746 the one-dimensional wave equation was introduced by d‘Alembert. Several years later the three-dimensional equation was discovered by Euler [63]. Since then, many mathematicians have made great contributions to differential equations. Nowadays, differential equations are not only the subject of study in pure mathematics where main focus is on the existence and uniqueness of solution but also applied mathematics with wide application in physics, engineering, biology, finance, economics, and chemistry. There are many works regarding differential equations such as [22], [65], [41], [43] and many other.

1.2 Introduction to difference calculus

Difference equations were discovered much earlier than differential equations, but little research has been done on them compared to differential equations. It is known that Isaac Newton made use of his calculus. After that, George Boole wrote a definite treatise on the calculus of finite difference in 1872. Difference equations are frequently useful in computer-aided problem solving because they lead naturally into recursive algorithms for computer solution. Computer programming, applications range and visualizations possibilities have changed a lot during last decades and it had also given the opportunity to compute several mathematical problems, for example, difference equations. In order to solve difference equation with computers, it requires not only a formulation of approximate difference but also a formulation of overall difference equation theory. Therefore, difference equations have recently come to light.

An equation which defines a value of a sequence as a function of other terms of the sequence is called a *difference equations* or *recurrence equation*. These equations occur in mathematics itself as well as its applications to statistics, computing, electrical circuit analysis, economics, biology, and other fields. For example, in economics difference equations are used to model gross domestic product, the inflation rate, the exchange rate etc.

1.2.1 Difference operator

For basic concepts in this section, we refer to [37], [1]. Analyzing and solving difference equations can be simplified by use of *difference calculus*, which is a collection of mathematical tools very similar to the differential calculus.

Definition 1. Let $y(t)$ be a function of a real or complex variable t. The *difference operator* Δ is defined by

$$\Delta y(t) = y(t+1) - y(t).$$

Usually, t is natural number $\mathbb{N} = \{1, 2, 3, ...\}$, however, sometimes t might be choose a continuous set as the interval $[0, \infty)$ or the complex plane. Step size is not always restricted to 1. Consider a difference operator with a step size $h > 0$

$$\Delta y(t) = y(t+h) - y(t).$$

Difference operator subscription is used to distinguish which variable is to be shifted. For example,

$$\Delta_t t^2 \sin(n) = (t+1)^2 \sin(n) - t^2 \sin(n)$$
$$\Delta_n t^2 \sin(n) = t^2 \sin(n+1) - t^2 \sin(n).$$

Higher order differences are defined by composing the difference operator with itself. The second order difference is

$$\Delta^2 y(t) = \Delta(\Delta(y(t))) = \Delta(y(t+1) - y(t)) =$$
$$= (y(t+2) - y(t+1)) - (y(t+1) - y(t)) = y(t+2) - 2y(t+1) + y(t).$$

The formula for n^{th} order difference equation is obtains by induction:

$$\begin{aligned} \Delta^n y(t) = y(t+n) - ny(t+n-1) + \tfrac{n(n-1)}{2!} y(t+n-2) + \\ + ... + (-1)^n y(t) = \\ = \sum_{k=0}^{n} (-1)^k \tbinom{n}{k} y(t+n-k). \end{aligned}$$

Definition 2. The *shift operator* E is defined by

$$Ey(t) = y(t+1).$$

Theorem 1. *The basic difference operator properties are as follows.*

- $\Delta^m(\Delta^n y(t)) = \Delta^{m+n} y(t)$ *for all positive integers* m *and* n.
- $\Delta(y(t) + z(t)) = \Delta y(t) + \Delta z(t)$.
- $\Delta(\alpha y(t)) = \alpha \Delta y(t)$*, where* α *is a constant.*

- $\Delta(y(t)z(t)) = y(t)\Delta z(t) + Ez(t)\Delta y(t)$, *where E is a shift operator.*
- $\Delta\left(\frac{y(t)}{z(t)}\right) = \frac{z(t)\Delta y(t) - y(t)\Delta z(t)}{z(t)Ez(t)}$.

In order to have more complete view of difference operator, we introduce its inverse operator anti-difference or sometimes called indefinite sum.

Definition 3. An *anti-difference* of $y(t)$ is any function $\sum y(t)$ that satisfies equation

$$\Delta(\sum y(t)) = y(t)$$

for all t in the domain of y.

Similar concept for differential equation is indefinite integral

$$\frac{d}{dt}(\int y(t)dt) = y(t).$$

Theorem 2. *If $z(t)$ is an anti-difference of $y(t)$ then any anti-difference of $y(t)$ is given by*

$$\sum y(t) = z(t) + C(t)$$

where $C(t)$ has the same domain as $y(t)$ and $\Delta C(t) = 0$.

If the domain is defined on a set of the type $\{a, a+1, a+2, ...\}$, where a is an any real number, then $C(t)$ is a constant function.

Theorem 3. *Let a be a constant. Then, for $\Delta C(t) = 0$,*

- $\sum a^t = \frac{a^t}{a-1} + C(t), \qquad a \neq 1.$
- $\sum \sin(at) = -\frac{\cos a(t-\frac{1}{2})}{2\sin\frac{a}{2}} + C(t), \qquad a \neq 2n\pi.$
- $\sum \cos(at) = \frac{\sin a(t-\frac{1}{2})}{2\sin\frac{a}{2}} + C(t), \qquad a \neq 2n\pi.$
- $\sum \log t = \log\Gamma(t) + C(t), \qquad t > 0.$
- $\sum \binom{t}{a} = \binom{t}{a+1} + C(t).$
- $\sum \binom{a+t}{t} = \binom{a+t}{t-1} + C(t).$

Some general anti-difference properties are given in the following theorem.

Theorem 4.

- $\sum(y(t) + z(t)) = \sum y(t) + \sum z(t)$.
- $\sum Dy(t) = D\sum y(t)$, *where D is a constant.*
- $\sum(y(t)\Delta z(t)) = y(t)z(t) - \sum Ez(t)\Delta y(t)$.
- $\sum(Ey(t)\Delta z(t)) = y(t)z(t) - \sum z(t)\Delta y(t)$.

Last two parts in the Theorem 4 is known as *summation by part* formulas.

1.2.2 Linear difference equation

Linear difference equations are special class of difference equations. Here the concept linear and nonlinear is similar to differential equations. Linear difference equations play very important role in overall difference equations analysis and there are several reasons for that:

- many types of problems are naturally formulated as linear equations (for example, population dynamics, the study of single commodity in economics, the study of the motion of single body in physics);
- certain linear difference equation sub-classes represent large families of equations that can be solved explicitly (for example, first order equation, equation with constant coefficients);
- it has algebraic properties that permits the use of matrix methods, operational methods transforms, generating functions, and other special techniques;
- certain methods of analysis for nonlinear equations depends on the properties of associated linear equations.

Definition 4. The *first order linear* difference equation is

$$y(t+1) - p(t)y(t) = r(t), \tag{1.21}$$

where $p(t)$ and $r(t)$ are given functions with $p(t) \neq 0$ for all t.

The equations 1.21 is said to be of first order because it contains first order difference operator $\Delta y(t) = y(t+1) - y(t)$ by having y at only t and $t+1$.

1.3 Time scales

Time scale calculus was first introduced in 1988 by Stefan Hilger in his PhD thesis [30] (supervised by Bernd Aulbach). Time scale is the unification of the theory of difference and differential equations, in other words it unifies differential and integral calculus with the calculus of finite difference and offers a formal study for hybrid discrete-continuous dynamical system. If we differentiate a function defined on the real numbers then definition of derivative is equal to that of standard differentiation but if the function is defined on the integers, then it is equivalent to the forward difference operator. In other words, time scale theory is like forceful theoretical tool which provides possibility to study differential and difference equations simultaneously. Martin Bohner and Allan Peterson published a book *Dynamic Equations on Time Scales-An Introduction with applications* [12] on the bases of S. Hilger's work, which has made an important contribution in the field of time scales. Many results which have been proved for differential equation might be carried over to difference equation quite easily but at the same time some results cannot be easily transformed from discrete to continuous case or vice verse. The study of time scales reveals such discrepancies. The overall idea is to prove result for dynamic equations in which the domain of the unknown function is so-called time scale or measure chain, which is an arbitrary non-empty closed subset of the real numbers. The following definitions and theorems and also a general introduction to the theory of time scales can be found in the book by M. Bohner and A.

Peterson [12] as well as many other literature afterwards, for example, [13], [17]. For basic concepts in this section, we refer to [12], [13].

Definition 5. A *time scale* $\mathbb{T}$ is an arbitrary non-empty closed subset of $\mathbb{R}$.

The set of real numbers, the integers, the natural numbers

$$\mathbb{R}, \quad \mathbb{Z}, \quad \mathbb{N}, \quad \mathbb{N}_0$$

are examples of time scales, while rational numbers, the irrational numbers, the complex numbers and open interval

$$\mathbb{Q}, \quad \mathbb{R}\backslash\mathbb{Q}\,, \quad \mathbb{C}, \quad (0,1)$$

are not time scales. Some other examples of time scales are as follows:

- $\mathbb{T} = [2,4]$;
- $\mathbb{T} = \{an : n \in \mathbb{Z}\}, a > 0$;
- $\mathbb{T} = \{p^n : n \in \mathbb{N}_0\}, p > 1$;
- $\mathbb{T} = \{6,7,8,9\}$;
- $\mathbb{T} = [2,3] \cup \{6\} \cup \{8\}$;
- The Cantor set.

We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from real numbers with the standard topology. Let‘s denote delta derivative for function f as f^Δ, where

1. $f^\Delta = f'$, if $\mathbb{T} = \mathbb{R}$ (derivative);
2. $f^\Delta = \Delta f$, if $\mathbb{T} = \mathbb{Z}$ (forward jump operator).

Definition 6. For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) =$inf $\{s \in \mathbb{T} : s > t\}$

Note that this inf always exists. Accordingly, we define backward jump operator.

Definition 7. For $t \in \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t)$ =sup $\{s \in \mathbb{T} : s < t\}$

If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Points that are *right-scattered* and *left-scattered* at the same time is called *isolated*. Also, if $t < sup\ \mathbb{T}$ and $\sigma(t) = t$ then t is called *right-dense*, and if $t > inf\ \mathbb{T}$ and $\rho(t) = t$ then t is called *left-dense*. Point that are *right-dense* and *left-dense* at the same time are called *dense*. If $\mathbb{T}$ has right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} - \{m\}$, otherwise $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has left-scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition 8. The graininess function μ is the distance from point to the closest point on the right and is given by $\mu(t) = \sigma(t) - t$

1.3.1 Examples of time scale

Example 1. Real numbers. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and graininess function $\mu(t) = 0$

Example 2. Whole numbers. If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\rho(t) = t - 1$ and $\mu(t) = 1$

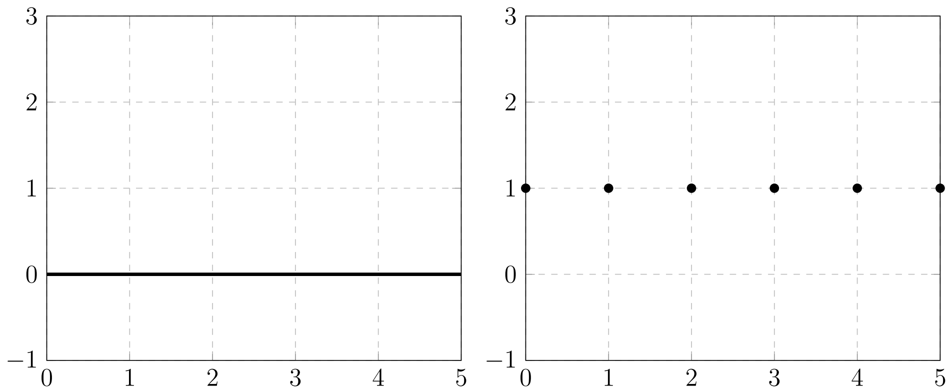

Figure 1.1: Graininess function μ. Left: $\mathbb{T} = \mathbb{R}$, right: $\mathbb{T} = \mathbb{Z}$

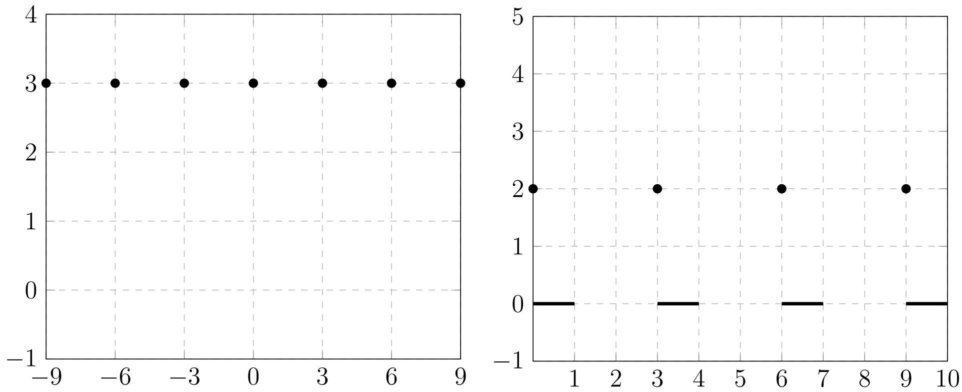

Figure 1.2: Graininess function μ. Left: $\mathbb{T} = 3\mathbb{Z}$, right: $\mathbb{T} = P_{1,2}$

Example 3. Let $c > 0$ be fixed real number. Define time scale $c\mathbb{Z}$ by

$$c\mathbb{Z} = \{cz : z \in \mathbb{Z}\} = \{..., -2c, -1c, 0, c, 1c, 2c, ...\} \tag{1.31}$$

Here $\sigma(t) = t + c$, $\rho(t) = t - c$ and $\mu(t) = t + c - t = c$.
If, for example, $c = 3$, then $3\mathbb{Z} = \{3z : z \in \mathbb{Z}\} = \{... - 9, -6, -3, 0, 3, 6, 9, ...\}$, $\sigma(t) = t + 3$, $\rho(t) = t - 3$, $\mu(t) = 3$.

Example 4. [17] Let $a, b > 0$ be fixed real numbers. Define the time scale $P_{a,b}$ as a collection of closed intervals starting form 0 by

$$P_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a] \tag{1.32}$$

Simple calculations show that:

$$\sigma(t) = \begin{cases} t, & t \in \bigcup\limits_{k=0}^{\infty} [k(a+b), k(a+b)+a) \\ t+b, & t \in \bigcup\limits_{k=0}^{\infty} \{k(a+b)+a\} \end{cases} \tag{1.33}$$

$$\rho(t) = \begin{cases} t, & t \in \bigcup\limits_{k=0}^{\infty} (k(a+b), k(a+b)+a] \\ t-b, & t \in \bigcup\limits_{k=0}^{\infty} \{k(a+b)\} \end{cases} \tag{1.34}$$

and

$$\mu(t) = \begin{cases} 0, & t \in \bigcup\limits_{k=0}^{\infty} [k(a+b), k(a+b)+a) \\ b, & t \in \bigcup\limits_{k=0}^{\infty} \{k(a+b)\} \end{cases} \tag{1.35}$$

If, for example, $a = 1$ and $b = 2$ then $P_{1,2} = \bigcup_{k=0}^{\infty} [3k, 3k+1]$,

$$\sigma(t) = \begin{cases} t, & t \in \bigcup_{k=0}^{\infty} [3k, 3k+1] \\ t + b, & t \in \bigcup_{k=0}^{\infty} [3k+1] \end{cases} \tag{1.36}$$

and

$$\mu(t) = \begin{cases} 0, & t \in \bigcup_{k=0}^{\infty} [3k, 3k+1] \\ b, & t \in \bigcup_{k=0}^{\infty} [3k] \end{cases} \tag{1.37}$$

1.3.2 Differentiation

The delta derivative f^{Δ} definition for a function f defined on $\mathbb{T}$ was for the first time introduced by Stefan Hilger in his PhD thesis [30]. In this section, the definition of Hilger or delta derivative, examples and some properties are described. We refer to [12].

Definition 9. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. $f^{\Delta}(t)$ is called the *delta or Hilger derivative* of f at t (provided it exists) if any $\epsilon > 0$, there is a neighborhood U of t (i.e. $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\mid [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] \mid \leqslant \epsilon \mid \sigma(t) - s \mid \tag{1.38}$$

We say f is *delta differentiable* on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the *delta derivative* of f on $\mathbb{T}^k$.

Example 5. If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = \alpha$, $\forall t \in \mathbb{T}$, where $\alpha \in \mathbb{R}$ is constant, then delta derivative of function f is 0 ($f^{\Delta} \equiv 0$). We can make sure of that from definition, because for any $\epsilon > 0$

$$\mid [f(\sigma(t)) - f(s)] - 0 * [\sigma(t) - s] \mid = \mid [\alpha - \alpha] - 0 * [\sigma(t) - s] \mid = 0 \leqslant \epsilon \mid \sigma(t) - s \mid$$

holds for all $s \in \mathbb{T}$

Example 6. If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then $f^{\Delta}(t) \equiv 1$. This follows since for any $\epsilon > 0$

$$\mid [f(\sigma(t)) - f(s)] - 1 * [\sigma(t) - s] \mid = \mid [\sigma(t) - s] - 1 * [\sigma(t) - s] \mid = 0 \leqslant \epsilon \mid \sigma(t) - s \mid$$

holds for all $s \in \mathbb{T}$

Theorem 5. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we have the following:*

- *If f is differentiable at t, then f is continuous at t;*
- *If f is continuous at t and t is right-scattered, then f is differentiable at t with*
$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)};$$
- *If f is right-dense then f is differentiable at t iff the limit*
$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$
exists as a finite number. In this case
$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s};$$
- *If f is differentiable at t,then*
$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

If $f : \mathbb{T} \to \mathbb{R}$ is a function, then $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e. $f^\sigma = f \circ \sigma$.

Lemma 6 ([80])**.** *Assume $f,\ g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$. The properties of delta derivatives are:*

$$(f+g)^\Delta(t) = f^\Delta(t) + g^\Delta(t);$$
$$(cf)^\Delta(t) = cf^\Delta(t), \quad c \in \mathbb{R};$$
$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t);$$
$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}, \quad g(t)g^\sigma(t) \neq 0.$$

From lemma we can obtain that

$$(f^2)^\Delta = (f * f)^\Delta = f^\Delta f + f^\sigma f^\Delta = (f + f^\sigma) f^\Delta.$$

Example 7. If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t^2$ for all $t \in \mathbb{T}$, then

$$(t^2)^{\Delta} = (t + t^{\sigma})t^{\Delta} = t + \sigma(t),$$

and from the delta derivative definition we can assure that for any $\epsilon > 0$

$$\begin{gathered} \mid [f(\sigma(t)) - f(s)] - (t + \sigma(t)) * [\sigma(t) - s] \mid = \\ = \mid [\sigma^2(t)) - s^2] - (t\sigma(t) + \sigma^2(t) - ts - s\sigma(t) \mid = \\ = \mid (s - t)(\sigma(t) - s) \mid \leqslant \epsilon \mid \sigma(t) - s \mid \end{gathered}$$

holds for all $s \in \mathbb{T}$.

Another consequence of the Lemma 6 is as follows:

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

Example 8. If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = \frac{1}{t}$, then

$$\left(\frac{1}{t}\right)^{\Delta} = -\frac{t^{\Delta}}{t * \sigma(t)} = -\frac{1}{t\sigma(t)}.$$

1.3.3 Integration

The basic concepts on integration on time scale are described in this section according to Bohner and Peterson [12], [13]. We present existence theorems, rd-continuity, regularity and examples.

Definition 10. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* if its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 11. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if

- it is continuous at right-dense points in $\mathbb{T}$ and
- its left-sided limits exist (finite) at left dense points in $\mathbb{T}$.

The set of rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ usually is denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The notability of rd-continuous functions for existence of solution under global Lipschitz-condition and the global existence theorem on dynamic equations is revealed by Hilger [32].

The following implications are immediate:

$$\text{continuous} \rightarrow \text{rd-continuous} \rightarrow \text{regulated}$$

Theorem 7. *Every differential function is continuous.*

The forward jump operator σ, is an example of rd-continuous function. The graininess function $\mu(t) = \sigma(t) - t$ is rd-continuous. In at the same time left-dense and right-scattered points, however, it is not continuous. Some results are summarized in the following theorem.

Theorem 8. *Assume $f : \mathbb{T} \rightarrow \mathbb{R}$.*

1. *If f is continuous, then f is rd-continuous.*
2. *If f is rd-continuous, then f is regulated.*
3. *The jump operator σ is rd-continuous.*
4. *If f is regulated or rd-continuous, then so is f^σ.*
5. *Assume f is continuous. If $g : \mathbb{T} \rightarrow \mathbb{R}$ is regulated or rd-continous, then $f \circ g$ has that property too.*

Definition 12. A continuous function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called pre-differentiable with (region of differentiable) $D \subset \mathbb{T}^k$, $\mathbb{T}^k \setminus D$ is countable and contains no right scattered elements of $\mathbb{T}$ and f is differentiable at each $t \in D$.

Theorem 9. *Every regulated function on a compact interval is bounded.*

Theorem 10 (Existence of pre-antiderivatives)**.** *Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t)$$

for all $t \in D$.

Definition 13. Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in theorem of existence of pre-antiderivatives is called a *pre-antiderivative* of f. We define *indefinite integral* of a regulated function f by

$$\int f(t)\Delta t = F(t) + C$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the *Cauchy integral* by

$$\int\limits_r^s f(t)\Delta t = F(s) - F(r), \forall r, s \in \mathbb{T}$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \text{ hold for all } t \in T^k.$$

Example 9. If $\mathbb{T} = \mathbb{Z}$, evaluate the indefinite integral

$$\int a^t \Delta t,$$

where $a \neq 1$ is a constant. Since

$$\left(\tfrac{a^t}{a-1}\right)^{\Delta} = \Delta\left(\tfrac{a^t}{a-1}\right) = \tfrac{a^{t+1}-a^t}{a-1} = a^t$$

we get that

$$\int a^t \Delta t = \tfrac{a^t}{a-1} + C,$$

where C is an arbitrary constant.

Theorem 11 (Existence of antiderivatives)**.** *Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int\limits_{t_0}^{t} f(\tau)\Delta\tau \textit{ for } t \in \mathbb{T}$$

is an antiderivative of f.

Two basic examples of time scale and their properties and conclusion from theory mention above is show in the table below (Table 1.1).

Theorem 12. *If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int\limits_t^{\sigma(t)} f(\tau)\Delta(\tau) = \mu(t)f(t).$$

Table 1.1: Two basic time scale examples.

	$\mathbb{R}$	$\mathbb{Z}$
Backward jump operator $\rho(t)$	t	$t-1$
Forward jump operator $\sigma(t)$	t	$t+1$
Graininess $\mu(t)$	0	1
Derivative $f^{\Delta}(t)$	$f'(t)$	$\Delta f(t)$
Integral $\int\limits_a^b f(t)\Delta t$	$\int\limits_a^b f(t)dt$	$\sum\limits_{t=a}^{b-1} f(t)$ (if $a<b$)
Rd-continuous f	continuous f	any f

Proof. According to existence of antiderivatives theorem there exists an antiderivative F of f, and

$$\int\limits_t^{\sigma(t)} f(\tau)\Delta\tau = F(\sigma(t)) - F(t) = \mu(t)F^{\Delta}(t) = \mu(t)f(t).$$

$\square$

Theorem 13. *If* $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$*, and* $f, g \in C_{rd}$*, then*

1. $\int\limits_a^b (f(t)+g(t))\Delta t = \int\limits_a^b f(t)\Delta(t) + \int\limits_a^b g(t)\Delta(t);$
2. $\int\limits_a^b (\alpha f)(t)\Delta(t) = \alpha \int\limits_a^b f(t)\Delta t;$
3. $\int\limits_a^b f(t)\Delta t = -\int\limits_a^b f(t)\Delta t;$
4. $\int\limits_a^b f(t)\Delta t = \int\limits_a^c f(t)\Delta t + \int\limits_c^b f(t)\Delta t;$
5. $\int\limits_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int\limits_a^b f^{\Delta}(t)g(t)\Delta(t);$
6. $\int\limits_a^b f(t)g^{\Delta}\Delta t = (fg)(b) - (fg)(a) - \int\limits_a^b f^{\Delta}(t)g(\sigma(t))\Delta t;$
7. $\int\limits_a^a f(t)\Delta t = 0;$

8. *If* $\mid f(t) \mid \leq g(t)$ *on* $[a, b)$, *then*

$$\mid \int_a^b f(t)\Delta t \mid \leq \int_a^b g(t)\Delta t;$$

9. *If* $f(t) > 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta t \geq 0$.

Since every rd-continuous function is also regulated, then all the formulas given in the theorem also holds for the case that f and g are only regulated. Also note that (5) and (6) are called *integration by parts* formulas.

Theorem 14. *Let* $a, b \in \mathbb{T}$ *and* $f \in C_{rd}$

1. *If* $\mathbb{T} = \mathbb{R}$, *then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.

2. *If interval* $[a, b]$ *consists only of isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t), & \text{if } a < b \\ 0, & \text{if } a = b \\ -\sum_{t\in[b,a)} \mu(t)f(t), & \text{if } a > b \end{cases}.$$

3. *If* $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, *where* $h > 0$, *then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h, & \text{if } a < b \\ 0, & \text{if } a = b \\ -\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h, & \text{if } a > b \end{cases}.$$

4. *If* $\mathbb{T} = \mathbb{Z}$, *then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum\limits_{t=a}^{b-1} f(t), & \textit{if } a < b \\ 0, & \textit{if } a = b \\ -\sum\limits_{t=a}^{b-1} f(t), & \textit{if } a > b \end{cases}.$$

Definition 14. If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$ and f is rd-continuous on $[a, \infty)$ then we define the improper integral by

$$\int_a^\infty f(t)\Delta(t) = \lim_{b\to\infty} \int_a^b f(t)\Delta t$$

provided this limit exists and we say that the improper integral converges. In case the limit does not exist, we say that the improper integral diverges.

1.3.4 Exponential function

Exponential function is convenient to describe some properties or to use it for some proofs. We introduce a generalized exponential function for an arbitrary time scale $\mathbb{T}$ by the cylinder transformation [12].

Definition 15. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $\mathbb{T}^k$

The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{R}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 16. If $p \in \mathcal{R}$, then the *exponential function* is defined by

$$e_p(t, s) = exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau)))\Delta\tau\right)$$

for $s, t \in \mathbb{T}$, where cylinder transformation

$$\xi_h(z) = \frac{1}{h} ln(1 + zh)$$

for $h > 0$ and for $h = 0$, $\xi_0(z) = z$, where $z \in \mathbb{C}$. ln is the principal logarithm function.

Lemma 15. *If $p \in \mathcal{R}$ then for all $r, s, t \in \mathbb{T}$*

$$e_p(t, r)e_p(r, s) = e_p(t, s).$$

Definition 17. If $p \in \mathcal{R}$, then the first order linear dynamic equation

$$y^{\Delta} = p(t)y \tag{1.39}$$

is called *regressive*.

Theorem 16. *If equation 1.39 is regressive and $t_0 \in \mathbb{T}$, then $e_p(\cdot, t_0)$ is a solution of the initial value problem $y^{\Delta} = p(t)y$, $y(t_0) = 1$.*

Some important properties of the exponential function are as follows.

Theorem 17. *If $p \in \mathcal{R}$, then*

- $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
- $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- $e_p(t, s) = \frac{1}{e_p(s,t)}$;
- $e_p(s, r)e_p(r, t) = e_p(s, t)$.

Many more properties of the exponential function can be found in [12] and [13].

Theorem 18 (Variation of Constants)**.** *Suppose $p \in \mathcal{R}$ and $f \in C_{rd}$. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. Then unique solution of the initial value problem*

$$y^{\Delta} = p(t)y + f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int\limits_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Example 10. Find solution for

$$y^{\Delta} = 2y + t$$

$$y(0) = 0.$$

Linear part of equation is $y^{\Delta} = 2y$, so $p = 2$. First, we find exponential function

$$e_2(t, t_0) = e_2(t, 0) = \exp\left(\int\limits_{0}^{t} 2\Delta\tau\right) =$$

$$= \exp\left(2\tau\Big|_0^t\right) = exp(2t-0) = e^{2t}$$

$$e_2(t,\sigma(t)) = e^{2(t-\sigma(t))}.$$

Next, using theorem 18 we find solution

$$y(t) = e_p(t,t_0)y_0 + \int\limits_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\Delta\tau =$$

$$= e_2(t,0)0 + \int\limits_0^t e_2(t,\sigma(\tau))\tau\Delta\tau = \int\limits_0^t e^{2(t-\sigma(\tau))}\tau\Delta\tau =$$

$$= e^{2t}\int\limits_0^t e^{-2\sigma(\tau)}\tau\Delta\tau$$

$$\begin{bmatrix} u=\tau & \Delta v = e^{-2\sigma(\tau)}\Delta\tau \\ \Delta u = d\tau & v = -\frac{1}{2}e^{-2\tau} \end{bmatrix}$$

$$= e^{2t}\left(-\frac{1}{2}\tau e^{-2\tau}\Big|_0^t + \frac{1}{2}\int\limits_0^t e^{-2\tau}\Delta\tau\right) =$$

$$= e^{2t}\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2\tau}\Big|_0^t\right) = e^{2t}\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}\right) =$$

$$= -\frac{1}{2}t - \frac{1}{4} + \frac{1}{4}e^{2t}.$$

We denote by $\Omega(t,s)$ the class of functions $k : \mathbb{T}\times\mathbb{T}\to\mathbb{R}$ which are continuous at (t,t), where $t\in\mathbb{T}^k$ with $t>t_0$, $t_0\in\mathbb{T}^k$ such that $k(t,\cdot)$ is rd-continuous on $[t_0,\sigma(t)]$ and for each $\varepsilon>0$ there exists a neighborhood U of t independent of $\tau\in[t_0,\sigma(t)]$ such that

$$|k(\sigma(t),\tau) - k(s,\tau) - k^{\Delta}(t,\tau)(\sigma(t)-s)| \le \varepsilon|\sigma(t)-s|.$$

Lemma 19 ([12],[45]). *Let* $k\in\Omega(t,s)$. *Then*

$$g(t) = \int\limits_{t_0}^{t} k(t,\tau)\Delta\tau$$

for $I_{\mathbb{T}}$, where $I_{\mathbb{T}} = I \cap \mathbb{T}$, $I = [t_0, \infty]$ implies

$$g^{\Delta}(t) = \int\limits_{t_0}^{t} k^{\Delta}(t, \tau)\Delta\tau + k(\sigma(t), t).$$

1.4 Metric Spaces

M. Fréchet was a French mathematician who made major contribution to the topology of points and defined and founded the theory of abstract spaces in his thesis [24]. The metric spaces was introduced by F. Hausdorff [29].

For metric space theory we refer to [38].

A metric space indicate that two given points of the space should have a real number that represents the distance between them. In order to discuss metric, it is naturally to begin with a pair (M, d), where M is a set and $d : M \times M \to \mathbb{R}^+$ is a mapping of Cartesian product $M \times M$ into the non-negative reals. $d(x, y)$ is the distance between points x and y from M that satisfies

- $d(x, y) = 0 \Leftrightarrow x = y$ and
- $d(x, y) = d(y, x)$.

In this case a pair (M, d) is called *semi-metric space*.

Definition 18. A semi-metric space (M, d) is called a *metric space* if it satisfies the triangle inequality, that is, for each three points $x, y, z \in M$,

$$d(x, y) \leq d(x, z) + d(z, y).$$

We say that two metric spaces are the same if there is a one-to-one distance preserving mapping from one onto other. Such mappings are called *isometries*.

Definition 19. Suppose (M, d) and (N, ρ) are metric spaces. A mapping $T : M \to N$ is said to be an isometry if $\rho(T(x), T(y)) = d(x, y)$ for each $x, y \in M$. If T is surjective (onto), then we say that M and N are isometric. A surjective isometry $T : M \to M$ is called motion of M.

Some common examples of metric spaces are as follows.

Example 11 (The discrete space). Let S be any set and for all $x, y \in S$ define

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}.$$

Definition 20. A *topology* on a set X is any family $\mathcal{F}$ of subsets of X which satisfies the following axioms:

- $\varnothing$ and X are in $\mathcal{F}$;
- The union of any subcollection of $\mathbb{F}$ is a member of $\mathcal{F}$;
- The intersection of any finite subcollection of $\mathcal{F}$ is a member of $\mathcal{F}$.

Together the pair $(X, \mathcal{F})$ is called a *topological space*.

A subset U of X is said to be an *open* set if $U \in \mathcal{F}$. A *closed* set in X is a set whose complement is open.

1.5 Banach fixed point theorem

The Banach fixed point theorem also called the contraction theorem concerns contraction of a complete metric space into itself. The theorem states sufficient condition for the existence and uniqueness of a fixed point. Fixed point is a point that is mapped to itself. Polish mathematician Stefan Banach (1982-1945) is widely regarded as the most powerful and influential mathematician to merge in Poland in the inter-war years. Abandoned at birth, he was brought up in Cracow by laundress. From the age of 15 he had to support himself by private coaching and was especially keen on teaching mathematics, a subject in which he was self-taught. Some of the notable mathematical concepts that bear Banach‘s name include Banach‘s space, Banach algebra, Banach measures, the Hahn-Banach theorem and the Banach fixed point theorem.
For fixed point theory, see [21].

Definition 21. Let (X, d) be a metric space. A sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ is called a Cauchy sequence if for any $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for any $m, n \geq n_\varepsilon$.

Theorem 20. *Any convergent sequence in a metric space is a Cauchy sequence.*

Definition 22. A metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 23. Let (X, d) be a metric space. A map $f : X \to X$ is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that $\forall x_1, x_2 \in X$

$$d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2).$$

The smallest λ for which the inequality hold is called Lipschitz constant of f. If $\lambda < 1$, f is said to be contraction.

Definition 24 (Fixed point)**.** The point $\overline{x} \in X$ is fixed point of the map f if $f(\overline{x}) = \overline{x}$.

Theorem 21 (Banach fixed point theorem)**.** *Let f be contraction on a complete metric space X. Then f has a unique fixed point $\overline{x} \in X$.*

1.6 Green‘s function

In 1828 British mathematical physicist George Green (1793-1841) wrote *An Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism* which introduced several important concepts such as theorem similar to the modern Green‘s theorem, the idea of potential functions and the concept of now called Green‘s function. There are several books written by Roach (1970), Stakgold (1979), and Sagan (1989) on the subject of Green‘s function and many other books written later, for example [39].

There is connection between differential and integral equations, some problems might be formulated ether way, for example, Green‘s function, Fredholm theory and Maxwell‘s equations. Green‘s functions are an important tool used in solving boundary value problems associated with ordinary and partial differential equations.

In order to explain Green‘s function concept, some basic results and definitions are needed. For next definition we refer to [39].

Definition 25. A mapping T of a linear space X into a linear space Y, where X and Y are vector spaces over some field F, is called *linear transformation* or *linear operator* if

- $T(x+y) = T(x) + T(y)$ for all $x, y \in X$;
- $T(\alpha x) = \alpha T(x)$ for all $x \in X$ and for all $\alpha \in F$.

We try to solve equation $Lu(x) = f(x)$. Green‘s function is a fundamental solution to the differential equations. It is very powerful tool and if Green‘s function can be computed then any solution for right hand side function $f(x)$ can be found. The solutions of this equations is $u(x) = L^{-1}f$. Green‘s function will help to answer the question how to calculate this inverse of L.

If function G can be found for operator L, then we obtain

$$\int LG(t,s)f(t)dt = f(x).$$

Since operator L is linear operator, L might be taken outside of the integration

$$L\int G(t,s)f(t)dt = f(x).$$

Therefore, we obtained

$$u(x) = \int G(t,s)f(t)dt.$$

Usually in the mathematical literature Green‘s function is associated with boundary value problems but further we will use analogy of this concept.

Let us consider system of differential equations $\dot{x} = Ax$, where A is $n \times n$ matrix with real part of eigenvalues not equal to 0 (Re$\lambda(A) \neq 0$). It is known that its nontrivial solution $\varphi(t) \to 0$, when $t \to +\infty$ and $|\varphi(t)| \to \infty$, when $t \to +\infty$

For example:

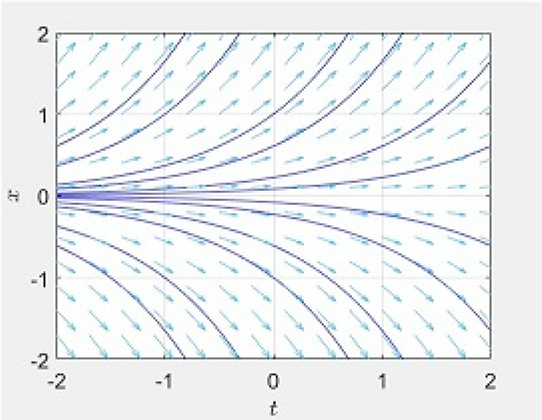

Figure 1.3: Green type map if $s = 0$

Example 12. $\dot{x} = ax$, where a is a constant.
Solutions is $x = Ce^{at}$, C-const.

- If $a > 0$ and $C \neq 0$ then $\varphi(t) \to 0$, when $t \to -\infty$ and $|\varphi(t)| \to \infty$, when $t \to +\infty$.
- If $a < 0$ and $C \neq 0$ then $\varphi(t) \to 0$, when $t \to +\infty$ and $|\varphi(t)| \to \infty$, when $t \to -\infty$.

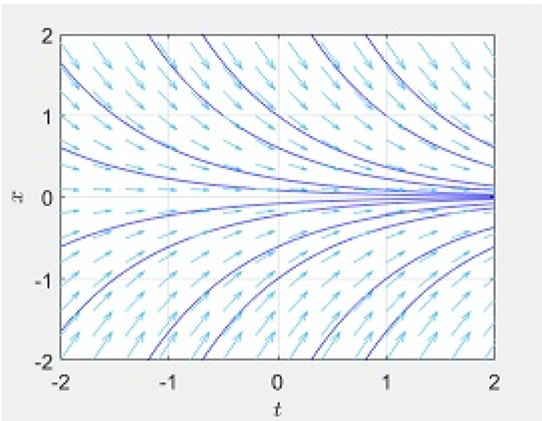

Figure 1.4: Green type map if $s = 0$

Therefore, there exists only one solution which is bounded, that is, $\varphi(t) \equiv 0$.

But what we can say about nonlinear system $\dot{x} = Ax + f(t)$ if we know that $\mathrm{Re}\lambda(A) \neq 0$ and $\sup\limits_{t \in \mathbb{R}} |f(t)| < +\infty$? Can we find bounded solution? Let us consider an example.

Example 13. $\dot{x} = ax + f(t)$, $x(s) = x_0$. Solution could be found using constant variation method $x(t) = x_0 e^{a(t-s)} + \int\limits_s^t e^{a(t-\tau)} f(\tau) d\tau$. In order to find bounded solution, multiply both sides by exponent and consider limit when $t \to +\infty$ and suppose that there is bounded solution $\sup\limits_{t \in \mathbb{R}} |x(t)| < +\infty$. Then

$$x(t) e^{a(s-t)} = x_0 e^{a(t-s)} e^{a(s-t)} + e^{a(s-t)} \int\limits_s^t e^{a(t-\tau)} f(\tau) d\tau$$

$$x(t) e^{a(s-t)} = x_0 e^0 + \int\limits_s^t e^{a(s-t)} e^{a(t-\tau)} f(\tau) d\tau$$

$$x(t) e^{a(s-t)} = x_0 + \int\limits_s^t e^{a(s-\tau)} f(\tau) d\tau.$$

(i) If $a > 0$, then

$$\lim_{t \to +\infty} x(t) e^{a(s-t)} = \lim_{t \to +\infty} (x_0 + \int\limits_s^t e^{a(s-\tau)} f(\tau) d\tau)$$

$$0 = x_0 + \lim_{t \to +\infty} \int\limits_s^t e^{a(s-\tau)} f(\tau) d\tau$$

$$x_0 = - \int\limits_s^\infty e^{a(s-\tau)} f(\tau) d\tau.$$

Therefore, bounded solutions for the given nonlinear differential equation is

$$x_b(t) = -e^{a(t-s)} \int\limits_s^\infty e^{a(s-\tau)} f(\tau) d\tau + \int\limits_s^t e^{a(t-\tau)} f(\tau) d\tau$$

$$x_b(t) = - \int\limits_s^\infty e^{a(t-\tau)} f(\tau) d\tau + \int\limits_s^t e^{a(t-\tau)} f(\tau) d\tau$$

$$x_b(t) = \cancel{- \int\limits_s^t e^{a(t-\tau)} f(\tau) d\tau} - \int\limits_t^\infty e^{a(t-\tau)} f(\tau) d\tau + \cancel{\int\limits_s^t e^{a(t-\tau)} f(\tau) d\tau}$$

$$x_b(t) = \int_t^{\infty} -e^{a(t-\tau)} f(\tau) d\tau.$$

We may introduce here a Green function as follows:

$$x_b(t) = \int_{-\infty}^{\infty} G(t,\tau) f(\tau) d\tau,$$

where

$$G(t,\tau) = \begin{cases} -e^{a(t-\tau)} & \tau \geq t \\ 0 & \tau < t \end{cases}.$$

(i) When $a < 0$ steps are similar.

$$\lim_{t \to -\infty} x(t) e^{a(s-t)} = \lim_{t \to -\infty} (x_0 - \int_t^s e^{a(s-\tau)} f(\tau) d\tau)$$

$$x_0 = \int_{-\infty}^{s} e^{a(s-\tau)} f(\tau) d\tau$$

$$x_b(t) = \int_{-\infty}^{t} e^{a(t-\tau)} f(\tau) d\tau + \cancel{\int_t^s e^{a(t-\tau)} f(\tau) d\tau} - \cancel{\int_t^s e^{a(t-\tau)} f(\tau) d\tau}$$

$$x_b(t) = \int_{-\infty}^{\infty} G(t,\tau) f(\tau) d\tau,$$

where

$$G(t,\tau) = \begin{cases} 0 & \tau \geq t \\ e^{a(t-\tau)} & \tau < t \end{cases}.$$

1.7 Green type map

Definition 26. A continuous map $G : \mathbb{R}^2 \setminus \{D\} \to \mathfrak{L}(\mathbb{R})$ where $D = \{(t,s) \in \mathbb{R}^2 \mid t = s\}$ is called *Green type map* if

(i) $G(\cdot, s)$ is continuous differentiable and it is solution of $\dot{x} = Ax$ except $t = s$;

(ii) $G(t+0,t) - G(t-0,s) = I$, where I is identity map.

Items (i) and (ii) are similar to Greens function properties associated with boundary value problems but here additional requirement $\int\limits_{-\infty}^{+\infty} |G(t,s)|dt < +\infty$ is added in order to obtain uniqueness of Green function.

Let us consider some examples.

Example 14. $\dot{x}(t) = -x(t)$
Solution is $x(t) = Ce^{-t}$. Let's define a Green type map as

$$G(t,s) = \begin{cases} e^{s-t}, & \text{if } t > s \\ 0, & \text{if } t < s \end{cases}$$

and verify Green type map definitions conditions:

(i) Function e^t is continuous thus $G(t,s)$ is continuous except $t = s$;

(ii) $G(s+0,s) - G(s-0,s) = e^{s-(s+0)} - 0 = e^{-0} = 1 = I$;

(iii)

$$\int\limits_{-\infty}^{+\infty} |G(t,s)|dt = \max\left(\int\limits_{-\infty}^{s} |0|dt, \int\limits_{s}^{+\infty} |e^{s-t}|dt \right) = \max(0, -e^{s-t}\Big|_{s}^{+\infty}) =$$

$$= -(e^{-\infty} - e^{-0}) = -(0-1) = 1 < +\infty$$

$\Rightarrow G(t,s)$ is a Green type map.

Example 15. Consider

$$\begin{cases} \dot{x}(t) = -x(t) \\ \dot{y}(t) = y(t) \end{cases}.$$

Solution is

$$\begin{cases} x(t) = C_1 e^{-t} \\ y(t) = C_2 e^{t} \end{cases},$$

fundamental matrix is

$$U = \begin{pmatrix} C_1 e^{-t} & 0 \\ 0 & C_2 e^t \end{pmatrix} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

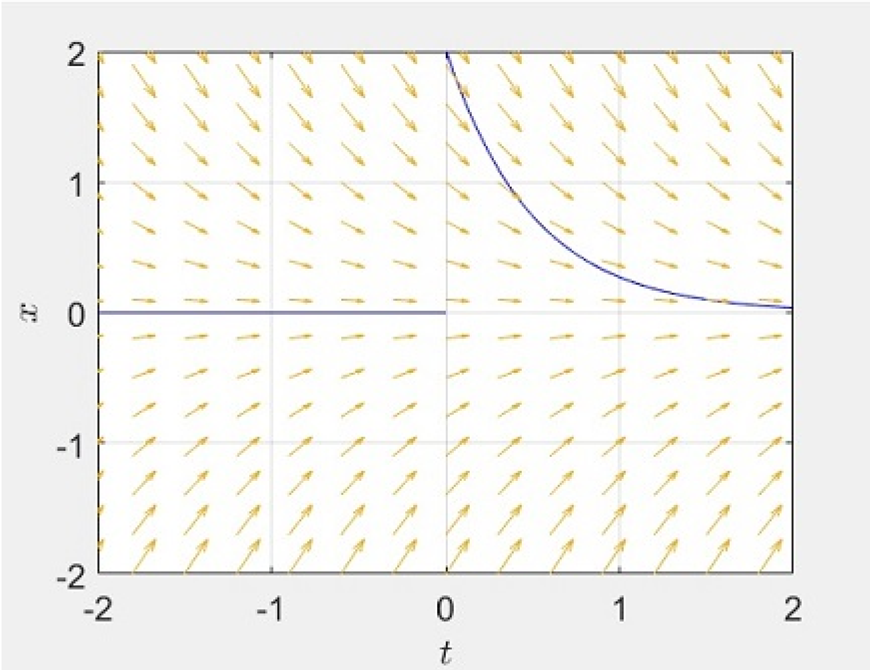

Figure 1.5: Green type map if $s = 0$

Let‘s define a Green type map as

$$G(t,s) = \begin{cases} \begin{pmatrix} e^{s-t} & 0 \\ 0 & 0 \end{pmatrix} & t > s \\ \\ \begin{pmatrix} 0 & 0 \\ 0 & -e^{-s+t} \end{pmatrix} & t < s \end{cases}$$

and verify (i), (ii) and (iii) from the definition.

- Function e^t is continuous therefore $G(t,s)$ is continuous for $t \in \mathbb{R}$, except $t = s$.
- $G(s+0,s) - G(s-0,s) = \begin{pmatrix} e^{-(s+0-s)} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -e^{(s-0-s)} \end{pmatrix} =$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E$$

- $$\int_{-\infty}^{+\infty} |G(t,s)|dt = \max\left(\int_{-\infty}^{s} |-e^{t-s}|dt,\ \int_{s}^{+\infty} |e^{-(t-s)}|dt\right) = \max(e^{t-s}\Big|_{-\infty}^{s}, -e^{-t+s}\Big|_{s}^{+\infty}) =$$

$$= \max(1-0, -(0-1)) = 1 < +\infty$$

$\Rightarrow$ Map $G(t,s)$ is a Green type map.

2 Grobman-Hartman theorem of quasi-linear equations

Most of the basic problems studied in mathematics and physics are linear and this is the result of simplifications of real problem application. For example, in mathematics finance Black Scholes‘ equation does not include transaction costs. But if one tries to incorporate transaction costs into the model then nonlinear differential equation is obtained. In the study of continuum mechanics, traffic flow models, groundwater flows, and gas dynamics the quasi-linear differential equations play important role (see [3], [33], [36]). Therefore, the theory of quasi-linear differential equations have become one of the most developing area in applied mathematics.

2.1 Grobman-Hartman theorem

In the study of local qualitative theory of dynamical system, the *Grobman-Hartman* theorem or also called *linearization theorem*, is very important result. The theorem is about local behavior of dynamical system in the neighborhood of a hyperbolic equilibrium point. The Grobman-Hartman theorem was proved independently by an American mathematician Philip Hartman [27] and a Russian mathematician D.M. Grobman [25] in 1959. H. Poincaré already in 1879 proved the theorem under the assumptions that the elementary divisors of A are simple and that the eigenvalues $\lambda_1, ..., \lambda_n$ of A lie in a half plane in C and satisfy $\lambda_j \neq m_1\lambda_1 + ... + m_n\lambda_n$

for all sets of non-negative integers $(m_1, ..., m_n)$ satisfying $m_1 + ... + m_n > 1$. A similar result for smooth function was established by S. Sternberg [64] in 1957. Further the linearization problem in the theory of ordinary differential equations was explored by D.M. Grobman [26], P. Hartman [28], K.J. Palmer [46] and other mathematicians [51], [52]. Variants of the Grobman-Hartman theorem to impulsive differential equations can be found in [6], [23], [56],[57], [58], [67]. We refer to [47] for in depth study of Grobman-Hartman theorem. The Grobman-Hartman theorem shows that near a hyperbolic equilibrium point x_0, the nonlinear system

$$\dot{x} = f(x) \tag{2.11}$$

has the same qualitative structure as the linear system

$$\dot{x} = Ax \tag{2.12}$$

with $A = Df(x_0)$, where Df is Jacobian matrix. Further we assume that the equilibrium point x_0 has been translated to the origin.
Two dynamical systems are *topologically conjugate* if they have the same dynamics or if there is a homeomorphism between the two systems which preserve orbits. If U, V are topological spaces with dynamics on them (continuous or discrete) then *topological conjugacy* is homeomorphism $H : U \to V$ between these two spaces such that it preserves orbits. If we use that topological conjugacy to map from U to V then resulting square of mappings is commutative. That means if firstly go forward on U and then jump down to V using H get the same if firstly jump down to V and only then the go forward on U.

$$\begin{tikzcd} U \arrow[r, bend left] \arrow[d, "H"] & U \arrow[d, "H"] \\ V \arrow[r, bend right] & V \end{tikzcd}$$

Let's formulate definition:

Definition 27. Two autonomous system of differential equations 2.11 and 2.12 are said to be *topologically equivalent* in a neighborhood of the origin or to have the *the same qualitative structure near the origin* if there is an open set V containing the origin which maps trajectories of 2.11 in U onto trajectories of 2.12 in V and preserves their orientation by time in the sense that if a trajectory is directed from x_1 to x_2 in U, then its image is directed from $H(x_1)$ to $H(x_2)$ in V. If the homeomorphism H preserves the parametrization by time, then the system 2.11 and 2.12 are said to be *topologically conjugate* in the neighborhood of the origin.

Example 16. Consider the linear systems $\dot{x} = Ax$ and $\dot{y} = By$ with

$$A = \begin{bmatrix} -1 & -3 \\ -3 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}.$$

Let $H(x) = Rx$, where the matrix

$$R = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } R^{-1} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then $B = RAR^{-1}$ and letting $y = H(x) = Rx$ or $x = R^{-1}y$ gives us

$$\dot{x} = RAR^{-1}y = By.$$

Thus, if $x(t) = e^{At}x_0$ is the solution of the first system through x_0, then $y(t) = H(x(t)) = Rx(t) = Re^{At}x_0 = e^{Bt}Rx_0$ is the solution of the second system through Rx_0, i.e, H maps trajectories of the first system onto trajectories of the second system and it preserves the parameterization since

$$He^{At} = e^{Bt}H.$$

The mapping $H(x) = Rx$ is simply a rotation through $45°$ and it is clearly a homeomorphism. The phase portraits of these two systems are shown in Figure 2.1.

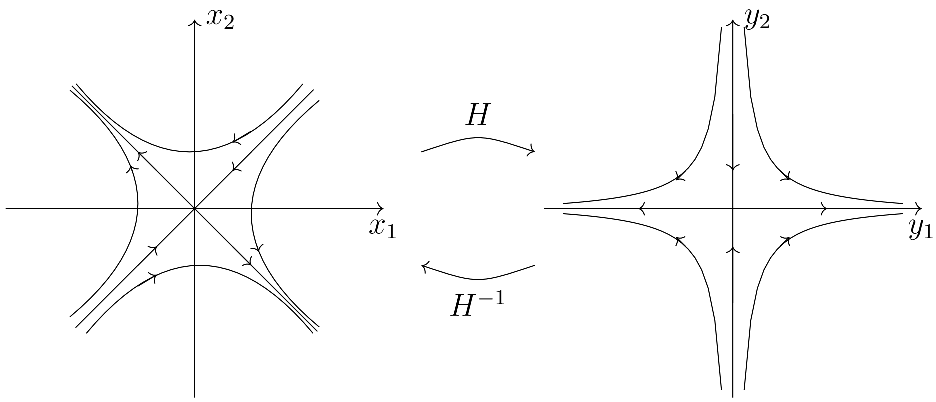

Figure 2.1: phase portraits

Theorem 22 (The Grobman-Hartman)**.** *Let E be an open subset of R^n containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the nonlinear system 2.11. Suppose that $f(0) = 0$ and that the matrix $A = Df(0)$ has no eigenvalue with zero real part. Then there exists a homeomorphism H of an open set U containing the origin onto an open set V containing the origin such that for each $x_0 \in U$, there is an open interval $I_0 \subset R$ containing zero such that for all $x_0 \in U$ and $t \in I_0$*

$$H \circ \phi_t(x_0) = e^{At} H(x_0),$$

i.e., H maps trajectories of 2.11 near the origin onto trajectories of 2.12 near the origin and preserves the parameterization by time.

Summary of the proof. Consider the nonlinear system 2.11 with $f \in C^1(E)$, $f(0) = 0$ and $A = Df(0)$.

1. The matrix A is grouped into two parts

$$A = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix},$$

where the eigenvalues of P have negative real part and the eigenvalues of Q have positive real part.

2. Suppose ϕ_t is the flow of the nonlinear system 2.11 with solution written in following form:

$$x(t, x_0) = \phi_t(x_0) = \begin{bmatrix} y(t, y_0, z_0) \\ z(t, y_0, z_0) \end{bmatrix},$$

where

$$x_0 = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \in R^n,$$

where y_0 is in the stable subspace of A and z_0 is in the unstable subspace of A.

3. Define the function

$$\tilde{Y}(y_0, z_0) = y(1, y_0, z_0) - e^P y_0$$
$$\tilde{Z}(y_0, z_0) = z(1, y_0, z_0) - e^Q z_0$$

And show that $\| D\tilde{Y}(y_0, z_0) \|$ and $\| D\tilde{Z}(y_0, z_0) \|$ is bounded by constant on the compact set $|y_0|^2 + |z_0|^2 \leq s_0^2$. Then by use of normalization obtain that

$$\| e^P \| < 1$$

and

$$\| e^{-Q} \| < 1.$$

4. For

$$x = \begin{bmatrix} x \\ y \end{bmatrix} \in R^n$$

define the transformation

$$L(y, z) = \begin{bmatrix} By \\ Cz \end{bmatrix}$$

and

$$T(y, z) = \begin{bmatrix} By + Y(y, z) \\ Cz + Z(y, z) \end{bmatrix},$$

where

$$Y(y_0,z_0)=\begin{cases}\tilde{Y}(y_0,z_0), & \text{if } |y_0|^2+|z_0|^2\le s_0^2/2\\ 0, & \text{if } |y_0|^2+|z_0|^2\ge s_0^2 y\end{cases}$$

and

$$Z(y_0,z_0)=\begin{cases}\tilde{Z}(y_0,z_0), & \text{if } |y_0|^2+|z_0|^2\le s_0^2/2\\ 0, & \text{if } |y_0|^2+|z_0|^2\ge s_0^2 y\end{cases}$$

i.e., $L(x)=e^{A}x$ and locally $T(x)=\phi_1(x)$.
Next, by using successive approximations, inductions and stable manifold theorem show that obtained continuous map

$$H_0(y,z)=\begin{bmatrix}\phi(y,z)\\ \psi(y,z)\end{bmatrix}$$

is a homeomorphism of R^n onto R^n such that

$$H_0\circ T=L\circ H_0.$$

5. Define one-parameter families of transformation T^t and L^t as follows.

$$L^t x_0=e^{At}x_0 \text{ and } T^t(x_0)=\phi_t(x_0)$$

And lastly show that H defined by

$$H=\int_0^1 L^{-1}H_0T^s ds$$

is a homeomorphism on R^n and equality $H\circ T^t=L^tH$ holds or equivalently

$$H\circ\phi_t(x_0)=e^{At}H(x_0).$$

$\square$

2.2 Palmer‘s generalization

Grobman-Hartman linearization theorem was extended by Kenneth J. Palmer [46] in 1973. The theorem was generalized to nonautonomous differential equations where the linear equation is assumed to have an exponential dichotomy.

In 1978 W.A. Cooper published his Lecture notes regarding Dichotomies covering basic definitions, summaries, and conclusions on the topic [15]. A dichotomy is a type of conditional stability. Recall some stability definitions.

Let $\tilde{x}(t)$ be a solutions of differential equation

$$\dot{x} = f(t, x). \tag{2.21}$$

Definition 28. The solution $\tilde{x}(t)$ is called *uniformly stable* if $\forall \epsilon > 0$ there is $\delta = \delta(\epsilon) > 0$ such that any solution $x(t)$ of 2.21 which satisfies the inequality $|x(s) - \tilde{x}(s)| < \delta$ for some $s \geq 0$ is defined and satisfies the inequality $|x(t) - \tilde{x}(t)| < \epsilon$ for all $t \geq s$.

Definition 29. The solution $\tilde{x}(t)$ is called *uniformly asymptotically stable* if in addition to uniformly stable there is a $\delta_0 > 0$ and for each $\epsilon > 0$ a corresponding $T = T(\epsilon) > 0$ such that if $|x(s) - \tilde{x}(s)| \leq \delta_0$ for some $s \geq 0$ then $|x(t) - \tilde{x}(t)| < \epsilon$ for all $t \geq s + T$.

Consider linear differential equations $\dot{x} = A(t)x$, where $A(t)$ is a continuous $n \times n$ matrix function for $0 \leq t < \infty$. Let $X(t)$ be the fundamental matrix for the linear differential equation. It may be easily shown that the solution $x = 0$ of $\dot{x} = A(t)x$ is uniformly stable $\Leftrightarrow$ there exists a constant $K > 0$ such that

$$|X(t)X^{-1}(s)| \leq K \quad for \quad 0 \leq s \leq t < \infty.$$

It is uniformly asymptotically stable $\Leftrightarrow \exists K > 0$, α:

$$|X(t)X^{-1}(s)| \leq Ke^{-\alpha(t-s)} \quad for \quad 0 \leq s \leq t < \infty.$$

Definition 30 (Ordinary dichotomy)**.** Let $A(t)$ be matrix function defined and continuous for all $t \in \mathbb{R}$. We say that the linear differential equation

$$\dot{x} = A(t)x$$

has an *ordinary dichotomy* if it has a fundamental matrix $X(t)$ such that

$$\|X(t)PX^{-1}(s)\| \leq K \text{ for } s \leq t,$$
$$\|X(t)(I-P)X^{-1}(s)\| \leq K \text{ for } s \geq t,$$

where $\|\cdot\|$ is operator norm, P is a projection ($P^2 = P$) and K is positive constant.

Definition 31 (Exponential dichotomy)**.** Let $A(t)$ be matrix function defined and continuous for all $t \in \mathbb{R}$. We say that the linear differential equation

$$\dot{x} = A(t)x$$

has an *exponential dichotomy* if it has a fundamental matrix $X(t)$ such that

$$\|X(t)PX^{-1}(s)\| \leq Ke^{-\alpha(t-s)} \text{ for } s \leq t,$$
$$\|X(t)(I-P)X^{-1}(s)\| \leq Ke^{-\alpha(s-t)} \text{ for } s \geq t,$$

where $\|\cdot\|$ is operator norm, P is a projection ($P^2 = P$) and K, α are positive constant.

If a linear system has an exponential dichotomy, then it has an ordinary dichotomy with asymptotically stable manifolds but the reverse does not hold. The non-homogeneous linear equation

$$\dot{x} = A(t)x + f(t)$$

has a unique bounded solution $x(t)$ given by

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s)ds - \int_{t}^{\infty} X(t)(I-P)X^{-1}(s)f(s)ds$$

and an elementary calculation yield the inequality

$$|x(t)| \leq \frac{2K}{\alpha} \sup_{-\infty<r<\infty} |f(t)|,$$

if $f(t)$ is bounded continuous vector function. If $A(t)$ is a constant matrix, then $\dot{x} = A(t)x$ has an exponential dichotomy if and only if none of the eigenvalues of A has zero real part.

Theorem 23. *Suppose $A(t)$ is a continuous matrix function such that the linear equation $\dot{x} = A(t)x$ has a fundamental matrix $X(t)$ satisfying exponential dichotomy. Suppose $f(t,x)$ is a continuous function of $R \times R$ into R^n such that*

$$|f(t,x)| \leq \mu$$

and

$$|f(t,x_1) - f(t,x_2)| \leq \gamma |x_1 - x_2|$$

for all t, x, x_1, x_2. Then if

$$\Delta \gamma K \leq \alpha,$$

there is a unique function $H(t,x)$ of $R \times R^n \to R^n$ satisfying

- *$H(t,x)$-x is bounded in $R \times R^n$;*
- *if $x(t)$ is any solution of the differential equations*

$$\dot{x} = A(t) + f(t,x),$$

then $H(t,x(t))$ is a solution of $\dot{x} = A(t)x$.

Moreover, H is continuous in $R \times R^n$ and

$$|H(t,x) - x| \leq 4K\mu\alpha^{-1}$$

for all t, x. For each fixed t, $H_t(x) = H(t,x)$ is a homeomorphism of R^n. $L(t,x) = H_t^{-1}(x)$ is continuous in $R \times R^n$ and if $y(t)$ is any solutions of $\dot{x} = A(t)x$ then $L(t,y(t))$ is a solution of $\dot{x} = A(t)x + f(t,x)$.

Although theorem is concerned with R^n, with minor changes the theorem is true when R^n is replaced by an arbitrary Banach space.

Recently Grobman-Hartman-Palmer linearization theorems have been extended to systems of differential equations in R^n with generalized exponential and ordinary dichotomy in papers [67], [34], [35], [68], [69], [70].

2.3 Dynamical equivalence of quasi-linear differential equations

Here we generalize Grobman-Hartman-Palmer results, even for R^n, by relaxing condition on the linear part A while strengthening condition on the nonlinear part f. We use Green type map and integral functional equation technique [52], [53], [54] to substantially simplify the proof. Moreover, we use a different method to prove the dynamical equivalence. Furthermore, for more general point of view, we consider nonautonomous differential equations and nonautonomous difference equations in an arbitrary Banach space. To highlight our improvement comparing to previous results, we use an example where the linear part of the differential equation even does not possess an ordinary dichotomy.

Using a suitable bump function and change of variables, it is possible to reduce the analysis of local equivalence of equations to investigation of global equivalence of equations.

Further in this section we refer to author‘s publication 1.

Let X be a Banach space and let $\mathfrak{L}(X)$ be the Banach space of bounded linear maps. Consider the quasi-linear differential equations

$$\dot{x} = A(t)x + f_1(t, x) \tag{2.31}$$

and

$$\dot{x} = A(t)x + f_2(t, x) \tag{2.32}$$

on X, where $A : \mathbb{R} \to \mathfrak{L}(X)$ is integrable in the Bochner‘s sense. In addition, suppose that maps $f_i : \mathbb{R}\times X \to X$, $i = 1, 2$ are locally integrable in the Bochner‘s sense with respect to t for fixed x, satisfy Lipschitz conditions

$$|f_i(t, x) - f_i(t, x')| \leq \varepsilon(t)|x - x'|, \quad i = 1, 2$$

and

$$\sup_x |f_1(t,x) - f_2(t,x)| \leq N(t) < +\infty,$$

where $N : \mathbb{R} \to \mathbb{R}_+$ and $\varepsilon : \mathbb{R} \to \mathbb{R}_+$ are integrable scalar functions. Let $x_i(\cdot, s, x) : \mathbb{R} \to X$, $i = 1, 2$, be the solutions of equations 2.31 and 2.32 respectively where $x_i(s, s, x) = x$.

Definition 32. The differential equations 2.31 and 2.32 are *globally dynamical equivalent* if there exists a continuous map $H : \mathbb{R} \times X \to X$ such that

(i) for each fixed $t \in \mathbb{R}$ the map $H(t, \cdot)$ is homeomorphism;

(ii) $\sup_{t,x} |H(t,x) - x| \leq +\infty$;

(iii) for all $t \in \mathbb{R}$

$$H(t, x_1(t, s, x)) = x_2(t, s, H(s, x)).$$

Let

$$D = \{(t, s) \in R^2 | t = s\}.$$

Definition 33 (Green type map)**.** A continuous map $G : R^2 \setminus D \to \mathfrak{L}(X)$ is a *Green type map* if

(i) it is a solution of linear differential equation

$$\dot{x} = A(t)x \tag{2.33}$$

except for $t = s$;

(ii)

$$G(t+0, t) - G(t-0, t) = I$$

where I is identity map.

Note that the differential equation 2.33 has infinitely many Green type maps. But if the linear differential equation 2.33 has an exponential dichotomy then, moreover, there exists an unique Green type map which satisfies the inequality

$$|G(t,s)| \le Kexp(-\lambda|t-s|), \quad K \ge 1, \lambda > 0.$$

Therefore, the Green type map can be represented in the form

$$G(t,s) = \begin{cases} U(t,s)P(s), & \text{if } t > s \\ U(t,s)(P(s) - I), & \text{if } t < s \end{cases},$$

where $U(t,s)$ is the evolution operator of 2.33. Let us note that

$$U(t,\tau)U(\tau,s) = U(t,s)$$

and the solutions of 2.31 and 2.32 can be represented in the form

$$x_i(t,s,x) = U(t,s)x + \int_s^t U(t,\tau)f_i(\tau, x_i(\tau,s,x))d\tau, \quad i = 1,2.$$

Theorem 24. *Suppose that the linear differential equations 2.33 has a Green type map $G(s,\tau) \in \mathfrak{L}(X)$ such that*

$$\sup_s \int_{-\infty}^{+\infty} |G(s,\tau)|N(\tau)d\tau < +\infty$$

$$\sup_s \int_{-\infty}^{+\infty} |G(s,\tau)|\epsilon(\tau)d\tau = q < 1.$$

Then the systems of differential equations 2.31 and 2.32 are globally dynamical equivalent.

Proof. Consider the set of continuous, bounded maps

$$\mathfrak{M} = \left\{ h : \mathbb{R} \times X \to X \,\middle|\, \sup_{s,x} |h(s,x)| < +\infty \right\}.$$

It is easy to see that $\mathfrak{M}$ is a Banach space with the supremum norm

$$\|h\| = \sup_{s,x} |h(s,x)|.$$

We will seek the map establishing the equivalence of 2.31 and 2.32 in the form $H(s,x) = x + h(s,x)$. We examine the following integro-functional equation

$$h(s,x) = \int_{-\infty}^{+\infty} G(s,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau. \tag{2.34}$$

Let us consider the map $h \mapsto \mathfrak{T}h$, $h \in \mathfrak{M}$ define by the equality

$$\mathfrak{T}h(s,x) = \int_{-\infty}^{+\infty} G(s,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau.$$

Because of boundedness, Lipschitz condition and condition of the Theorem 24, also $\mathfrak{T}h(s,x) \in \mathfrak{M}$.

(i) Since $\sup_x |f_1(t,x) - f_2(t,x)| \le N(t) < +\infty$, also

$$\sup_x |f_2(t, x_1(t,s,x) + h(t, x_1(t,s,x))) - f_1(t, x_1(t,s,x))| \le N(t) < +\infty.$$

(ii)

$$\sup_x |\mathfrak{T}h(s,x)| =$$

$$= \sup_x \left| \int_{-\infty}^{+\infty} G(s,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau \right|$$

$$\le \sup_x \left| \int_{-\infty}^{+\infty} G(s,\tau) N(\tau) d\tau \right|.$$

(iii) From condition of the Theorem 24

$$\sup_s \int_{-\infty}^{+\infty} |G(s,\tau)| N(\tau) d\tau < +\infty$$

$$\Rightarrow \sup_{s,x} |\mathfrak{T}h(s,x)| < +\infty \Rightarrow \mathfrak{T}h(s,x) \in \mathfrak{M}.$$

Next, we get

$$|\mathfrak{T}h(s,x)-\mathfrak{T}h'(s,x)|=$$

$$=\left|\int\limits_{-\infty}^{+\infty}G(s,\tau)(f_2(\tau,x_1(\tau,s,x)+h(\tau,x_1(\tau,s,x)))-f_1(\tau,x_1(\tau,s,x)))d\tau-\right.$$

$$\left.-\int\limits_{-\infty}^{+\infty}G(s,\tau)(f_2(\tau,x_1(\tau,s,x)+h'(\tau,x_1(\tau,s,x)))-f_1(\tau,x_1(\tau,s,x)))d\tau\right|=$$

$$=\left|\int\limits_{-\infty}^{+\infty}G(s,\tau)(f_2(\tau,x_1(\tau,s,x)+h(\tau,x_1(\tau,s,x)))-\cancel{f_1(\tau,x_1(\tau,s,x))}-\right.$$

$$\left.-f_2(\tau,x_1(\tau,s,x)+h'(\tau,x_1(\tau,s,x)))+\cancel{f_1(\tau,x_1(\tau,s,x))})d\tau\right|=$$

$$\left|\int\limits_{-\infty}^{+\infty}G(s,\tau)(f_2(\tau,x_1(\tau,s,x)+h(\tau,x_1(\tau,s,x)))-f_2(\tau,x_1(\tau,s,x)+h'(\tau,x_1(\tau,s,x))))d\tau\right|\leq$$

$$\leq\int\limits_{-\infty}^{+\infty}|G(s,\tau)|\,\epsilon(\tau)\,|h(\tau,x_1(\tau,s,x))-h'(\tau,x_1(\tau,s,x))|\,d\tau\leq$$

$$\leq\sup_s\int\limits_{-\infty}^{+\infty}|G(s,\tau)|\,\epsilon(\tau)\,|h(\tau,x_1(\tau,s,x))-h'(\tau,x_1(\tau,s,x))|\,d\tau\leq$$

From a Banach space supremum norm obtain

$$\leq\sup_s\int\limits_{-\infty}^{+\infty}|G(s,\tau)|\,\epsilon(\tau)d\tau\,||h-h'||=$$

and from condition of the Theorem 24 follows

$$=q\,||h-h'||,$$

where $q<1$. Thus, the map $\mathfrak{T}$ is a contraction and consequently the integro-functional equation 2.34 has a unique solution in $\mathfrak{M}$. We have

$$h(t,x_1(t,s,x))=\int\limits_{-\infty}^{+\infty}G(t,\tau)(f_2(\tau,x_1(\tau,s,x)+h(\tau,x_1(\tau,s,x)))-f_1(\tau,x_1(\tau,s,x)))d\tau=$$

$$= \int\limits_{-\infty}^{t} G(t,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau +$$

$$+ \int\limits_{t}^{+\infty} G(t,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau =$$

$$= \int\limits_{-\infty}^{s} U(t,s)G(s,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau +$$

$$+ \int\limits_{s}^{t} U(t,\tau)P(\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau +$$

$$+ \int\limits_{s}^{+\infty} U(t,s)G(s,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau +$$

$$+ \int\limits_{t}^{s} U(t,\tau)(P(\tau) - I)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau =$$

$$U(t,s) \int\limits_{-\infty}^{+\infty} G(s,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau +$$

$$\int\limits_{s}^{t} U(t,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau =$$

$$= U(t,s)h(s,x) +$$

$$\int\limits_{s}^{t} U(t,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau,$$

because

$$\int\limits_{s}^{t} U(t,\tau)P(\tau)(...)d\tau + \int\limits_{t}^{s} U(t,\tau)(P(\tau) - I)(...)d\tau =$$

$$= \int\limits_{s}^{t} U(t,\tau)P(\tau)(...)d\tau - \int\limits_{s}^{t} U(t,\tau)(P(\tau) - I)(...)d\tau =$$

$$= \int_s^t \cancel{U(t,\tau)P(\tau)}(...)d\tau - \int_s^t \cancel{U(t,\tau)P(\tau)}(...)d\tau + \int_s^t U(t,\tau)(I)(...)d\tau =$$

$$= \int_s^t U(t,\tau)(...)d\tau$$

Consequently, we have

$$h(t, x_1(t,s,x)) = \int_s^t U(t,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x))) - f_1(\tau, x_1(\tau,s,x)))d\tau.$$

By separating integral into two parts obtain

$$h(t, x_1(t,s,x)) = \int_s^t U(t,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x)))d\tau - \int_s^t U(t,\tau) f_1(\tau, x_1(\tau,s,x)))d\tau.$$

And then by taking one integral to the other side and adding $U(t,s)x$ to both sides obtain

$$U(t,s)x + h(t, x_1(t,s,x)) + \int_s^t U(t,\tau) f_1(\tau, x_1(\tau,s,x)))d\tau =$$

$$= U(t,s)x + \int_s^t U(t,\tau)(f_2(\tau, x_1(\tau,s,x) + h(\tau, x_1(\tau,s,x)))d\tau.$$

Thus,

$$x_1(t,s,x) + h(t, x_1(t,s,x)) = x_2(t,s,x + h(s,x)).$$

Changing the roles of f_1 and f_2, we prove in the same way the existence of $h'(s,x)$ that satisfies the equality

$$x_2(t,s,x) + h'(t, x_2(t,s,x)) = x_1(t,s,x + h'(s,x)).$$

Designing $H(s,x) = x + h(s,x)$, $H'(s,x) = x + h'(s,x)$, we get

$$H'(t, H(t, x_1(t,s,x))) = x_1(t,s,H'(s, H(s,x))),$$

$$H(t, H'(t, x_2(t,s,x))) = x_2(t,s,H(s, H'(s,x))).$$

Taking into account uniqueness of maps $H'(t, H(t,\cdot)) - I$ and $H(t, H'(t,\cdot)) - I$ in $\mathfrak{M}$ we have $H'(t, H(t,\cdot)) = I$ and $H(t, H'(t,\cdot)) = I$ and therefore $H(t,\cdot)$ is a homeomorphism establishing a dynamical equivalence of the 2.31 and 2.32. $\square$

Let $f_2(t,x) = 0$. Then Theorem 24 implies that differential equations 2.31 and 2.33 are globally dynamical equivalent.

2.3.1 Example

Consider the system of quasi-linear differential equation in $\mathbb{R}^3$

$$\dot{x} = A(t)x + f(t,x) \tag{2.35}$$

where

$$A(t) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{2t}{1+t^2} \end{pmatrix}$$

$$|f(t,x) - f(t,x')| \leq \epsilon(t)|x - x'|$$

and

$$\sup_x |f(t,x)| \leq N(t) < +\infty.$$

Then the fundamental matrix of the linear differential equations system

$$\dot{x} = A(t)x$$

takes the form

$$U(t,s) = \begin{pmatrix} cos(t-s) & -sin(t-s) & 0 \\ sin(t-s) & cos(t-s) & 0 \\ 0 & 0 & -\frac{1+s^2}{1+t^2} \end{pmatrix}.$$

The corresponding Green type map can be represented in the form

$$G(t,s) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1+s^2}{1+t^2} \end{pmatrix}, \quad if \quad t > s.$$

$$G(t,s)=\begin{pmatrix} cos(t-s) & -sin(t-s) & 0 \\ sin(t-s) & cos(t-s) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad if \quad t<s.$$

If

$$\sup_t \int_{-\infty}^{+\infty} |G(t,s)|\varepsilon(s)ds \le \int_{-\infty}^{+\infty} (1+s^2)\varepsilon(s)ds < 1$$

and

$$\sup_t \int_{-\infty}^{+\infty} |G(t,s)|N(s)ds \le \int_{-\infty}^{+\infty} (1+s^2)N(s)ds < +\infty$$

then, in accordance with the Theorem 24, the system 2.35 is globally dynamically equivalent to the linear one

$$\dot{x} = A(t)x. \tag{2.36}$$

The system 2.36 does not even have an ordinary dichotomy [15].

2.4 Dynamical equivalence of invertible difference equations

In this section we refer to author‘s publication 1.

We generalize this result of invertible semilinear difference equations

$$x(t+1) = A(t)x(t) + f_1(t,x(t)) \tag{2.41}$$

and

$$x(t+1) = A(t)x(t) + f_2(t,x(t)) \tag{2.42}$$

on X, where the map $A(t) \in \mathfrak{L}(X)$ is invertible, the maps $f_i : \mathbb{Z} \times X \to X$, $i = 1, 2$, satisfy the estimates

$$|f_1(t,x) - f_2(t,x)| \le N(t) < +\infty$$

$$|f_i(t,x) - f_i(t,x')| \le \epsilon(t)\,|x - x'| \quad i = 1, 2$$

and $\sup_t(||A^{-1}(t)||\epsilon(t)) < 1$.

Definition 34. The difference equations 2.41 and 2.42 are *globally dynamical equivalent* if there exists a continuous map $H : \mathbb{Z} \times X \to X$ such that

(i) for each fixed $t \in \mathbb{Z}$ map $H(t, \cdot) : X \to X$ is a homeomorphism;

(ii) $\sup_{t,x} |H(t, x) - x| < +\infty$;

(iii) for all $t \in \mathbb{Z}$
$$H(t, x_1(t, s, x_s)) = x_2(t, s, H(s, x_s)).$$

Definition 35. The map $G : \mathbb{Z}^2 \to \mathfrak{L}(X)$ is a *Green type map* if

(i) $G(t + 1, s) = A(t)G(t, s), \quad t \neq s - 1$;

(ii) $G(t, t) = A(t)G(t - 1, t) + I$.

Theorem 25. *Suppose that the linear difference equation*

$$x(t + 1) = A(t)x(t) \tag{2.43}$$

has a Green type map $G(t, s) \in \mathfrak{L}(X)$ such that

$$\sup_{t\in\mathbb{Z}} \sum_{s=-\infty}^{+\infty} |G(t, s + 1)|\, N(s) < +\infty$$

$$\sup_{t\in\mathbb{Z}} \sum_{s=-\infty}^{+\infty} |G(t, s + 1)|\, \epsilon(s) = q < 1,$$

then the difference equations 2.41 and 2.42 are globally dynamical equivalent. To prove Theorem 25, we prove that the following functional equation

$$h(s, x_s) = \sum_{i=-\infty}^{+\infty} G(s, i+1)(f_2(i, x_1(i, s, x_s)+h(i, x_1(i, s, x_s)))-f_1(i, x_1(i, s, x_s)))$$

has unique continuous and bounded solution. The proof is similar to the proof for differential equations.

Proof. Consider the set of continuous, bounded maps

$$\mathfrak{M} = \left\{ h : \mathbb{R} \times X \to X \,\middle|\, \sup_{s,x} |h(s, x_s)| < +\infty \right\}$$

$\mathfrak{M}$ is a Banach space with the supremum norm

$$\|h\| = \sup_{s,x} |h(s, x_s)|.$$

We will seek the map establishing the equivalence of 2.41 and 2.42 in the form $H(s, x_s) = x_s + h(s, x_s)$. We examine the following functional equation

$$h(s,x_s) = \sum_{i=-\infty}^{+\infty} G(s,i{+}1)(f_2(i,x_1(i,s,x_s){+}h(i,x_1(i,s,x_s))){-}f_1(i,x_1(i,s,x_s)))d\tau. \tag{2.44}$$

Let us consider the map $h \mapsto \mathfrak{T}h$, $h \in \mathfrak{M}$ define by the equality

$$\mathfrak{T}h(s,x_s) = \sum_{i=-\infty}^{+\infty} G(s,i{+}1)(f_2(i,x_1(i,s,x_s){+}h(i,x_1(i,s,x_s))){-}f_1(i,x_1(i,s,x_s)))d\tau.$$

Because of boundedness, Lipschitz condition and condition of the Theorem 25, also $\mathfrak{T}h(s, x_s) \in \mathfrak{M}$. Next, we get

$$|\mathfrak{T}h(s,x_s) - \mathfrak{T}h'(s,x_s)| =$$

$$= \left| \sum_{i=-\infty}^{+\infty} G(s,i+1)(f_2(i,x_1(i,s,x_s) + h(i,x_1(i,s,x_s))) - f_1(i,x_1(i,s,x_s))) - \right.$$

$$\left. - \sum_{i=-\infty}^{+\infty} G(s,i+1)(f_2(i,x_1(i,s,x_s) + h'(i,x_1(i,s,x_s))) - f_1(i,x_1(i,s,x_s))) \right| =$$

$$= \left| \sum_{i=-\infty}^{+\infty} G(s,i+1)(f_2(i,x_1(i,s,x_s) + h(i,x_1(i,s,x_s))) - \cancel{f_1(i,x_1(i,s,x_s))} - \right.$$

$$\left. -f_2(i,x_1(i,s,x_s) + h'(i,x_1(i,s,x_s))) + \cancel{f_1(i,x_1(i,s,x_s))}) \right| =$$

$$\left| \sum_{i=-\infty}^{+\infty} G(s,i+1)(f_2(i,x_1(i,s,x_s) + h(i,x_1(i,s,x_s))) - f_2(i,x_1(i,s,x_s) + h'(i,x_1(i,s,x_s)))) \right| \le$$

$$\le \sum_{i=-\infty}^{+\infty} |G(s,i+1)|\, \epsilon(i)\, |h(i,x_1(i,s,x_s)) - h'(i,x_1(i,s,x_s))| \le$$

$$\leq \sup_{s\in\mathbb{Z}} \sum_{i=-\infty}^{+\infty} |G(s,i+1)|\, \epsilon(i)\, |h(i,x_1(i,s,x_s)) - h'(i,x_1(i,s,x_s))| \leq$$

$$\leq \sup_{s\in\mathbb{Z}} \sum_{i=-\infty}^{+\infty} |G(s,i+1)|\, \epsilon(i)\, ||h-h'||\,| = q\, ||h-h'||\,|,$$

where $q < 1$. Thus, the map $\mathfrak{T}$ is a contraction and consequently the functional equation 2.44 has a unique solution in $\mathfrak{M}$. We have

$$h(t,x_1(t,s,x_s)) = \sum_{i=-\infty}^{+\infty} G(t,i+1)(f_2(i,x_1(i,s,x_s)+h(i,x_1(i,s,x_s)))-f_1(i,x_1(i,s,x_s))) =$$

$$= \sum_{i=-\infty}^{t} G(t,i+1)(f_2(i,x_1(i,s,x_s)+h(i,x_1(i,s,x_s)))-f_1(i,x_1(i,s,x_s)))+$$

$$+\sum_{i=t}^{+\infty} G(t,i+1)(f_2(i,x_1(i,s,x_s)+h(i,x_1(i,s,x_s)))-f_1(i,x_1(i,s,x_s))) =$$

$$= \sum_{i=-\infty}^{s} U(t,s)G(s,i+1)(f_2(i,x_1(i,s,x_s)+h(i,x_1(i,s,x_s)))-f_1(i,x_1(i,s,x_s)))+$$

$$+\sum_{i=s}^{t} U(t,i)P(i)(f_2(i,x_1(i,s,x_s)+h(i,x_1(i,s,x_s)))-f_1(i,x_1(i,s,x_s)))+$$

$$+\sum_{i=s}^{+\infty} U(t,s)G(s,i+1)(f_2(i,x_1(i,s,x_s)+h(i,x_1(i,s,x_s)))-f_1(i,x_1(i,s,x_s)))+$$

$$+\sum_{i=t}^{s} U(t,i)(P(i)-I)(f_2(i,x_1(i,s,x_s)+h(i,x_1(i,s,x_s)))-f_1(i,x_1(i,s,x_s))) =$$

$$U(t,s)\sum_{i=-\infty}^{+\infty} G(s,i+1)(f_2(i,x_1(i,s,x_s)+h(i,x_1(i,s,x_s)))-f_1(i,x_1(i,s,x_s)))+$$

$$\sum_{i=s}^{t} U(t,i)(f_2(i,x_1(i,s,x_s)+h(i,x_1(i,s,x_s)))-f_1(i,x_1(i,s,x_s)))$$

$$= U(t,s)h(s,x)+$$

$$\sum_{i=s}^{t} U(t,i)(f_2(i,x_1(i,s,x_s)+h(i,x_1(i,s,x_s)))-f_1(i,x_1(i,s,x_s))).$$

Consequently, we have

$$h(t, x_1(t, s, x_s)) = \sum_{i=s}^{t} U(t, i)(f_2(i, x_1(i, s, x_s) + h(i, x_1(i, s, x_s))) - f_1(i, x_1(i, s, x_s))).$$

By splitting sum into two parts obtain

$$h(t, x_1(t, s, x_s)) = \sum_{i=s}^{t} U(t, i)(f_2(i, x_1(i, s, x_s) + h(i, x_1(i, s, x_s))) - \sum_{i=s}^{t} U(t, i) f_1(i, x_1(i, s, x_s)))$$

and then by taking one sum to the other side and adding $U(t, s)x$ to both sides obtain

$$U(t, s)x_s + h(t, x_1(t, s, x_s)) + \sum_{i=s}^{t} U(t, i) f_1(i, x_1(i, s, x_s))) =$$

$$= U(t, s)x_s + \sum_{i=s}^{t} U(t, i)(f_2(i, x_1(i, s, x_s) + h(i, x_1(i, s, x_s))).$$

Thus,

$$x_1(t, s, x_s) + h(t, x_1(t, s, x_s)) = x_2(t, s, x_s + h(s, x_s)).$$

Changing the roles of f_1 and f_2, we prove in the same way the existence of $h'(s, x_s)$ that satisfies the equality

$$x_2(t, s, x_s) + h'(t, x_2(t, s, x_s)) = x_1(t, s, x_s + h'(s, x_s)).$$

Designing $H(s, x_s) = x_s + h(s, x_s)$, $H'(s, x_s) = x_s + h'(s, x_s)$, we get

$$H'(t, H(t, x_1(t, s, x_s))) = x_1(t, s, H'(s, H(s, x_s))),$$

$$H(t, H'(t, x_2(t, s, x_s))) = x_2(t, s, H(s, H'(s, x_s))).$$

Taking into account uniqueness of maps $H'(t, H(t, \cdot)) - I$ and $H(t, H'(t, \cdot)) - I$ in $\mathfrak{M}$ we have $H'(t, H(t, \cdot)) = I$ and $H(t, H'(t, \cdot)) = I$ and therefore $H(t, \cdot)$ is a homeomorphism establishing a dynamical equivalence of the 2.41 and 2.42. $\square$

3 Grobman-Hartman theorem of impulsive equations

Many evolution processes are characterized by the fact that at certain moment of time they experience rapid change of state. To describe it mathematically sometimes it is natural to assume that this change is instantaneous, that is impulse. Example of such processes can be found in biology (many biological phenomena involving thresholds, medicine (bursting patterns in neurology or movement of drugs within the body), economics (optimization in logistics or production). Therefore, differential equations involving impulse effect appear as a natural description of observed evolution of several real world problems. The theory of impulsive differential equations are wider comparing to the corresponding theory of differential equations without impulse effect. For example, impulsive differential equation may indicate several new ideas and interpretation such as rhythmical beating, merging of solutions, and non-continuability of solutions. The interest of ordinary differential equations with impulses are not new. In 1937 N.M. Kruylov and N.N. Bogolyubov showed example about model of clock involving impulses in their monograph *Introduction to Nonlinear Mechanics*. The first investigators of differential equations with impulses were Myshkis and Milman [44] in 1960. A systematic study on impulsive differential equations was presented in the monographs-Lakshmikantham, Bainov and Simeonov (1989) [40], Bainov, Kostadinov, Minh (1994) [6] and Samoilenko, Perestyuk (1995) [61]. Many math-

ematicians made great contributions to the dynamical equivalence problem of impulsive differential equation.
Let us consider linear homogeneous impulsive differential system

$$\begin{aligned} \dot{x} &= A(t)x, \quad t \neq \tau_i \\ \triangle x|_{t=\tau_i} &= x(\tau_i + 0) - x(\tau_i - 0) = B_i x. \end{aligned} \tag{3.01}$$

Matrix $A(t)$ is $n \times n$ constant matrix, B_i are constant matrices, $\tau_i \in L$ are fixed times indexed by a set or subset of integers such that $\tau_i < \tau_{i+1}$. The following theorem is not only important result for linear system of type 3.01 but also by it's proof the construction of solutions is represented.

Theorem 26. *Let an interval [$t_0, t_0 + h$] $\subset L$ contain a finite number of points τ_i. Then for any $x_0 \in R^n$, a solution of system 3.01 $x(t, x_0)$, $x(t_0, x_0) = x_0$, exists for all $t \in [t_0, t_0 + h]$. Moreover if, for all i such that $\tau_i \in [t_0, t_0 + h]$, the matrices $I + B_i$ (where I is identity map) are nonsingular, then $x(t, x_0) \neq x(t, y_0)$ for all $t \in [t_0, t_0 + h]$ if $x_0 \neq y_0$.*

Let $t_0 < \tau_j < \tau_{j+1} < \ldots < \tau_{j+k} \leq t_0 + h$. By Picard theorem, a solution $x = \varphi_j(t)$, $\varphi_j(t_0) = x_0$, of system

$$\dot{x} = A(t)x \tag{3.02}$$

exists on $[t_0, \tau_j]$ for any $x_0 \in \mathbb{R}^n$ and is unique.

- For $t \in [t_0, \tau_j]$ set $x(t, x_0) = \varphi_j(t)$.
- For $t = \tau_j$ from system 3.01 obtain

$$x(\tau_j + 0, x_0) = (I + B_j)x(\tau_j, x_0) = x_j^+.$$

- For $t \in [\tau_j, \tau_{j+1}]$ again by the Picard theorem there exists a unique solution of system 3.01

$$x = \varphi_{j+1}(t), \varphi_{j+1}(\tau_j) = x_j^+.$$

- For $t = \tau_{j+1}$

$$x(\tau_{j+1} + 0, x_0) = (I + B_{j+1})x_j^+ = (I + B_{j+1})(I + B_j)x(\tau_j, x_0) = x_{j+1}^+.$$

- And so on.

On the sequel, we will investigate only the systems, for which the following condition hold:

1) any compact interval $[a, b]$ contains only a finite number of the points τ_i;

2) for all i such that $\tau_i \in L$, the matrices $I + B_i$ are nonsingular.

Under these assumptions, the following theorem is true:

Theorem 27. *The set X of all the solutions of linear homogeneous impulsive differential system 3.01, which are defined on the interval $[a, b]$ form n-dimensional vector space.*

A basis in the space X is called a fundamental system of solution of 3.01. This theorem has a number of important corollaries.

1. System 3.01 has a fundamental system of n solutions of $\varphi_1(t)$, $\varphi_2(t)$, ..., $\varphi_n(t)$.

2. Every solution of system 3.01 is a linear combination of solutions of the fundamental system.

3. Any $n + 1$ solutions of 3.01 are linearly dependent.

Matrix $X(t)$ is called fundamental matrix. Notation $X(t, t_0)$ is called matriciant and satisfies the condition $X(t_0) = I$. Let $U(t, \tau)$ be a solution of the matrix Cauchy problem

$$\dot{U} = A(t)U, \quad U(\tau, \tau) = I.$$

Then any solution can be represented as

$$X(t) = U(t, \tau_{j+k})(I + B_{j+k})U(\tau_{j+k}, \tau_{j+k-1})(I + B_{j+k-1})...$$
$$...(I + B_j)U(\tau_j, t_0)X(t_0),$$
$$\tau_{j-1} < t_0 \leq \tau_j < \tau_{j+k} < t < \tau_{j+k-1}.$$

For $X(t, x_0)$ we have

$$X(t, t_0) = U(t, \tau_{j+k})(I + B_{j+k})U(\tau_{j+k}, \tau_{j+k-1})(I + B_{j+k-1})...$$
$$...(I + B_j)U(\tau_j, t_0),$$
$$\tau_{j-1} < t_0 \leq \tau_j < \tau_{j+k} < t < \tau_{j+k-1}.$$

or

$$X(t, t_0) = U(t, \tau_{j+k})(I + B_{j+k}) \prod_{\nu=k}^{1} U(\tau_{j+\nu}, \tau_{j+\nu-1}) \times (I + B_{j+\nu-1})U(\tau_j, t_0),$$
$$\tau_{j-1} < t_0 \leq \tau_j < \tau_{j+k} < t < \tau_{j+k-1}.$$

Further we consider impulsive equation of the form

$$\dot{x} = A(t)x + f(t, x), \quad t \neq \tau_i$$
$$\triangle\, x|_{t=\tau_i} = x(\tau_i + 0) - x(\tau_i - 0) = \tag{3.03}$$
$$= B_i x(\tau_i - 0) + I_i x(\tau_i - 0),$$

where impulse might be into linear and nonlinear part and where $i \in \mathbb{Z}$, $x \in X$ and X is a complex Banach space.

$$...\tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < ...$$

and $\lim\limits_{i \to +\infty} \tau_i = +\infty$ and $\lim\limits_{i \to -\infty} \tau_i = -\infty$. We assume that there exists uniformly in $t \in \mathbb{R}^+$

$$\lim_{T \to +\infty} \frac{m(i, i+T)}{T} = p < +\infty, \tag{3.04}$$

where $m(a, b)$ is defined as a number of points t_i belonging to (a, b).

Definition 36. By solution of 3.03 we understand a piecewise continuous function $x(t)$, $t \in \mathbb{R}^+$ with discontinuities of the first kind at the points τ_i, $i \in \mathbb{Z}$ such that for every $t \neq \tau_n$ we have

$$\dot{x} = A(t)x(t) + f(t, x(t))$$

and $x(t)$ satisfies 3.04 for every $t = \tau_i$

Solution can be written in the form

$$x(t, t_0, x_0) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, \tau)f(x(\tau, t_0, x_0))d\tau + \sum_{t_0 < \tau_i < t} X(t, \tau_i)I_i(x(\tau_i, t_0, x_0)),$$

where

$$X(t, t_0) = U(t, \tau_{j+k})(I + B_{j+k}) \prod_{\nu=k}^{1} U(\tau_{j+\nu}, \tau_{j+\nu-1}) \times (I + B_{j+\nu-1})U(t_j, t_0),$$

$$\tau_{j-1} < t_0 \leq \tau_{j+k} < t \leq \tau_{j+k+1},$$

$U(t, \tau)$ is fundamental matrix, $U(\tau, \tau) = I$, where I- identity map.

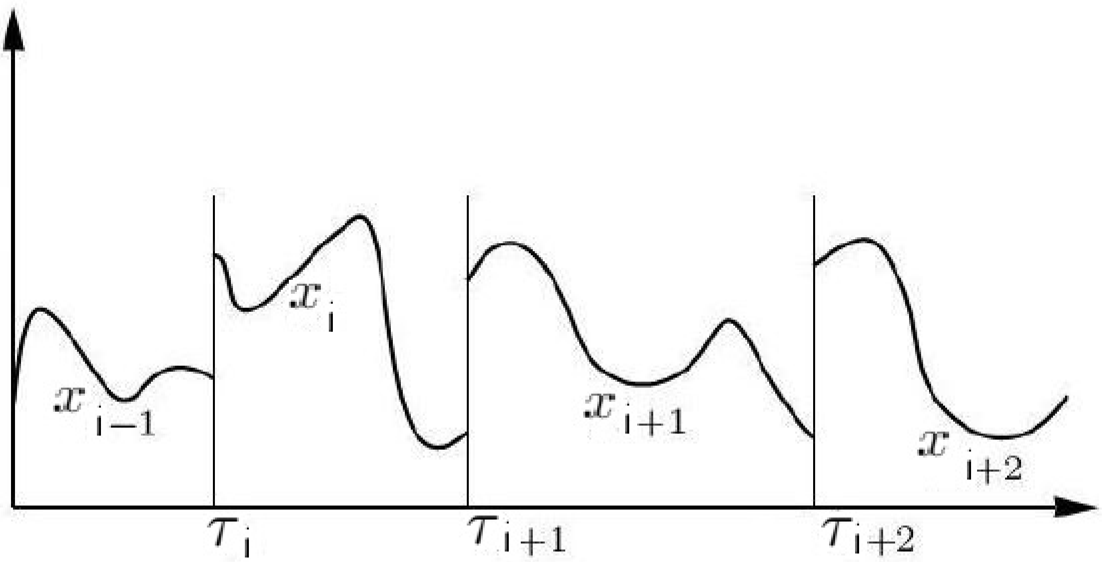

Figure 3.1: Sketch of solution of impulsive differential equation

Example 17. As an example, let us solve linear impulsive differential equation with constant coefficient.

$$\begin{cases} \dot{x}_1 = 2x_2 & t \neq \tau_i \\ \dot{x}_2 = -x_1 + 3x_2 & t \neq \tau_i \\ \triangle x_1 \Big|_{t=\tau_i} = -\frac{3}{2}x_1 - \frac{2}{3}x_2 & \\ \triangle x_2 \Big|_{t=\tau_i} = -\frac{1}{3}x_1 + \frac{1}{2}x_2 & \end{cases}.$$

Or in matrix form

$$\begin{cases} \dot{x} = A(t)x & t \neq \tau_i \\ \triangle x|_{t=\tau_i} = B_i x & \end{cases},$$

where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -\frac{3}{2} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{2} \end{pmatrix}$.

$$AB = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -\frac{3}{2} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & -1 \\ \frac{3}{2} - 1 & -\frac{2}{3} - \frac{3}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{2}{3} & -1 \\ \frac{1}{2} & -\frac{13}{6} \end{pmatrix}$$

and

$$BA = \begin{pmatrix} -\frac{3}{2} & -\frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & -1 \\ \frac{1}{2} - 1 & -\frac{2}{3} - \frac{3}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} -\frac{2}{3} & -1 \\ \frac{1}{2} & -\frac{13}{6} \end{pmatrix}.$$

Next, we find eigenvalues

$$\begin{vmatrix} -\lambda & 2 \\ -1 & 3 - \lambda \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0,$$

$$\lambda_1 = 2, \lambda_2 = 1$$

and corresponding eigenvectors

$\lambda_1 = 2$ $\qquad\qquad$ $\lambda_2 = 1$

$$\begin{pmatrix} -2 & 2 \\ -1 & 3-2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 2 \\ -1 & 3-1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2a+2b \\ -a+b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} -a+2b \\ -a+2b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Since A and B commute, then solutions of equation can be written as $x(t, x_0) = e^{A(t-t_0)}(I+B)^{i(t,t_0)}x_0$, where $i(t, t_0)$ is the number of points τ_i which are in the segment $[t_0, t)$ and furthermore

$$x(t, x_0) = e^{\lambda t}(I+B)^{i(t,0)}x_0,$$

where λ-eigenvalue and x_0-corresponding eigenvector.

$$x^{(1)}(t) = e^{2t}(I+B)^{i(0,t)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad x^{(2)}(t) = e^{t}(I+B)^{i(0,t)} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

where

$$I+B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\frac{3}{2} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{2} \end{pmatrix}.$$

Therefore, solutions of the given example for $t > 0$ is as follows:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (I+B)^{i(0,t)}(C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}).$$

Denote $I + B$ by M and compute first several powers

$$M^2 = \begin{pmatrix} -\frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{36} & 0 \\ 0 & \frac{1}{36} \end{pmatrix}$$

$$M^3 = M^2 M = \begin{pmatrix} -\frac{1}{36} & 0 \\ 0 & \frac{1}{36} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{36*2} & \frac{2}{36*3} \\ -\frac{1}{36*3} & \frac{1}{36*2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{6^2*2} & \frac{2}{6^2*3} \\ -\frac{1}{6^2*3} & \frac{1}{6^2*2} \end{pmatrix}$$

$$M^4 = M^3 M = \begin{pmatrix} -\frac{1}{6^2*2} & \frac{2}{6^2*3} \\ -\frac{1}{6^2*3} & \frac{1}{6^2*2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{6^4} & 0 \\ 0 & \frac{1}{6^4} \end{pmatrix}$$

$$M^5 = M^4 M = \begin{pmatrix} \frac{1}{6^4} & 0 \\ 0 & \frac{1}{6^4} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{6^4*2} & \frac{2}{6^4*3} \\ -\frac{1}{6^4*3} & \frac{1}{6^4*2} \end{pmatrix}$$

...

$$M^{2k} = \begin{pmatrix} \frac{1}{6^{2k}} & 0 \\ 0 & \frac{1}{6^{2k}} \end{pmatrix}$$

$$M^{2k+1} = \begin{pmatrix} -\frac{1}{6^{2k}*2} & \frac{2}{6^{2k}*3} \\ -\frac{1}{6^{2k}*3} & \frac{1}{6^{2k}*2} \end{pmatrix}.$$

If $\tau_{2k} < t \leq \tau_{2k+1}$ then

$$x_1(t) = \frac{1}{6^{2k}}(C_1e^{2t} + C_2e^t),$$
$$x_2(t) = \frac{1}{6^{2k}}(C_1e^{2t} + 2C_2e^t).$$

If $\tau_{2k+1} < t \leq \tau_{2k+2}$ then

$$x_1(t) = -\frac{1}{2*6^{2k}}(C_1e^{2t} + C_2e^t) + \frac{2}{3*6^{2k}}(C_1e^{2t} + 2C_2e^t),$$
$$x_2(t) = -\frac{1}{3*6^{2k}}(C_1e^{2t} + C_2e^t) + \frac{2}{2*6^{2k}}(C_1e^{2t} + 2C_2e^t).$$

Example shows that even for simplified case of impulsive equations solutions is quite complicated. Also, other results for impulsive equations can get very complicated, for example, the linearization problem. The equivalence problem involving impulse effect was first considered by A. Reinfelds and L. Sermone [56], [57], [62]. Other generalizations of the linearization problem are reported in [55], [67], [69] and [70]. In previous papers uniform dichotomy is required but in this these results are generalized with nonuniform dichotomy. Nonuniform hyperbolic dynamics for Grobman-Hartman theorem was consider by Barreira and Valls [9], [10] but not for impulsive equations. Therefore, nonuniform hyperbolic dynamics for impulsive equations is novelty.

In our research we generalize these results. We use Green type map and integral functional equation technique [52] to substantially simplify the proof. For more general point of view we consider impulsive differential equations in arbitrary Banach space. To highlight our improvement comparing to previous results, we use an example, where the linear part of the differential equations even does not possess an ordinary dichotomy. Using the suitable bump function and the change of variables it is possible to reduce the analysis of local equivalence of equations to investigation of the global equivalence of equations.

3.1 Main result and proof

In this section we refer to author's publication 2.

Let **X** be a Banach space and let $\mathfrak{L}(\mathbf{X})$ be the Banach space of bounded linear maps. Consider the following impulsive differential equations

$$\begin{aligned} \dot{x} &= A(t)x + f_1(t,x), \quad t \neq \tau_i \\ \Delta x|_{t=\tau_i} &= x(\tau_i+0) - x(\tau_i-0) = C_i x(\tau_i-0) + p_{1i}(x(\tau_i-0)) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \dot{x} &= A(t)x + f_2(t,x), \quad t \neq \tau_i \\ \Delta x|_{t=\tau_i} &= x(\tau_i+0) - x(\tau_i-0) = C_i x(\tau_i-0) + p_{2i}(x(\tau_i-0)) \end{aligned} \tag{3.12}$$

where:

(i) the map $A\colon \mathbb{R} \to \mathfrak{L}(\mathbf{X})$ is locally integrable in the Bochner's sense;

(ii) the maps $f_j\colon \mathbb{R} \times \mathbf{X} \to \mathbf{X}$, $j = 1,2$ are locally integrable in the Bochner's sense with respect to t for fixed x, and, in addition they satisfy the Lipschitz conditions

$$|f_j(t,x) - f_j(t,x')| \leq \varepsilon(t)|x - x'|, \quad j = 1,2,$$

and the estimate

$$\sup_x |f_1(t,x) - f_2(t,x)| \leq N(t) < +\infty,$$

where $N\colon \mathbb{R} \to \mathbb{R}_+$ and $\varepsilon\colon \mathbb{R} \to \mathbb{R}_+$ are integrable scalar functions;

(iii) for $i \in Z$, $C_i \in \mathfrak{L}(X)$, the maps $p_{1i}\colon \mathbf{X} \to \mathbf{X}$, $p_{2i}\colon \mathbf{X} \to \mathbf{X}$ satisfy the Lipschitz conditions and the estimates

$$|p_{ji}(x) - p_{ji}(x')| \leq \varepsilon(\tau_i - 0)|x - x'|, \quad j = 1,2,$$

$$|p_{1i}(x) - p_{2i}(x)| \leq N(\tau_i - 0) < +\infty;$$

(iv) the maps $x \to x+C_i x$ are homeomorphisms and $\sup_i |(I+C_i)^{-1}\varepsilon(\tau_i-0)| < 1$, where I is the identity map;

(v) the moments τ_i of impulse effect form a strictly increasing sequence

$$\ldots < \tau_{-2} < \tau_{-1} < \tau_0 < \tau_1 < \tau_2 < \ldots$$

where the limit point may be only $\pm\infty$.

Note that condition (iv) implies continuability of solutions (3.11) and (3.12) in the negative direction. Furthermore, condition (v) together with the Lipschitz property with respect to x of the right hand side ensures that there is a unique solution defined on $\mathbb{R}$. Let $x_j(\cdot, s, x)\colon \mathbb{R} \to \mathbf{X}$, $j = 1, 2$, be the solutions of impulsive differential equations (3.11), (3.12) respectively where $x_j(s, s, x) = x$. At the break points τ_i the value for solutions are taken at $\tau_i + 0$. For short, we will use the notation $x_j(t) = x_j(t, s, x)$.
Using the suitable bump function, it is possible to reduce the analysis of local equivalence in a fixed radius tabular neighbourhood of the origin to investigation of the global equivalence.

Definition 37. The impulsive differential equations (3.11) and (3.12) are *globally dynamical equivalent* if there exists a map $H\colon \mathbb{R} \times \mathbf{X} \to \mathbf{X}$ such that

(i) for each fixed $t \in \mathbb{R}$ the map $H(t, \cdot)\colon \mathbf{X} \to \mathbf{X}$ is a homeomorphism;

(ii) $\sup\limits_{t,x} |H(t, x) - x| < +\infty$;

(iii) for all $t \in \mathbb{R}$
$$H(t, x_1(t, s, x)) = x_2(t, s, H(s, x)).$$

Let
$$D = \{(t, s) \in \mathbb{R}^2 \mid t = s\}.$$

Definition 38. A continuous map $G\colon \mathbb{R}^2 \backslash D \to \mathfrak{L}(\mathbf{X})$ is *Green type map* if

(i) $G(\cdot, s)$ is solution of linear impulsive differential equation

$$\begin{aligned} \dot{x} &= A(t)x \\ \Delta x|_{t=\tau_i} &= C_i x(\tau_i - 0) \end{aligned} \tag{3.13}$$

except of $t = s$;

(ii)

$$G(t+0, t) - G(t-0, t) = I.$$

Note that the linear impulsive differential equation (3.13) has infinitely many Green type maps. But if the linear impulsive differential equation (3.13) has an exponential dichotomy, then moreover there exists a unique Green type map which satisfies the inequality

$$|G(t, s)| \le K \exp(-\lambda |t - s|), \quad K \ge 1, \lambda > 0.$$

Therefore, the Green type map can be represented in the form

$$G(t, s) = \begin{cases} U(t, s)P(s), & \text{if } t > s \\ U(t, s)(P(s) - I), & \text{if } t < s \end{cases},$$

where $U(t, s)$ is evolution operator of (3.13). Let us note that

$$U(t, \tau)U(\tau, s) = U(t, s).$$

The solutions of (3.11) and (3.12) for $t \ge s$ can be represented in the form

$$x_j(t, s, x) = U(t, s)x + \int_s^t U(t, \tau) f_j(\tau, x_j(\tau, s, x))\, d\tau,$$

$$+ \sum_{s < \tau_i \le t} U(t, \tau_i) p_{ji}(x_j(\tau_i - 0, s, x)), \quad j = 1, 2.$$

Theorem 28. *Suppose that the linear impulsive differential equation* (3.13) *has a Green type map* $G(s, \tau) \in \mathfrak{L}(\mathbf{X})$ *such that*

$$\sup_s \left(\int_{-\infty}^{+\infty} |G(s, \tau)| N(\tau)\, d\tau + \sum_{i=-\infty}^{+\infty} |G(s, \tau_i)| N(\tau_i - 0) \right) < +\infty,$$

$$\sup_s \left(\int_{-\infty}^{+\infty} |G(s,\tau)|\varepsilon(\tau)\, d\tau + \sum_{i=-\infty}^{+\infty} |G(s,\tau_i)|\varepsilon(\tau_i - 0) \right) = q < 1.$$

Then the impulsive differential equations (3.11) *and* (3.12) *are globally dynamical equivalent.*

Proof. Let $\mathbf{PC}(\mathbb{R}\times\mathbf{X},\mathbf{X})$ be a set of maps $h\colon \mathbb{R}\times\mathbf{X}\to\mathbf{X}$ that are bounded, continuous for $(s,x)\in[\tau_i,\tau_{i+1})\times\mathbf{X}$ and have discontinuities of the first kind for $s=\tau_i$. The set

$$\mathfrak{M} = \left\{ h \in \mathbf{PC}(\mathbb{R}\times\mathbf{X},\mathbf{X}) \;\middle|\; \sup_{s,x} |h(s,x)| < +\infty \right\}$$

is Banach space with the supremum norm

$$\|h\| = \sup_{s,x} |h(s,x)|.$$

We will seek the map establishing the equivalence of (3.11) and (3.12) in the form $H(s,x) = x + h(s,x)$. We examine the following integro-functional equation

$$h(s,x) = \int_{-\infty}^{+\infty} G(s,\tau)\Bigg(f_2\Big(\tau, x_1(\tau) + h(\tau, x_1(\tau))\Big) - f_1(\tau, x_1(\tau)) \Bigg) d\tau \tag{3.14}$$

$$+ \sum_{i=-\infty}^{+\infty} G(s,\tau_i)\Bigg(p_{2i}\Big(x_1(\tau_i - 0) + h(\tau_i - 0, x_1(\tau_i - 0))\Big) - p_{1i}(x_1(\tau_i - 0)) \Bigg).$$

Let us consider the map $h \mapsto \mathfrak{T}h$, $h\in\mathfrak{M}$ defined by the equality

$$\mathfrak{T}h(s,x) = \int_{-\infty}^{+\infty} G(s,\tau)\Bigg(f_2\Big(\tau, x_1(\tau) + h(\tau, x_1(\tau))\Big) - f_1(\tau, x_1(\tau)) \Bigg) d\tau$$

$$+ \sum_{i=-\infty}^{+\infty} G(s,\tau_i)\Bigg(p_{2i}\Big(x_1(\tau_i - 0) + h(\tau_i - 0, x_1(\tau_i - 0))\Big) - p_{1i}(x_1(\tau_i - 0)) \Bigg).$$

Because of Lipschitz condition and conditions of the Theorem 28, also $\mathfrak{T}h \in \mathfrak{M}$. Next, we get

$$|\mathfrak{T}h(s,x) - \mathfrak{T}h'(s,x)|$$

$$= \Bigg| \int_{-\infty}^{+\infty} G(s,\tau)\Bigg(f_2\Big(\tau, x_1(\tau) + h(\tau, x_1(\tau))\Big) - f_2\Big(\tau, x_1(\tau) + h'(\tau, x_1(\tau))\Big)\Bigg) d\tau$$

$$+\sum_{i=-\infty}^{+\infty} G(s,\tau_i)\Big(p_{2i}(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-$$

$$-p_{2i}\Big(x_1(\tau_i-0)+h'(\tau_i-0,x_1(\tau_i-0))\Big)\Big|$$

$$\le \int_{-\infty}^{+\infty} |G(s,\tau)|\varepsilon(\tau)|h(\tau,x_1(\tau))-h'(\tau,x_1(\tau))|\,d\tau$$

$$+\sum_{i=-\infty}^{+\infty} |G(s,\tau_i)|\varepsilon(\tau_i-0)|h(\tau_i-0,x_1(\tau_i-0))-h'(\tau_i-0,x_1(\tau_i-0))|$$

$$\le \sup_s \left(\int_{-\infty}^{+\infty} |G(s,\tau)|\varepsilon(\tau)\,d\tau+\sum_{i=-\infty}^{+\infty} |G(s,\tau_i)|\varepsilon(\tau_i-0)\right)\|h-h'\|$$

$$=q\|h-h'\|,$$

where $q<1$. Thus, the map $\mathfrak{T}$ is a contraction and consequently the integro-functional equation (3.14) has a unique solution in $\mathfrak{M}$.
We have

$$h(t,x_1(t))$$

$$=\int_{-\infty}^{+\infty} G(t,\tau)\bigg(f_2\Big(\tau,x_1(\tau)+h(\tau,x_1(\tau))\Big)-f_1(\tau,x_1(\tau))\bigg)\,d\tau$$

$$+\sum_{i=-\infty}^{+\infty} G(t,\tau_i)\bigg(p_{2i}\Big(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-p_{1i}(x_1(\tau_i-0))\bigg)$$

$$=\int_{-\infty}^{t} G(t,\tau)\bigg(f_2\Big(\tau,x_1(\tau)+h(\tau,x_1(\tau))\Big)-f_1(\tau,x_1(\tau))\bigg)\,d\tau$$

$$+\sum_{\tau_i\le t} G(t,\tau_i)\bigg(p_{2i}\Big(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-p_{1i}(x_1(\tau_i-0))\bigg)$$

$$+\int_{t}^{+\infty} G(t,\tau)\bigg(f_2\Big(\tau,x_1(\tau)+h(\tau,x_1(\tau))\Big)-f_1(\tau,x_1(\tau))\bigg)\,d\tau$$

$$+\sum_{t<\tau_i} G(t,\tau_i)\bigg(p_{2i}\Big(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-p_{1i}(x_1(\tau_i-0))\bigg)$$

$$= \int_{-\infty}^{s} U(t,s)G(s,\tau)\Bigg(f_2\Big(\tau, x_1(\tau)+h(\tau,x_1(\tau))\Big) - f_1(\tau,x_1(\tau))\Bigg)\,d\tau$$

$$+\sum_{\tau_i\le s} U(t,s)G(s,\tau_i)\Bigg(p_{2i}\Big(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-p_{1i}(x_1(\tau_i-0))\Bigg)$$

$$+\int_{s}^{t} U(t,\tau)P(\tau)\Bigg(f_2\Big(\tau, x_1(\tau)+h(\tau,x_1(\tau))\Big) - f_1(\tau,x_1(\tau))\Bigg)\,d\tau$$

$$+\sum_{s<\tau_i\le t} U(t,\tau_i)P(\tau_i)\Bigg(p_{2i}\Big(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-p_{1i}(x_1(\tau_i-0))\Bigg)$$

$$+\int_{s}^{+\infty} U(t,s)G(s,\tau)\Bigg(f_2\Big(\tau, x_1(\tau)+h(\tau,x_1(\tau))\Big) - f_1(\tau,x_1(\tau))\Bigg)\,d\tau$$

$$+\sum_{s<\tau_i} U(t,s)G(s,\tau_i)\Bigg(p_{2i}\Big(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-p_{1i}(x_1(\tau_i-0))\Bigg)$$

$$+\int_{t}^{s} U(t,\tau)(P(\tau)-I)\Bigg(f_2\Big(\tau, x_1(\tau)+h(\tau,x_1(\tau))\Big) - f_1(\tau,x_1(\tau))\Bigg)\,d\tau$$

$$+\sum_{t<\tau_i\le s} U(t,\tau_i)(P(\tau_i)-I)\Bigg(p_{2i}\Big(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-p_{1i}(x_1(\tau_i-0))\Bigg)$$

$$= U(t,s)\left(\int_{-\infty}^{+\infty} G(s,\tau)\Bigg(f_2\Big(\tau, x_1(\tau)+h(\tau,x_1(\tau))\Big) - f_1(\tau,x_1(\tau))\Bigg)\,d\tau\right.$$

$$\left.+\sum_{i=-\infty}^{+\infty} G(s,\tau_i)\Bigg(p_{2i}\Big(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-p_{1i}(x_1(\tau_i-0))\Bigg)\right)$$

$$+\int_{s}^{t} U(t,\tau)\Bigg(f_2\Big(\tau, x_1(\tau)+h(\tau,x_1(\tau))\Big) - f_1(\tau,x_1(\tau))\Bigg)\,d\tau$$

$$+\sum_{s<\tau_i\le t} U(t,\tau_i)\Bigg(p_{2i}\Big(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-p_{1i}(x_1(\tau_i-0))\Bigg)$$

$$= U(t,s)h(s,x)$$

$$+\int\limits_s^t U(t,\tau)\Bigg(f_2\Big(\tau, x_1(\tau)+h(\tau,x_1(\tau))\Big)-f_1(\tau,x_1(\tau))\Bigg)\,d\tau$$

$$+\sum_{s<\tau_i\le t} U(t,\tau_i)\Bigg(p_{2i}\Big(x_1(\tau_i-0)+h(\tau_i-0,x_1(\tau_i-0))\Big)-p_{1i}(x_1(\tau_i-0))\Bigg).$$

Consequently, we have

$$x_1(t,s,x)+h(t,x_1(t,s,x))=x_2(t,s,x+h(s,x)).$$

Changing the roles of f_1 and f_2, we prove in the same way the existence of $h'(s,x)$ that satisfies the equality

$$x_2(t,s,x)+h'(t,x_2(t,s,x))=x_1(t,s,x+h'(s,x)).$$

Designing

$$H(s,x)=x+h(s,x),\ H'(s,x)=x+h'(s,x),$$

we get

$$H'\Big(t,H(t,x_1(t,s,x))\Big)=x_1\Big(t,s,H'(s,H(s,x))\Big),$$
$$H\Big(t,H'(t,x_2(t,s,x))\Big)=x_2\Big(t,s,H(s,H'(s,x))\Big).$$

Taking into account uniqueness of maps $H'(t,H(t,\cdot))-I$ and $H(t,H'(t,\cdot))-I$ in $\mathfrak{M}$ we have $H'(t,H(t,\cdot))=I$ and $H(t,H'(t,\cdot))=I$ and therefore $H(t,\cdot)$ is a homeomorphism establishing a dynamical equivalence of the (3.11) and (3.12). □

Let $f_2(t,x)=0$. Then Theorem 28 implies that the impulsive differential equations (3.11) and (3.13) are globally dynamical equivalent.

3.2 Example

In this section we refer to author‘s publication 2.

Consider an impulsive differential equation in $\mathbb{R}^2$

$$\begin{aligned}\dot{x}&=A(t)x+f(t,x),\quad t\neq\tau_i\\ \Delta x|_{t=\tau_i}&=C_i x(\tau_i-0)+p_i(x(\tau_i-0)),\end{aligned}\tag{3.21}$$

where

$$A(t) = \begin{pmatrix} \ln 2 & 0 \\ 0 & -\frac{2t}{1+t^2} \end{pmatrix},$$

$$C_i = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$|f(t,x) - f(t,x')| \leq \varepsilon(t)|x - x'|,$$

$$|p_i(x) - p_i(x')| \leq \varepsilon(\tau_i - 0)|x - x'|,$$

$$\sup_x |f(t,x)| \leq N(t) < +\infty,$$

$$|p_i(x)| \leq N(\tau_i - 0) < +\infty$$

and $\tau_i = i$, $i \in \mathbb{Z}$.

Then the fundamental matrix of the impulsive differential equation takes the forms

$$U(t,s) = \begin{pmatrix} 2^{t-s-k(t,s)} & 0 \\ 0 & \frac{1+s^2}{1+t^2} \end{pmatrix},$$

where $k(t,s)$ is the number of points τ_i which belong to the segment $(s,t]$. Corresponding Green type map can be represented in the form

$$G(t,s) = \begin{pmatrix} 2^{t-s-k(t,s)} & 0 \\ 0 & 0 \end{pmatrix} \text{ if } t > s$$

and

$$G(t,s) = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1+s^2}{1+t^2} \end{pmatrix}, \text{ if } t < s.$$

If

$$\int\limits_{-\infty}^{+\infty} (1+\tau^2)N(\tau)\,d\tau + \sum_{i=-\infty}^{+\infty} (1+\tau_i^2)N(\tau_i - 0) < +\infty$$

and

$$\int\limits_{-\infty}^{+\infty} \max(2, 1+\tau^2)\varepsilon(\tau)\,d\tau + \sum_{i=-\infty}^{+\infty} \max(2, 1+\tau_i^2)\varepsilon(\tau_i - 0) = q < 1,$$

then in accordance with Theorem 28 the impulsive differential equation (3.21) is globally dynamical equivalent to the linear one

$$\begin{aligned} \dot{x} &= A(t)x, \quad t \neq \tau_i, \\ \Delta x|_{t=\tau_i} &= C_i x(\tau_i - 0). \end{aligned} \tag{3.22}$$

The impulsive differential equation (3.22) does not even have an ordinary dichotomy.

4 Grobman-Hartman theorem of quasi-linear dynamic equations on time scales

It is of interest to understand what is the most general class of systems for which the linearization problem can be solved. There are several papers considering Hartman-Grobman theorem on time scales [31], [49], [71], [72], [73], [75]. In our research we generalize these results, even for $\mathbb{R}^n$, by relaxing conditions on linear part A and strengthening conditions on nonlinear part f. We use Green type map and integral functional equation technique [52] to substantially simplify the proof. Moreover, our method to prove the dynamical equivalence used in this paper is completely different from previous papers. Furthermore, for more general point of view we consider differential equations in arbitrary Banach space. To highlight our improvement comparing to previous results, we use an example where the linear part of the differential equation even does not possess an ordinary dichotomy.

4.1 Main result and proof

In this section we refer to author‘s publication 3.

Let $\mathbb{T}$ be a unbounded above and below time scale. Let $\mathbf{X}$ be a Banach space and let $\mathfrak{L}(\mathbf{X})$ be the Banach space of linear bounded endomorphisms. Consider the following dynamic equations

$$x^{\Delta} = A(t)x + f_1(t, x) \tag{4.11}$$

and

$$x^{\Delta} = A(t)x + f_2(t, x) \tag{4.12}$$

where:

(i) the map $A\colon \mathbb{T} \to \mathfrak{L}(\mathbf{X})$ is rd-continuous and the map $A(t)\colon \mathfrak{L}(\mathbf{X}) \to \mathfrak{L}(\mathbf{X})$ is regressive;

(ii) the maps $f_j\colon \mathbb{T} \times \mathbf{X} \to \mathbf{X}$, $j = 1, 2$ are rd-continuous with respect to t for fixed x, and, in addition they satisfy the Lipschitz conditions

$$|f_j(t, x) - f_j(t, x')| \leq \varepsilon(t)|x - x'|, \quad j = 1, 2,$$

and the estimate

$$\sup_x |f_1(t, x) - f_2(t, x)| \leq N(t) < +\infty,$$

where $N\colon \mathbb{T} \to \mathbb{R}_+$ and $\varepsilon\colon \mathbb{T} \to \mathbb{R}_+$ are integrable scalar functions;

(iii) the maps $I + \mu(t)A(t) + \mu(t)f_j(t, \cdot)\colon \mathbf{X} \to \mathbf{X}$, $j = 1, 2$ are invertible, where I is the identity map.

Here $\sigma\colon \mathbb{T} \to \mathbb{T}$ is the *forward jump operator* defined by equality

$$\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\}$$

and $\mu\colon \mathbb{T} \to [0, +\infty)$ is the *graininess function* defined by

$$\mu(t) = \sigma(t) - t.$$

[For details, see [12]].

Note that condition (iii) implies continuability of solutions (4.11) and (4.12) in the negative direction. Furthermore, this together with the Lipschitz property with respect to x of the right hand side ensures that there is a unique solution for initial value problem defined on $\mathbb{T}$.

Let $x_j(\cdot, s, x)\colon \mathbb{T} \to \mathbf{X}$, $j = 1, 2$, be the solutions of dynamic equations (4.11), (4.12) respectively satisfying the initial conditions $x_j(s) = x$. So $x_j(s, s, x) = x$ and, because of uniqueness of solutions, for $t, \tau, s \in \mathbb{T}$ we have

$$x_j(t, s, x) = x_j(t, \tau, x_j(\tau, s, x)).$$

For short, we will use the notation $x_j(t) = x_j(t, s, x)$.

Local results, which hold under more realistic assumptions on the nonlinearity, can be deduced using standard bump function technique.

Definition 39. The dynamic equations (4.11) and (4.12) are *globally dynamical equivalent* if there exists a map $H\colon \mathbb{T} \times \mathbf{X} \to \mathbf{X}$ such that

(i) for each fixed $t \in \mathbb{T}$ the map $H(t, \cdot)\colon \mathbf{X} \to \mathbf{X}$ is a homeomorphism;

(ii) $\sup\limits_{t,x} |H(t, x) - x| < +\infty$;

(iii) for all $t \in \mathbb{T}$
$$H(t, x_1(t, s, x)) = x_2(t, s, H(s, x)).$$

Green type map can be represented in the form

$$G(t, s) = \begin{cases} e_A(t, s)P(s), & \text{if } t > s \\ e_A(t, s)(P(s) - I), & \text{if } t < s \end{cases},$$

where $e_A(t, s)$ is the *exponential map* of linear dynamic equation

$$x^{\Delta} = A(t)x \tag{4.13}$$

and $P(s) \in \mathfrak{L}(\mathbf{X})$ is rd-continuous with respect to $s \in \mathbb{T}$. Note that the linear dynamic equation (4.13) has infinitely many Green type maps. But if $\mathbb{T} = \mathbb{R}$ and the linear dynamic equation (4.13) has an exponential dichotomy then moreover there exists a unique Green type map which satisfies the inequality

$$|G(t,s)| \leq K \exp(-\lambda|t-s|), \quad K \geq 1, \lambda > 0.$$

Let us note that

$$e_A(t,\tau)e_A(\tau,s) = e_A(t,s).$$

The solutions of (4.11) and (4.12) can be represented in the form

$$x_j(t,s,x) = e_A(t,s)x + \int\limits_s^t e_A(t,\sigma(\tau))f_j(\tau,x_j(\tau,s,x))\,\Delta\tau,\ j = 1,2.$$

Theorem 29. *Suppose that the linear dynamic equation* (4.13) *has an* rd*-continuous Green type map* $G(s,\tau) \in \mathfrak{L}(\mathbf{X})$ *such that*

$$\sup_s \int\limits_{-\infty}^{+\infty} |G(s,\sigma(\tau))|N(\tau)\,\Delta\tau < +\infty,$$

$$\sup_s \int\limits_{-\infty}^{+\infty} |G(s,\sigma(\tau))|\varepsilon(\tau)\,\Delta\tau = q < 1.$$

Then the dynamic equations (4.11) *and* (4.12) *are globally dynamical equivalent.*

Proof. Let $\mathbf{C}_{rd}(\mathbb{T}\times\mathbf{X},\mathbf{X})$ be a set of maps $h\colon \mathbb{T}\times\mathbf{X}\to\mathbf{X}$ that are rd-continuous with respect to t for fixed x and continuous with respect to x. The set

$$\mathfrak{M} = \left\{ h \in \mathbf{BC}_{rd}(\mathbb{T}\times\mathbf{X},\mathbf{X}) \;\middle|\; \sup_{s,x}|h(s,x)| < +\infty \right\}$$

is Banach space with the supremum norm

$$\|h\| = \sup_{s,x}|h(s,x)|.$$

We will seek the map establishing the equivalence of (4.11) and (4.12) in the form $H_1(s,x) = x + h_1(s,x)$. We examine the following integro-functional equation

$$h_1(s,x) = \int\limits_{-\infty}^{+\infty} G(s,\sigma(\tau))\Bigg(f_2\Big(\tau,x_1(\tau)+h_1(\tau,x_1(\tau))\Big) - f_1(\tau,x_1(\tau))\Bigg)\,\Delta\tau. \tag{4.14}$$

Let us consider the map $h_1 \mapsto \mathfrak{T}h_1$, $h_1 \in \mathfrak{M}$ defined by the equality

$$\mathfrak{T}h_1(s,x) = \int_{-\infty}^{+\infty} G(s,\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau) + h_1(\tau, x_1(\tau))\Big) - f_1(\tau, x_1(\tau))\Bigg)\,\Delta\tau.$$

Because of Lipschitz condition and conditions of the Theorem 29, also $\mathfrak{T}h_1 \in \mathfrak{M}$.
Next, we get

$$|\mathfrak{T}h_1(s,x) - \mathfrak{T}h_1'(s,x)|$$

$$= \left|\int_{-\infty}^{+\infty} G(s,\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau) + h_1(\tau, x_1(\tau))\Big) - f_2\Big(\tau, x_1(\tau) + h_1'(\tau, x_1(\tau))\Big)\Bigg)\,\Delta\tau\right|$$

$$\leq \int_{-\infty}^{+\infty} |G(s,\sigma(\tau))|\varepsilon(\tau)|h_1(\tau, x_1(\tau)) - h_1'(\tau, x_1(\tau))|\,\Delta\tau$$

$$\leq \sup_s \int_{-\infty}^{+\infty} |G(s,\sigma(\tau))|\varepsilon(\tau)\,\Delta\tau\|h_1 - h_1'\| = q\|h_1 - h_1'\|,$$

where $q < 1$. Thus, the map $\mathfrak{T}$ is a contraction and consequently the integro-functional equation (4.14) has a unique solution in $\mathfrak{M}$.
We have

$$h_1(t, x_1(t))$$

$$= \int_{-\infty}^{+\infty} G(t,\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau) + h_1(\tau, x_1(\tau))\Big) - f_1(\tau, x_1(\tau))\Bigg)\,\Delta\tau$$

$$= \int_{-\infty}^{t} G(t,\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau) + h_1(\tau, x_1(\tau))\Big) - f_1(\tau, x_1(\tau))\Bigg)\,\Delta\tau$$

$$+ \int_{t}^{+\infty} G(t,\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau) + h_1(\tau, x_1(\tau))\Big) - f_1(\tau, x_1(\tau))\Bigg)\,\Delta\tau$$

$$= \int_{-\infty}^{s} e_A(t,s)G(s,\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau) + h_1(\tau, x_1(\tau))\Big) - f_1(\tau, x_1(\tau))\Bigg)\,\Delta\tau$$

$$+ \int_{s}^{t} e_A(t,\sigma(\tau))P(\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau) + h_1(\tau, x_1(\tau))\Big) - f_1(\tau, x_1(\tau))\Bigg)\,\Delta\tau$$

$$+\int\limits_{s}^{+\infty} e_A(t,s)G(s,\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau)+h_1(\tau,x_1(\tau))\Big)-f_1(\tau,x_1(\tau))\Bigg)\,\Delta\tau$$

$$+\int\limits_{t}^{s} e_A(t,\sigma(\tau))(P(\sigma(\tau))-I)\Bigg(f_2\Big(\tau, x_1(\tau)+h_1(\tau,x_1(\tau))\Big)-f_1(\tau,x_1(\tau))\Bigg)\,\Delta\tau$$

$$=e_A(t,s)\int\limits_{-\infty}^{+\infty} G(s,\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau)+h_1(\tau,x_1(\tau))\Big)-f_1(\tau,x_1(\tau))\Bigg)\,\Delta\tau$$

$$+\int\limits_{s}^{t} e_A(t,\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau)+h_1(\tau,x_1(\tau))\Big)-f_1(\tau,x_1(\tau))\Bigg)\,\Delta\tau$$

$$=e_A(t,s)h_1(s,x)$$

$$+\int\limits_{s}^{t} e_A(t,\sigma(\tau))\Bigg(f_2\Big(\tau, x_1(\tau)+h_1(\tau,x_1(\tau))\Big)-f_1(\tau,x_1(\tau))\Bigg)\,\Delta\tau.$$

Consequently, we have

$$x_1(t,s,x)+h_1(t,x_1(t,s,x))=x_2(t,s,x+h_1(s,x)).$$

Changing the roles of f_1 and f_2, we prove in the same way the existence of $h_2\in\mathfrak{M}$ satisfying integro-functional equation

$$h_2(s,x)=\int\limits_{-\infty}^{+\infty} G(s,\sigma(\tau))\Bigg(f_1\Big(\tau, x_2(\tau)+h_2(\tau,x_2(\tau))\Big)-f_2(\tau,x_2(\tau))\Bigg)\,\Delta\tau \tag{4.15}$$

that satisfies the equality

$$x_2(t,s,x)+h_2(t,x_2(t,s,x))=x_1(t,s,x+h_2(s,x)).$$

Designing $H_2(s,x)=x+h_2(s,x)$, we get

$$H_2\Big(t,H_1(t,x_1(t,s,x))\Big)=x_1\Big(t,s,H_2(s,H_1(s,x))\Big),$$

$$H_1\Big(t,H_2(t,x_2(t,s,x))\Big)=x_2\Big(t,s,H_1(s,H_2(s,x))\Big).$$

Taking into account uniqueness of maps $H_2(t,H_1(t,\cdot))-I$ and $H_1(t,H_2(t,\cdot))-I$ in $\mathfrak{M}$ we have $H_2(t,H_1(t,\cdot))=I$ and $H_1(t,H_2(t,\cdot))=I$ and therefore $H_1(t,\cdot)$ is a homeomorphism establishing a dynamical equivalence of the (4.11) and (4.12). $\square$

Let $f_2(t,x) = 0$. Then Theorem 29 implies that the dynamic equations (4.11) and (4.13) are globally dynamical equivalent.

4.2 Example

In this section we refer to author's publication 3.

Consider a dynamic equation in $\mathbb{R}^2$

$$x^{\Delta} = A(t)x + f(t,x) \tag{4.21}$$

where

$$A(t) = \begin{pmatrix} \ln 2 & 0 \\ 0 & -\frac{2t}{1+t^2} \end{pmatrix},$$

$$|f(t,x) - f(t,x')| \le \varepsilon(t)|x - x'|,$$

$$\sup_x |f(t,x)| \le N(t) < +\infty.$$

Then the exponential map of the dynamic equation takes the form

$$e_A(t,s) = \begin{pmatrix} 2^{t-s} & 0 \\ 0 & \frac{1+s^2}{1+t^2} \end{pmatrix}.$$

Corresponding Green type map can be represented in the form

$$G(t,s) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1+s^2}{1+t^2} \end{pmatrix}, \text{ if } t > s$$

and

$$G(t,s) = \begin{pmatrix} -2^{t-s} & 0 \\ 0 & 0 \end{pmatrix}, \text{ if } t < s.$$

If

$$\int\limits_{-\infty}^{+\infty} (1+\tau^2)N(\tau)\,\Delta\tau < +\infty$$

and

$$\int_{-\infty}^{+\infty} (1+\tau^2)\varepsilon(\tau)\,\Delta\tau = q < 1,$$

then in accordance with Theorem 29 the dynamic equation (4.21) is globally dynamical equivalent to the linear one

$$x^{\Delta} = A(t)x. \tag{4.22}$$

Let us note that $|G(t,s)| = \frac{1+s^2}{1+t^2}$ for $t > s$. It means that $|G(t,s)|$ is not globally bounded. The dynamic equation (4.22) does not even have the uniform ordinary dichotomy.

5 Bounded solutions and Hyers-Ulam stability of quasi-linear dynamic equations on time scales

Important result regarding existence of a bounded solution in the theory of dynamical systems is Bohl-Perron type theorem [14] [48] [18] [16]. One example of Bohl-Perron type theorem for differential equation state that if the bounded input implies bounded output then the system is exponentially stable. P. Bohl in his dissertation [14] studied the problem of existence of a bounded solution at $x \in \mathbb{R}^n$ for the quasi-linear differential equation

$$\dot{x} = Ax + f(t, x),$$

where real parts of matrix A eigenvalues are not equal to zero. P. Bohl proved the theorem for the cases $n = 1$ and $n = 2$. Later B.P. Demidovish proved the theorem in the general case [19] (see also [20] p.300) and he called it the Bohl theorem.

We consider the quasi-linear dynamic equation in a Banach space on unbounded above and below time scales $\mathbb{T}$ with rd-continuous, regressive right hand side. We define the corresponding Green's type map. Using the integral functional technique, we find a new simpler, but at the same time more general sufficient condition for the existence of a bounded solution on the time scales, expressed in terms of integrals of the Green's type map. We construct previously unknown linear

scalar differential equation which does not possess exponentially dichotomous, but for which the integral of the corresponding Green's type map is uniformly bounded. The existence of such example allows on the one hand to obtain the new sufficient condition for the existence of bounded solution and on the other hand to prove Hyers-Ulam stability for a much broader class of linear dynamic equations even in the classical case.
Further is this chapter we refer to author‘s publication 8 and 12 .

5.1 Bounded solution

Let $\mathfrak{L}(\mathbf{X})$ be the Banach space of linear bounded endomorphisms. Let‘s look closely to the following quasi-linear regressive dynamic equation

$$x^{\Delta} = A(t)x + f(t,x), \tag{5.11}$$

where:

(i) the map $A : \mathbb{T} \to \mathfrak{L}(\mathbf{X})$ is *rd*-continuous and the corresponding linear dynamic equation

$$x^{\Delta} = A(t)x \tag{5.12}$$

is regressive;

(ii) the map $f : \mathbb{T} \times (\mathbf{X}) \to (\mathbf{X})$ is rd-continuous with respect to t for fixed x and it satisfies the Lipschitz conditions

$$|f(t,x) - f(t,x')| \leq \varepsilon(t)|x - x'|$$

and in addition, it satisfies the following inequality

$$|f(t,0)| \leq N(t) < +\infty,$$

where $N : \mathbb{T} \to \mathbb{R}_+$ and $\varepsilon : \mathbb{T} \to \mathbb{R}_+$ are integrable scalar functions.

From conditions (i) and (ii) we obtained that the solutions of (5.11) are continuable in the negative direction. Moreover, the solution for initial value problem defined on $\mathbb{T}$ is unique because of Lipschitz condition with respect to x of the right hand. The solution of dynamic equation (5.11) is denoted by $x(\cdot, s, x) \colon \mathbb{T} \to \mathbf{X}$ with initial condition $x(s) = x$. So $x(s, s, x) = x$ and, because of uniqueness of solutions, for $t, \tau, s \in \mathbb{T}$ we have

$$x(t, s, x) = x(t, \tau, x(\tau, s, x)).$$

For short, we will use the notation $x(t) = x(t, s, x)$. The exponential operator (transition operator) $e_A(\cdot, s) \colon \mathbb{T} \to \mathfrak{L}(\mathbf{X})$ is solution of the corresponding operator-valued initial value problem

$$X^{\Delta} = A(t)X,\ X(s) = I,$$

where $t, s \in \mathbb{T}$, $e_A(s, s) = I$ and $I \in \mathfrak{L}(\mathbf{X})$ is identity operator. Let us note that linear cocycle property holds for $\tau, s, t \in \mathbb{T}$

$$e_A(t, \tau)e_A(\tau, s) = e_A(t, s).$$

The solution of (5.11) can be represented in the form

$$x(t, s, x) = e_A(t, s)x + \int_s^t e_A(t, \sigma(\tau))f(\tau, x(\tau, s, x))\,\Delta\tau.$$

Green type map can be represented in the form

$$G(t, s) = \begin{cases} e_A(t, s)P(s), & \text{if } s \leq t \\ e_A(t, s)(P(s) - I_{\mathbf{X}}), & \text{if } t < s \end{cases},$$

where $P(s) \in \mathfrak{L}(\mathbf{X})$ is rd-continuous with respect to $s \in \mathbb{T}$. Note that the linear dynamic equation (5.12) has infinitely many Green type maps. But if $\mathbb{T} = \mathbb{R}$ and the linear regressive dynamic equation (5.12) has uniform exponential dichotomy then moreover there exists a unique Green type map which satisfies the inequality

$$|G(t,s)| \leq K \exp(-\lambda|t-s|), \quad K \geq 1, \lambda > 0.$$

Let us formulate sufficient conditions for the existence of bounded solution to the quasi-linear dynamic equation (5.11). Note that the weaker condition of the nonlinear member and its Lipschitz coefficient play an important role in the formulation of the theorem.

Theorem 30. *Suppose that the linear dynamic equation* (5.12) *has an rd-continuous Green type map $G(s,\tau) \in \mathfrak{L}(\mathbf{X})$ such that*

$$\sup_{t\in\mathbb{T}} \int\limits_{-\infty}^{+\infty} |G(t,\sigma(\tau))| N(\tau)\,\Delta\tau < +\infty, \tag{5.13}$$

$$\sup_{t\in\mathbb{T}} \int\limits_{-\infty}^{+\infty} |G(t,\sigma(\tau))| \varepsilon(\tau)\,\Delta\tau = q < 1. \tag{5.14}$$

Then the quasi-linear equation (5.11) *has a bounded solution.*

Proof. Let $\mathbf{C}_{rd}(\mathbb{T},X)$ be a set of maps $\eta\colon \mathbb{T} \to X$ that are rd-continuous with respect to t. The set

$$\mathfrak{B} = \left\{ \eta \in \mathbf{C}_{rd}(\mathbb{T},X) \;\middle|\; \sup_{t\in\mathbb{T}} |\eta(t)| < +\infty \right\}$$

is Banach space with the supremum norm

$$\|\eta\| = \sup_{t\in\mathbb{T}} |\eta(t)|.$$

Let us consider the following map $\eta \mapsto \mathfrak{T}\eta$, $\eta \in \mathfrak{B}$ defined by the equality

$$\mathfrak{T}\eta(t) = \int\limits_{-\infty}^{+\infty} G(t,\sigma(\tau)) f(\tau,\eta(\tau))\,\Delta\tau.$$

Note that

$$|f(t,x)| \leq |f(t,0)| + |f(t,x) - f(t,0)| \leq N(t) + \varepsilon(t)|x|.$$

We have

$$|\eta(t)| \leq \int_{-\infty}^{+\infty} |G(t,\sigma(\tau))||f(\tau,\eta(\tau))|\,\Delta\tau \leq$$

$$\leq \int_{-\infty}^{+\infty} |G(t,\sigma(\tau))|\varepsilon(\tau)\|\eta\|\,\Delta\tau + \int_{-\infty}^{+\infty} |G(t,\sigma(\tau))||N(\tau)|\,\Delta\tau$$

$$\leq q\|\eta\| + \int_{-\infty}^{+\infty} |G(t,\sigma(\tau))||N(\tau)|\,\Delta\tau < +\infty.$$

It follows that $\mathfrak{T}\eta \in \mathfrak{B}$. Next, we get

$$|\mathfrak{T}\eta(t) - \mathfrak{T}\eta'(t)|$$

$$= \left| \int_{-\infty}^{+\infty} G(s,\sigma(\tau))(f(\tau,\eta(\tau)) - f(\tau,\eta'(\tau)))\,\Delta\tau \right|$$

$$\leq \int_{-\infty}^{+\infty} |G(s,\sigma(\tau))|\varepsilon(\tau)|\eta(\tau) - \eta'(\tau)|\,\Delta\tau$$

$$\leq \sup_t \int_{-\infty}^{+\infty} |G(s,\sigma(\tau))|\varepsilon(\tau)\,\Delta\tau\|\eta - \eta'\| = q\|\eta - \eta'\|,$$

where $q < 1$. Thus, the map $\mathfrak{T}$ is a contraction and consequently the integro-functional equation (5.15) has a unique solution in $\mathfrak{B}$.

$$\eta(t) = \int_{-\infty}^{+\infty} G(t,\sigma(\tau))f(\tau,\eta(\tau))\,\Delta\tau. \tag{5.15}$$

Next, we will prove that the bounded solution of integro-functional equation (5.15) is also the solution of (5.11). First, consider the case when $s < t$. Note that

$$\int_{-\infty}^{s} G(t,\sigma(\tau))f(\tau,\eta(\tau))\,\Delta\tau = \int_{-\infty}^{s} e_A(t,\sigma(\tau))P(\sigma(\tau))f(\tau,\eta(\tau))\,\Delta\tau =$$

$$= \int_{-\infty}^{s} e_A(t,s)e_A(s,\sigma(\tau)P(\sigma(\tau))f(\tau,\eta(\tau))\,\Delta\tau = \int_{-\infty}^{s} e_A(t,s)G(s,\sigma(\tau))f(\tau,\eta(\tau))\,\Delta\tau.$$

Then

$$\eta(t) = \int\limits_{-\infty}^{+\infty} G(t,\sigma(\tau)) f(\tau,\eta(\tau))\,\Delta\tau$$

$$= \int\limits_{-\infty}^{t} G(t,\sigma(\tau)) f(\tau,\eta(\tau))\,\Delta\tau$$

$$+ \int\limits_{t}^{+\infty} G(t,\sigma(\tau)) f(\tau,\eta(\tau))\,\Delta\tau$$

$$= \int\limits_{-\infty}^{s} e_A(t,s) G(s,\sigma(\tau)) f(\tau,\eta(\tau))\,\Delta\tau$$

$$+ \int\limits_{s}^{t} e_A(t,\sigma(\tau)) P(\sigma(\tau)) f(\tau,\eta(\tau))\,\Delta\tau$$

$$+ \int\limits_{s}^{+\infty} e_A(t,s) G(s,\sigma(\tau)) f(\tau,\eta(\tau))\,\Delta\tau$$

$$+ \int\limits_{t}^{s} e_A(t,\sigma(\tau)) (P(\sigma(\tau)) - I) f(\tau,\eta(\tau))\,\Delta\tau$$

$$= e_A(t,s) \int\limits_{-\infty}^{+\infty} G(s,\sigma(\tau)) f(\tau,\eta(\tau))\,\Delta\tau$$

$$+ \int\limits_{s}^{t} e_A(t,\sigma(\tau)) f(\tau,\eta(\tau))\,\Delta\tau$$

$$= e_A(t,s)\eta(s) + \int\limits_{s}^{t} e_A(t,\sigma(\tau)) f(\tau,\eta(\tau))\,\Delta\tau.$$

Analogously, we consider the case when $s > t$.
The bounded solution η is unique in $\mathfrak{B}$ and $\|\eta\| = \sup_{t\in\mathbb{T}} |\eta(t)| < +\infty$. The theorem is proved. $\square$

Remark. Note that conditions (5.13) and (5.14) can be simplified if the improper integral of the Green's type map is uniformly bounded, that is, if

$$\sup_{t\in\mathbb{T}} \int\limits_{-\infty}^{+\infty} |G(t,\sigma(\tau))|\,\Delta\tau < +\infty. \tag{5.16}$$

The condition (5.16) is satisfied if the linear dynamic equation (5.12) admits a uniform exponential dichotomy [18], [16]. In the case $\mathbb{T} = \mathbb{R}$ it can be concluded that there exists a projector $P\colon \mathbb{R} \to \mathfrak{L}(\mathbf{X})$ such that $P^2(s) = P(s)$ and

$$e_A(t,s)P(s) = P(t)e_A(t,s) \tag{5.17}$$

holds, and there exist constants $K \geq 1$ and $\lambda > 0$, such that

$$|e_A(t,s)P(s)| \leq Ke^{-\lambda(t-s)}, \text{ if } s \leq t,$$

$$|e_A(t,s)(I - P(s))| \leq Ke^{\lambda(t-s)}, \text{ if } t \leq s.$$

Linear dynamic equation (5.12) is said to have an exponential dichotomy [49, 50, 72, 73, 76, 77, 78] on $\mathbb{T}$ if there exists a projector $P\colon \mathbb{T} \to \mathfrak{L}(\mathbf{X})$ such that $P^2(s) = P(s)$ and (5.17) holds, and there exist positive constants K_i and λ_i, $i = 1, 2$ such that

$$|e_A(t,s)P(s)| \leq K_1 e_{\ominus\lambda_1}(t,s) \text{ for } s \leq t,\ s,t \in \mathbb{T},$$

$$|e_A(t,s)(I - P(s))| \leq K_2 e_{\ominus\lambda_2}(s,t) \text{ for } t \leq s,\ s,t \in \mathbb{T}.$$

Example 18. We construct the linear scalar differential equation, the solution of which on the one hand has infinite Lyapunov exponent [18, 16], but on the other hand integral of corresponding Green's type map is uniformly bounded. Consider

$$\dot{x} = -\left(\frac{a'(t)}{a(t)} + a(t)\right)x,\ a(t) = \alpha + \kappa(t),\ \alpha > 0,\ \kappa(t) \geq 0, \tag{5.18}$$

where the function $\kappa\colon \mathbb{R} \to \mathbb{R}$ is sawtooth piecewise linear and satisfies the estimates

$$\int\limits_{-\infty}^{+\infty} \kappa(t)\,dt < +\infty, \quad \lim\sup_{t\to+\infty} \frac{\ln(\kappa(t))}{t} = +\infty.$$

Then the Cauchy initial value problem $x(s) = 1$ of equation (5.18) has solution

$$x(t,s) = \exp\left(-\int\limits_s^t a(\tau)\,d\tau\right)\frac{a(s)}{a(t)} > 0$$

with property

$$\limsup_{t\to+\infty}\frac{\ln(x(t,s))}{t} = -\alpha - \limsup_{t\to+\infty}\frac{\ln(a(t))}{t}$$
$$= -\alpha - \limsup_{t\to+\infty}\frac{\ln(\kappa(t))}{t} = -\infty.$$

For example, the function $\kappa\colon \mathbb{R} \to \mathbb{R}$ can be taken as follows. κ is piecewise linear continuous scalar function, $\kappa(-t) = \kappa(t)$ and $n \in \mathbb{N}$

$$\kappa(t) = \begin{cases} 0, & \text{if } t \in [0, 1/2) \\ 0, & \text{if } t \in [n-1/2, n-2^{-1}e^{-n^2}n^{-2}] \\ \text{linear}, & \text{if } t \in (n-2^{-1}e^{-n^2}n^{-2}, n) \\ e^{n^2}, & \text{if } t = n \\ \text{linear}, & \text{if } t \in (n, n+2^{-1}e^{-n^2}n^{-2}) \\ 0, & \text{if } t \in [n+2^{-1}e^{-n^2}n^{-2}, n+1/2). \end{cases}$$

Then

$$\int\limits_{-\infty}^{+\infty}\kappa(t)\,dt = 2\sum_{n=1}^{+\infty}\frac{1}{2n^2} = \frac{\pi^2}{6} =< +\infty$$

and

$$\limsup_{t\to+\infty}\frac{\ln(\kappa(t))}{t} = \lim_{n\to+\infty}\frac{\ln(\kappa(n))}{n} = \lim_{n\to+\infty} n = +\infty.$$

Function $k(t)$ is saw-tooth type function where each subsequent vertex grows very fast. Next, figure represents sketch of function $k(t)$ for n=1,2,3. We choose Green's type map

$$G(t,s) = \begin{cases} x(t,s), & \text{if } s \le t \\ 0, & \text{if } t < s. \end{cases}$$

Let us note that

$$\int\limits_{-\infty}^{+\infty}|G(t,s)|ds = \int\limits_{-\infty}^{t}x(t,s)\,ds = \frac{1}{a(t)} \le \frac{1}{\alpha}.$$

If $\sup\limits_{t,x\in\mathbb{R}}|f(t,x)| < +\infty$ and $\sup\limits_{t\in\mathbb{R}}\varepsilon(t) < \alpha$ then corresponding quasi-linear equation has bounded solution although the solutions of the corresponding linear differential equation has infinite Lyapunov exponent.

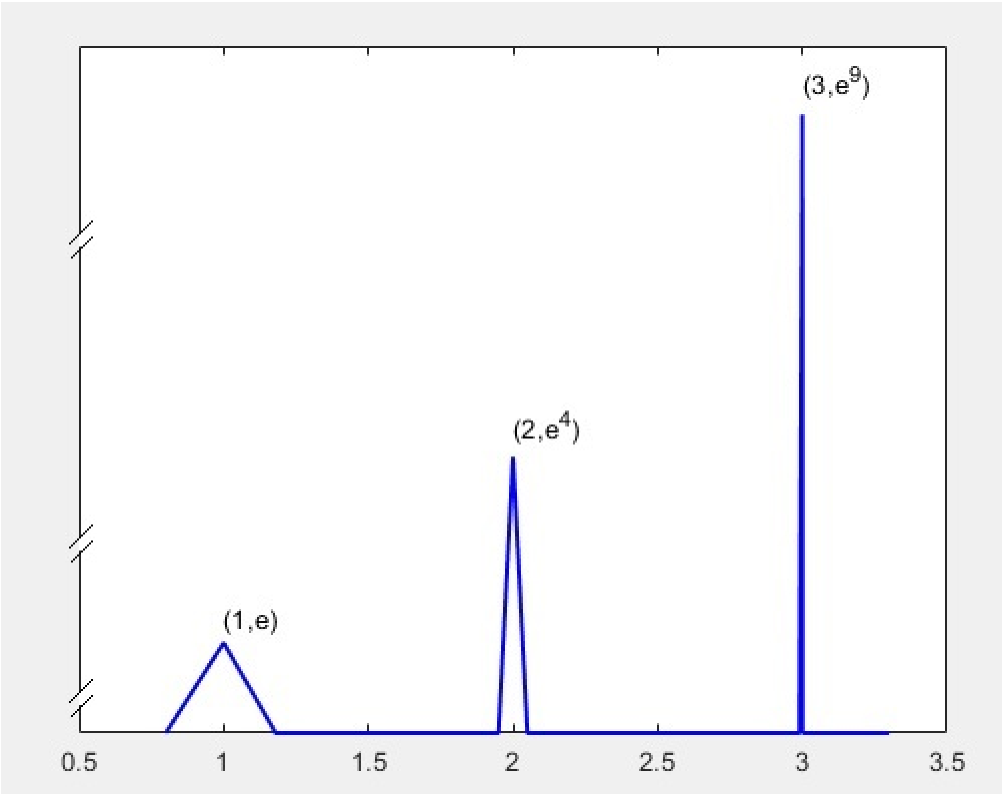

Figure 5.1: Sketch of function $k(t)$

Example 19. Let's look at the case where $\mathbb{T} = \mathbb{R}$. Suppose that linear dynamic equation (5.12) admits a nonuniform exponential dichotomy [7, 74, 75, 79]. Then there exists a projector $P\colon \mathbb{T} \to \mathfrak{L}(\mathbf{X})$ and (5.17) holds, and there exist constants $K \geq 1$, $\lambda > 0$ and $\delta > 0$, such that

$$|e_A(t,s)P(s)| \leq Ke^{-\lambda(t-s)}e^{\delta|s|}, \text{ if } s \leq t,$$

$$|e_A(t,s)(I - P(s))| \leq Ke^{\lambda(t-s)}e^{\delta|s|}, \text{ if } t \leq s.$$

When $\delta \equiv 0$ we obtain the classical notion of uniform exponential dichotomy. If $\delta \equiv 0$ and $\lambda \equiv 0$ we obtain a uniform dichotomy. If $N(t) \leq \mu e^{-\delta|t|}$, $\varepsilon(t) \leq re^{-\delta|t|}$ and $2Kr < \lambda$ then dynamic equation (5.11) in the case of nonuniform exponential dichotomy has bounded solution [74].

If $\int\limits_{-\infty}^{+\infty} N(t) < +\infty$ and

$$\sup_{t,s\in\mathbb{R}} |G(t,s)| \int\limits_{-\infty}^{+\infty} \varepsilon(t)\,dt = q < 1$$

then dynamic equation (5.11) in the case of uniform dichotomy has bounded solution [35].

Example 20. For simplicity let us consider nonhomogeneous linear differential equation on $\mathbb{R}$

$$x' = A(t)x + f(t) = -\frac{2t}{1+t^2}x - \frac{2te^{-t^2}}{1+t^2}.$$

with nonuniform dichotomy. In our case exponential map is

$$e_A(t,s) = \frac{1+s^2}{1+t^2}$$

and choose the Green type map in the following form

$$G(t,s) = \begin{cases} \frac{1+s^2}{1+t^2}, & \text{if } s \le t \\ 0, & \text{if } s > t \end{cases}.$$

It follows that bounded solution is

$$\eta(t) = -\int_{-\infty}^{t} \frac{1+s^2}{1+t^2}\frac{2se^{-s^2}}{1+s^2}\,ds = \frac{e^{-t^2}}{1+t^2}.$$

5.2 Periodic solution

Let the time scale be periodic, that is if $t \in \mathbb{T}$, then $t + \omega \in \mathbb{T}$, where $\omega > 0$ is the period of the time scale $\mathbb{T}$. Then the graininess function is also periodic $\mu(t) = \mu(t+\omega)$.

Let us consider the quasi-linear dynamic equation (5.11) where the right hand side is periodic with period $\omega > 0$, consequently, $A(t+\omega) = A(t)$ and $f(t+\omega, x) = f(t,x)$. We obtain that exponential map satisfies equality $e_A(t+\omega, s+\omega) = e_A(t,s)$. If $P(s+\omega) = P(s)$ it follows that $G(t+\omega, s+w) = G(t,s)$

It is important to note that

$$\eta(t+\omega) = \int_{-\infty}^{+\infty} G(t+\omega, \sigma(\tau)) f(\tau, \eta(\tau))\,\Delta\tau$$

$$= \int_{-\infty}^{+\infty} G(t+\omega, \sigma(r+\omega)) f(r+\omega, \eta(r+\omega))\, \Delta r$$

$$= \int_{-\infty}^{+\infty} G(t, \sigma(r)) f(r, \eta(r+\omega))\, \Delta r.$$

Because the integro-functional equation (5.15) has a unique bounded solution, we get that unique bounded solution is periodic

$$\eta(t+w) = \eta(t).$$

5.3 Hyers-Ulam stability

Stanislaw Marcim Ulam (1909-1984) was a Polish-American scientist in the fields of mathematics and nuclear physics. In 1940 S.M. Ulam proposed the following stability problem of functional equations [66] at the University of Wisconsin. Let E and E' be Banach spaces and let δ be a positive number. A transformation $f(x)$ of E into E' will be called δ-linear if $\|f(x+y) - f(x) - f(y)\| < \delta$ for all x and y in E. Does there exist for each $\varepsilon > 0$ a $\delta > 0$ such that, to each δ-linear transformation $f(x)$ of E into E' there corresponds a linear transformation $l(x)$ of E into E' satisfying the inequality $\|f(x) - l(x)\| \leq \varepsilon$ for all x in E? Or in other words, when a solution of an equation differing slightly from a given one, it must be somehow near to the exact solution of the given equation. In the next year Donald H. Hyers, Professor Emeritus from the University of Southern California, gave partial answer to the question.

Since then, a large number of papers have been published in connection with various generalization of Ulam‘s problem and Hyers‘s theorem.

Definition 40. We say that dynamic equation $x^\Delta = f(t,x)$ is Hyers-Ulam stable if there exists a positive constant $C > 0$ such that for each real number $\varepsilon > 0$ and for each solution x of the inequality

$$|x^\Delta - f(t,x)| \leq \varepsilon$$

there exists a solution x_0 of the dynamic equation $x^{\Delta} = f(t, x)$ with the property

$$\sup_{t \in \mathbb{T}} |x(t) - x_0(t)| \leq C\varepsilon.$$

It should be pointed out that the Lyapunov stability is different from the Hyers-Ulam satbility. The Hyers-Ulam stability means that a differential equation has a close exact solution generated by the apporxiamte solutions and the error of the approxiamte solution can be estimated. Nevertheless, the Lyapunov stability means that if a solution which starts out near an equilibrium point, will stay near the equilibrium point forever. The Hyers-Ulam stability can be widely used to the fields where it is difficult to find exact solutions, e.g., numerical analysis, optimization, biology, economics, and others [2].

We will give a new sufficient condition for the Hyers-Ulam stability of a linear dynamic equation (5.12) in the case when the integral of Green's type map is uniformly bounded. Example 18 shows that there are cases where, on the one hand, the solution of a linear dynamical system has an infinite Lyapunov exponent, but on the other hand, the integral from the Green type map is uniformly bounded.

Theorem 31. *Suppose that the linear dynamic equation* (5.12) *has an rd-continuous Green type map $G(s, \tau) \in \mathfrak{L}(\mathbf{X})$ such that*

$$\sup_{t \in \mathbb{T}} \int_{-\infty}^{+\infty} |G(t, \sigma(\tau))| \, \Delta\tau < +\infty.$$

Then the linear dynamic equation (5.12) *is Hyers-Ulam stable.*

Proof. Let

$$\eta(t) = x^{\Delta} - A(t)x.$$

Then $|\eta(t)| \leq \varepsilon$ and general solution of

$$x^{\Delta} = A(t)x + \eta(t).$$

is

$$x(t) = e_A(t, s)C_1 + \int_{-\infty}^{+\infty} G(t, \sigma(\tau))\eta(\tau) \, \Delta\tau.$$

Let us take

$$x_0(t) = e_A(t,s)C_1,$$

where $C_1 \in \mathbf{X}$. Then

$$|x(t) - x_0(t)| \leq \left| \int\limits_{-\infty}^{+\infty} G(t, \sigma(\tau))\eta(\tau)\,\Delta\tau \right|$$

$$\leq \varepsilon \int\limits_{-\infty}^{+\infty} |G(t, \sigma(\tau))|\,\Delta\tau$$

$$\leq \varepsilon \sup_{t \in \mathbb{T}} \int\limits_{-\infty}^{+\infty} |G(t, \sigma(\tau))|\,\Delta\tau,$$

where

$$C = \sup_{t \in \mathbb{T}} \int\limits_{-\infty}^{+\infty} |G(t, \sigma(\tau))|\,\Delta\tau.$$

The theorem is proved. $\square$

Conclusions

The thesis presents more general results for the Grobman-Hartman theorem of quasi-linear differential and difference equation (even for R^n), quasi-linear dynamic equations on time scale, impulse equations on time scale, as well as the existence of bounded solutions, periodic solutions, and Hyers-Ulam stability.

Our research to included representative examples to highlight the novelty of equations to which the Grobman-Hartman theorem now can be applied.

To achieve the overall goal and the objectives of the thesis investigation on Grobman-Hartamn theorem and its Palmer's generalization was done. Also, time scale calculus, introduction to a Green type map, the Banach fixed point theorem, bounded solutions and Hyers-Ulam stability theory was captured.

Several citations have been made on our publications. 11 citations might be found in Scopus data base and 9 citations in Web of Science data base. In this thesis obtained result regarding authors publication Nr. 1 is rather general result that can be applied to situation when linear equation does not possess any type of dichotomy, as mentioned in article [4] of Journal of Differential Equations (2021) . One of the latest citation in year 2022 [5] strengthens the belief that the problem discussed in this work is modern, relevant and contain new ideas.

It is my belief that the overall goal of the thesis has been achieved.

List of conferences

- The international conference on difference equations and applications 2016 (ICDEA 2016), ”Bounded solution of dynamic system on time scale”, July 24.-29., 2016, Osaka, Japan.
- 10th International conference on Progress on Difference Equations, ”Bohl-Perron principle for dynamic systems on time scale”, May 17.-20., 2016, Riga, Latvia.
- 11th Latvian Conference on Mathematics, ”Dynamical system on time scales”, April 15.-16., 2016 Daugavpils, Latvia
- 74th Conference of University of Latvia, ”Dinamiskās sistēmas laika skalā”, February 2016, Riga, Latvia.
- The international conference on difference equations and applications 2015 (ICDEA 2015), ”Dynamical equivalence of quasi-liner dynamic equations on time scale”, July 19.-25., 2015, Bialystok, Poland.
- 73th Conference of University of Latvia, ”Kvazilineāru vienādojumu ekvivalence laika skalā”, February 2015, Riga, Latvia.
- Conference on Differential and Difference Equations and Applications, ”Conjugacy of dynamical systems on time scale”, June 23. – 27., 2014, Jasna, Slovak Republic.

- 72th Conference of University of Latvia, ”Kvazilineāru vienādojumu ekvivalence”, February 2014, Riga, Latvia.
- 10th Latvian Conference on Mathematics, ”Apgriežamu kvazilineāru diferenču vienādojumu ekvivalence”, April 11.-12., 2014 Liepaja, Latvia.
- 18th International Conference Mathematical Modelling and Analysis, ”Conjugancy of quasilinear equations”, May 27.-30., 2013, Tartu, Estonia.

Author‘s publications

1. A. Reinfelds and Dz. Steinberga. Dynamical equivalence of quasilinear equations *International Journal of Pure and Applied Mathematics*, 98(3), 355-364, 2015. Scopus

2. A. Reinfelds and Dz. Steinberga. Dynamical equivalence of impulsive quasilinear equations *Tatra Mountains Mathematical Publications*, 63(1), 237-246, 2015. Scopus.

3. A. Reinfelds and Dz. Steinberga. Dynamical equivalence of quasilinear dynamic equations on time scales, *Journal of Mathematical Analysis*, 7 (2016), no. 1, 115 – 120. Web of Science.

4. A. Reinfelds and Dz. Steinberga. Dynamical equivalence of quasilinear dynamical equations on time scales, *Abstracts of the 21st International Conference on Difference Equations and Applications*, 19 – 25 July 2015, Bialystok, Poland.

5. Dz. Steinberga. Dynamic equation on time scale, *Abstracts of the 11th Latvian Mathematical Conference*, April 15-16, 2016, Daugavpils, Latvia.

6. A. Reinfelds and Dz. Steinberga. Dynamical equivalence of quasilinear dynamic equations on time scale *Journal of Mathematical Analysis. Special Issue associated with The Cape Verde International Days on Mathematics*, 2015.

7. A. Reinfelds and Dz. Steinberga. Bohl-Perron principle for dynamic systems on time scales *Abstracts of PODE 2016*, May 17–20, 2016, Riga, Latvia.

8. A. Reinfelds and Dz. Steinberga. Bounded solution of dynamic system on time scale, *Abstracts of ICDEA 2016*, July 24-29, 2016, Osaka, Japan.

9. A. Reinfelds and Dz. Steinberga. Conjugacy of an invertible quaslinear difference equations, *Abstracts of the 10th Latvian Mathematical Conference*, April 11-12, 2014, Liepaja, Latvia.

10. A. Reinfelds and Dz. Steinberga. Conjugacy of quasiliner equations, *Abstracts of the 18th International Conference Mathematical Modelling and Analysis and 4th International Conference Approximation Methods and Orthogonal Expansions, MMA2013 and AMOE2013*, May 27 – 30, 2013, Tartu, Estonia.

11. A. Reinfelds and Dz. Steinberga. Conjugacy of dynamical systems on time scale, *Conference on Differential and Difference Equations and Applications*, June 23-27, 2014, Jasna, Slovak Republic.

12. A. Reinfelds and Dz. Steinberga. Bounded solutions and Hyers–Ulam stability of quasilinear dynamic equations on time scales, *Nonlinear Analysis: Modelling and Control*. Web of Science. Published Online, February 22 2023, pp. 1-15, https://doi.org/10.15388.2023.28.31603.

Bibliography

[1] R.P. Agarwal *Difference equations and inequalities*. Theory, Methods, and Applications, Marcel Dekker Inc., New York, Basel, Second Edition, (2000).

[2] E. Ahmed, A.M.A. El-Sayed, H.A.A. El-Saka, G.A. Ashry, *On applications of Ulam-Hyers stability in biology and economics*, arXiv:1004.1354 [nlin.CD], pp.1-6, (2010).

[3] H. Amann *Linear and Quasilinear Parabolic Problems* Vol. I. Abstract Linear Theory, Monographs in Mathematics, 89, Birkhäuser, (1995).

[4] L. Backes, D. Dragičevic, K.J. Palmer *Linearization and Hölder continuity for nonautonomous systems*, Journal of Differential Equations, 297, 536-574, (2021).

[5] L. Backes, D. Dragičevic, *A generalized Grobman–Hartman theorem for nonautonomous dynamics*, Collectanea Mathematica, Springer, (2022).

[6] D.D. Bainov, S.I. Kostadinov and N.V. Minh, *Dichotomies and Integral Manifolds of Impulsive Differential Equations*, Science Culture Tech. Publishing, Singapore, (1994).

[7] L. Barreira, C. Valls *Stability of Nonautonomous Differential Equations. Lecture Notes in Mathematics*, vol. 1926, Springer, Berlin, Heidelberg, (2008). DOI: 10.1007/978-3-540-74775-8.

[8] L. Barreira and C. Valls *A simple proof of the Grobman-Hartman theorem for nonuniformly hyperbolic flows*, Nonlinear Analysis, Theory Methods Appl., 74, No. 18 (2011), 7210-7225.

[9] L. Barreira, C. Valls *A Grobman-Hartman theorem for nonunifomly hyperbolic dynamics*, J.Differential Equations 228 (2006), 285-310.

[10] L. Barreira, C. Valls *A Grobman-Hartman theorem for general nonuniform exponential dichotomies*, J.Func. Anal. 257 (2009), 1976-1993.

[11] J. Bernoulli, *Explicationes, Annotationes and Additiones ad ea, quae in Actis sup. de Curva Elastica, Isochrona Paracentrica, and Velaria, hinc inde memorata, and paratim controversa legundur; ubi de Linea mediarum directionum, alliisque novis*, Acta Eruditorum, (1965)

[12] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*. An Introduction with Applications, Birkhäuser, Boston, Basel, Berlin, 2001.

[13] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*. Birkhäuser, Boston, Basel, Berlin, 2003.

[14] P. Bohl, *Über einige Diferentialgleichungen allgemeinen Character*, welche in der Mechanik anwendbar Sind. Dorpat, (1900). (in Russian).

[15] W.A. Coppel, *Dichotomies in Stability Theory*, Springer-Verlag, Berlin Heidelberg New York (1978).

[16] W.A. Coppel *Dichotomies in Stability Theory. Lecture Notes in Mathematics*, vol. 629, Springer-Verlag, Berlin, Heidelberg, New York, 1978. DOI: 10.1007/BFb0067780

[17] J.J. DaCunha *Lyapunov Stability and Floquet Theory for Nonautonomous Linear Dynamic Systems on Tim e Scales* Dissertation, Baylor University, Texas 2004.

[18] Ju.L. Daleckii, M.G. Krein *Stability of Solutions of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, RI, (1974).

[19] B.P. Demidovch *On bounded solutions of a certain nonlinear system of ordinary differential equations*, Matematicheskii Sbornik, vol. 40, no. 1, 73-94, (1956) (in Russian)

[20] B.P. Demidovich *Lectures on stablity theory*, Nauka, Moscow, (1967) (in Russian).

[21] J. Dugundji, A. Granas *Fixed point theory*, I, vol. 61 Monogra

e Matematyczne, Mathematical Monographs, Warsaw, (1982).

[22] S. Elaydi *An Introduction to Difference Equations*, Texas, USA, (2000).

[23] J.L. Fenner and M. Pinto, *On a Hartmans linearization theorem for a class of ODE with impulse effect*, Nonlinear Anal., Theory Methods Appl. 38, No. 3 (1999), 307-325.

[24] M. Fréchet *Sur Quelques Points Du Calcul Fonctionnel* Doctoral thesis, Rendiconti del Circolo Matematico Palermo, (1906).

[25] D. Grobman *Homeomorphisms of systems of differential equations*, Dokl Akad. Nauk SSSR, 128, 880-881, 1959.

[26] D.M. Grobman, *Topological classification of neighbourhoods of a singularity in n-space*,Mat. Sb., 56, No. 1 (1962), 77-94.

[27] P. Hartman *A lemma in the theory of structural stability of differential equations.*, Proc. A.M.S., 11, 610-620, 1960.

[28] P. Hartman, *On the local linearization of differential equations*, Proc. Amer. Math. Soc., 14, No. 4 (1963), 568-573.

[29] F. Hausdorff *Grundzüge der Mengenlehre* Leipzig, (1914).

[30] S. Hilger *A dimensional chain calculus with application to behavioral manifolds* Doctoral thesis, University Würzburg, (1988).

[31] S. Hilger *Generalized theorem of Hartman-Grobman on measure chains*, J. Aust. Math. Soc. Ser. A 60, No. 2 (1996), 157-191.

[32] S. Hilger *Analysis on measure chains. A unified approach to continuous and discrete calculus.* Results in Mathematics 18,no.1-2, 18-56, (1990).

[33] A. Jeffrey *Applied partial differential equations*, Academic Press,USA, (2003).

[34] L. Jiang *Generalized exponential dichotomy and global linearization*, Math. Anal. Appl., 315, No. 2 (2006), 474-490.

[35] L. Jiang, *Ordinary dichotomy and global linearization*, Nonlinear Anal., Theory Methods Appl., 70, No. 7 (2009), 2722-2730.

[36] T. Kato *Quasi-linear equations of evolution, with applications to partial differential equations*, Spectral theory and differential equations, pp. 25-70. Lecture Notes in Math., Vol. 448, Springer, Berlin, (1975).

[37] W.G. Kelley and A.C. Peterson, *Difference equations.* . An Introduction with Applications,Second edition (2001).

[38] M.A. Khamsi and W.A. Kirk *An Introduction to Metric Spaces and Fixed Point Theory*. John Wiley and Sons, Inc. (2001).

[39] P.K. Kythe, *Green's functions and linear differential equations* Theory, Applications, and Computation, University of New Orleans Louisiana, USA, (2011).

[40] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations* in: Ser. Modern Appl. Math., Vol. 6, World Scientific, Singapore, (1989).

[41] R.J. LeVeque *Finite Difference Methods for Ordinary and Partial Differential Equations*, Seattle, Washington, (2007)

[42] I. Newton *Methodus Fluxionum et Serierum Infinitarum (The Method of Fluxions and Infinite Series)*, published in 1736, Opuscula, 1744, Vol. I. p. 66, (c.1671)

[43] L.C. Maximon *Differential and Difference Equations*,Comparison of Methods of Solution, Washington, 2016

[44] V.D. Milman, A.D.Myshkis *On the stability of motion in the presence of impulses* (in Russian) Siberian Mathematical Journal, Vol. 1, No. 2, pp. 233237, (1960).

[45] D.B. Pachpatte, *On a nonstandard Volterra type dynamic integral equation on time scales*, Electronic Journal of Qualitative Theory of Differential Equations, 72, (2009), 1-14.

[46] K.J. Palmer *A generalization of Hartman's linearization theorem*, Math. Anal. Appl., 41, No. 3 (1973), 752-758.

[47] L. Perko *Differential Equations and Dynamical Systems* Third Edition, Springer-Verlang, New York, (2001)

[48] O. Perron *Die Stabilititätsfrage bei Differentialgleichungen*, Math. Z., 32(5):703-728, (1930), https://doi.org/10.1007/BF01194662

[49] Ch. Pötzsche *Topological decoupling, linearization and perturbation on homogeneous time scales*, J. Differential Equations 245, No. 5 (2008), 1210-1242.

[50] C. Pötzsche *Exponential Dichotomies for Dynamic Equations on Measure Chains*, Nonlinear Anal. TMA, vol. 479, no. 2, 873-884, (2001), DOI: 10.1016/s0362-546X(01)00230-9.

[51] C.C. Pugh *On a theorem of P. Hartman*, Amer. J. Math., 91, No. 2 (1969), 363-367.

[52] A. Reinfelds, *On generalized Grobman-Hartman theorem*, Latv. Mat. Ezhegodnik, 29 (1985), 84-88 (in Russian).

[53] A. Reinfelds *Grobman's-Hartman's theorem for time dependent difference equations*, Latv. Univ. Zinātniskie Raksti, 605 (1997), 9-13.

[54] A. Reinfelds *Asymptotic equivalence of difference equations in Banach space*, Theory and Applications of Difference Equations and Discrete Dynamical Systems, Springer Proc. Math. Statist.,, 102 (2014), pp. 215-222, doi:10.1007/978-3-662-44140-4−12

[55] A. Reinfelds *A reduction theorem for systems of differential equations with impulse effect in a Banach space*, J. Math. Anal. Appl, 203, No. 1 (1996), 187-210.

[56] A. Reinfelds and L. Sermone *Equivalence of differential equations with impulse action*, Latv. Univ. Zināt. Raksti 553 (1990), 124-130 (in Russian).

[57] A. Reinfelds and L. Sermone, *Equivalence of nonlinear differential equations with impulse effect in Banach space*, Latv. Univ. Zināt. Raksti 577 (1992), 68-73.

[58] A. Reinfelds and D.Šteinberga *Dynamical equivalence of impulsive quasilinear equations*, Tatra Mt. Math. Publ., 63, (2015), 237-246.

[59] A. Reinfelds, D.Šteinberga *Dynamical Equivalence of Quasilinear Dynamic Equations on Time Scales* J. Math. Analysis, vol. 7, no. 1, 115-120, (2016). http://www.ilirias.com

[60] A. Reinfelds and D.Šteinberga *Dynamical equivalence of quasilinear equations*, Int. J. Pure Appl. Math., 98, No. 3 (2015), 355-364.

[61] A.M. Samoilenko, N.A. Perestyuk *Impulsive Differential Equation* World Sci. Ser. Nonlinear Sci. Ser. A, Vol. 14, World Scientific, Singapore, (1995).

[62] L. Sermone *Equivalence of linear differential equations with impulse effect*, Proc. Latv. Acad. Sci. Sect. B, No. 2(559), (1994), 78-80.

[63] D. Speiser *Discovering the Principles of Mechanics 1600-1800*,p.191 Birkhäuser, (2008)

[64] S. Sternberg *On local C^n contractions of the real line*, Duke Math. J.,24 (1957), 97-102.

[65] G. Strang *Differential equations and linear algebra*, Wellesley-Cambridge Press, United States of America, 2014

[66] S.M. Ulam *A Collection of the Mathematical Problems. Interscience*, New York, (1960).Ulam S.M, A Collection of the Mathematical Problems. Interscience, New York, 1960.

[67] Y. Xia, X. Chen and V. Romanovski, *On the linearization theorem of Fenner and Pinto*, J. Math. Anal. Appl., 400, No. 2 (2013), 439-451.

[68] Y. Xia and X. Chen *Topological conjugacy between two kinds of nonlinear differential equations via generalized exponential dichotomy*, Int. J. Differ. Equ., (2011), doi: 10.1155/2011/871574.

[69] Y. Xia, Y. Gao, X. Yuan and P.J.Y. Wong, *Linearization of impulsive differential equations with ordinary dichotomy*, Abstr. Appl. Anal., (2014), doi: 10.1155/2014/632109.

[70] Y. Xia, Y.G.X. Yuan and P.J.Y. Wong, *Linearization of nonautonomous impulsive systems with nonuniform exponential dichotomy*, Abstr. Appl. Anal., (2014), doi: 10.1155/2014/860378.

[71] Y. Xia, *Linearization for systems with partially hyperbolic linear part*, Bull. Malays. Math. Sci. Soc. (2) 37, No. 4 (2014), 1195-1207.

[72] Y. Xia, J. Cao and M. Han *A new analytical method for the linearization of dynamic equation on measure chains*, J. Differential Equations 235, No. 2 (2007) 527-543.

[73] Y. Xia, J. Li and P.J.Y. Wong *On the topological classification of dynamic equations on time scales*, Nonlinear Analysis: Real World Applications 14, No. 6 (2013) 2231-2248.

[74] Y.H. Xia, X. Yuan, K.I. Kou ,P.J.Y.Wong *Existence and Uniqueness of Solution for Perturbed Nonautonomous Systems with Nonuniform Exponential*

Dichotomy. Abstract and Applied Analysis, vol. (2014), Article ID 725098, 10 pages. DOI: 10.1155/2014/725098

[75] J. Zhang, M. Fan and X. Chang *Nonlinear perturbations of nonuniform exponential dichotomy on measure chains*, Nonlinear Anal., Theory Methods Appl., 75, No. 2 (2012), 670-683.

[76] J.Zhang , M. Fan ,H. Zhu *Existence and Roughness of Exponential Dichotomies of Linear Dynamic Equations on Time Scales*, Comput. Math. Appl., vol. 59, 2658-2675, (2010), DOI: 10.1016/j.camwa.2010.01.035.

[77] J. Zhang, Y. Song, Z. Zhao *General Exponential Dichotomies on Time Scales and Parameter Dependence of Roughness. Advances in Difference Equations*, vol. (2013), Article number 339, 2013. DOI 10.1186/1687-1847-2013-339.

[78] J. Zhang, M. Fan, H. Zhu *Necessary and Sufficient Criteria for the Existence of Exponential Dichotomy on Time Scales*, Comput. Math. Appl., vol. 60, 2387-2398, (2010), DOI: 10.1016/j.camwa.2010.08.034.

[79] J. Zhang, M. Fan, X. Chang *Nonuniform (*h, k, μ, ν*)-dichotomy with applications to nonautonomous dynamical systems*, J. Math. Anal. Appl., vol. 452, no. 1, 505-551, (2017), DOI: 10.1016/j.jmaa.2017.02.064.

[80] Y. Zhang, X. Tian *Time-scales Herglotz type Noether theorem for delta derivatives of Birkhoffian systems*, Royal Society open science 6, (2019), 191-248.